



H.P. Sculp^t.

Multi pertransibunt et augebitur Scientia.

Bacon.



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SECOND VOLUME
OF THE
INSTRUCTIONS
GIVEN IN THE
DRAWING SCHOOL

ESTABLISHED BY THE
DUBLIN-SOCIETY,

Pursuant to their RESOLUTION of the Fourth
of FEBRUARY, 1768;

To enable Youth to become PROFICIENTS in the different
Branches of that Art, and to pursue with Success, GEOGRA-
PHICAL, NAUTICAL, MECHANICAL, COMMERCIAL, and
MILITARY STUDIES.

Under the direction of JOSEPH FENN, heretofore Professor of
PHILOSOPHY in the University of NANTS.

*Quid munus Reipublicæ majus aut melius offerre possumus, quam si Ju-
ventutem bene Erudiamus?*
CICERO,

DUBLIN:

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SECOND VOLUME

INSTRUCTIONS

DRAWING SCHOOL

DUBLIN SOCIETY

Printed to their Resolution of the Fourth
of February 1768;



To enable Youth to be instructed in the different
Branches of the Arts, viz. Surveying, Geometry,
Drawing, Writing, and other useful Arts, and
to promote the Education of the Poor.

Printed by J. O'SHEA, at the University of Dublin.

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DUBLIN

Printed by Alex. McCulloch, in Fleet Street.

HISTORY OF MATHEMATICKS,

Containing an Account of the Progress of MATHEMATICKS from the Origin of that Science to the present Time; wherein are exhibited the principal mathematical Discoveries, the Disputes they have given Rise to, with an interesting Detail of the Lives of the most eminent MATHEMATICIANS:

CHAR. I.

Preliminary Discourse on the Nature, Divisions, and Utility of Mathematicks.

THE Ancients are generally thought to have called this Science, *Matheſis*, or *Mathemata*, by which is meant the Sciences, on Account of the luminous Certitude that characterises it above all the other Branches of human Knowledge.

This Etymology does so much Honour to this Science, that we could wish to establish it on a more solid Foundation; it must be confessed, that it is not supported by the Testimony of any Author of Antiquity (a), and without Doubt is the Conjecture of some modern Panegyrist. *Proclus*, who gives such great Encomiums to Mathematicks (b), and who relates with such Care the Sentiments of his Predecessors concerning their Nature, Divisions, &c. says nothing of this Etymology, but gives a metaphysical Derivation of it, more subtle than solid (c), which we therefore omit.

Some among the Moderns, not satisfied with this Etymology, have given another, which they thus account for: That at the Time Mathematicks

(a) Scapula in his Greek Dictionary, at the word *μαθημα*, cites *Philon*, or the author of the book *De Mundo*, in proof of this ety-

mology, but no such thing is found there.

(b) Comm. in 1 Euc. l. v.

(c) Ibid c. 15.

took its Rise, and for some Ages after, they were the only Sciences taught in the Schools of Philosophers. Rhetorick, Dialectick, Grammar, Morality, which in subsequent Ages had so great a Share in the Improvement of the Mind, being then unknown; Mathematicks and Natural Philosophy alone occupied the human Understanding; the latter being always preceded by the Study of the former, as being the Avenue and Introduction to it. it is well known that in the School of *Pythagoras*, the Class of Mathematicians preceded that of Natural Philosophers, that *Plato*, in Times much posterior, excluded from his Physical and Metaphysical Lectures, those who were not initiated in Geometry: it was, in fine, what gave Occasion to the Speech of *Xenocrates* to one, who, ignorant of Geometry and Arithmetick, attended his Lectures, withdraw, said the Philosopher to him harshly *Anfas Philosophia non habes* (d).

Hence it appears that the Study of Mathematicks was the first pursued in the Schools of Philosophers; and from this Priority of Time, not from its superior Perfection, say the Authors we mentioned (e), has this Branch of Knowledge been called *Mathefsis*, that is, *Science*, or rather Instruction.

II.

As to the Nature of Mathematicks, they may be defined, the Science of *the Relations of Magnitude and Number, that all Things susceptible of Augmentation or Diminution can have to each other.*

Thus Geometry, for Example, considers the Relations of the different Parts of Extension; since to *measure*, which is the principal Object of Geometry, is nothing more than to determine the Ratio of a certain Portion of Extension to another taken as a fixed Measure. Astronomy is wholly employed in discovering the Order and Position of the Stars, that is, their Distances from each other, in computing the Times in which they perform their Revolutions, in foretelling their Conjunctions, Oppositions, &c. Mechanicks, in comparing the Weights and Motions of Bodies with one another, in calculating the Powers acting in opposite Directions, &c. In all those Considerations, there are Relations of Magnitude, and to those alone Mathematicks are confined. If the Mind attempts to reason on the Nature of the Stars, of Extension, of Motion, or on the Cause of Gravity, &c. it refers those Researches to Physicks.

III.

Mathematicks are naturally divided into two Classes; one comprehending pure or abstract, the other, mixed Mathematicks. The first considers the Properties of Quantities after an abstract Manner, precisely as it is susceptible of Augmentation or Diminution; and as the Mind distinguishes two

(d) Diog. Laert. in *Xenocrat.*

(e) Ramus *Præm in math.* Barrow *lect. math. lect. i.*

Sorts of Magnitude, one consisting in *Number* or *Multitude*, the other in *Space* or *Extension*, thence arise the two principal Branches of the first Division, viz. Arithmetick and Geometry. Numbers are the Object of the former; Extension, its Relations, its Measure, that of the latter.

Mixed Mathematicks consist of certain Parts of Physicks, to which abstract Mathematicks may be applied: thus in Opticks, the Effects and Properties of Light are investigated by Means of certain Principles, which reduce their Consideration to pure Geometry. It is first assumed as a Principle, that the Rays of Light are propagated in straight Lines, if no Obstacle is opposed to their Passage; that they are reflected making the Angles of Reflection equal to those of Incidence; that passing thro' one Medium into another of different Density, they are deflected from their first Direction, observing however in their Deflections a certain geometrical Law. These Principles once established, the Mathematician examines no further the Nature of Light, or of the Mediums it passes through, or that reflect it. He considers the Rays of Light as straight Lines, the reflecting or refracting Mediums as purely Mathematical Surfaces, the Form of which is alone attended to. Thus he determines the Path of the Rays of Light on Mirrors, through Optical Glasses, their Effects on the Sight, &c. It cannot be denied, but that those Researches are, properly speaking, Physical; but as they are intimately connected with and dependant upon abstract Mathematicks, from which they derive their Certainty, they are in some Sense raised thereby to the Rank of Mathematicks, of which they form the second Division. In this Respect, they hold a middle Place between Enquiries purely Physical, which are commonly involved in Obscurity, and pure Mathematicks, which shine with unclouded Perspicuity. They cannot have a greater Degree of absolute Certainty, than the Principle upon which they are founded, and in this View they are Physical: on the other Hand, they have an hypothetical Evidence equal to that of abstract Mathematicks, that is, their Principle being supposed true, they are not less certain than this latter. They have even the Advantage of having a Kind of metaphysical Certainty, tho' their Principle is not existent in Nature, provided that Principle is not repugnant to Reason. What *Archimedes* has demonstrated concerning the Ratio of Weights that are in Equilibrio at the Extremity of a Balance, is equally true whether the Directions of heavy Bodies are parallel or converge to a Point; in the latter Case only, the Theory of *Archimedes* can be applied to Weights that gravitate on the Surface of the Earth, but it is likewise to those which gravitate, or are conceived to gravitate, in parallel Lines, which is not metaphysically impossible, and by Means of this Principle purely hypothetical, and that does not take Place in the present Order of the Universe. The Mathematician of *Syracusa*, squared the Parabola. The Physico-mathematical Discoveries of *Newton*, on the Form of the Orbits, that the Planets should describe, ac-

according to the different Laws of Attraction, would not be less true, tho' it had been demonstrated that this Attraction did not exist, they would then stand in the same Rank with the Properties of a Triangle or Circle, if none such really existed in Nature.

From what has been said concerning mixed Mathematicks, it follows, that their different Branches cannot be fixed and determined, as those of abstract Mathematicks. In Proportion as experimental Philosophy acquiring new Riches, has ascertained certain Facts that may serve as first Principles, mixed Mathematicks have been extended. This the illustrious Chancellor Bacon observed, with that Sagacity wherewith he foresaw the future Improvement of Natural Knowledge: *Prout Physica, says he (f), majora in dies Incrementa capiet, et nova Axiomata educt, eo mathematica nova Opera in multis Indigebit, et plures demum fient Mathematicæ mixtæ.*

It is therefore no way surprising, that mixed Mathematicks made so slow a Progress among the Ancients, whilst abstract Mathematicks advanced so rapidly by a Variety of important Discoveries. The human Mind need only descend into itself to improve in pure Mathematicks, but to advance in the other, a quite contrary Method is to be pursued; it requires a Series of Experiments and Observations: in this the ancient Philosophers were deficient; in general they neglected Observation too much, they laid too great a Stress on Reasoning and Metaphysics. Excited by an impatient, and, after all, a very excusable Curiosity, they wanted to explain Nature before they were acquainted with her first Operations: thus the Structure they had raised, like that erected by ignorant Architects on a weak Foundation, soon fell to the Ground.

The successive Rise of the different Branches of Mathematicks, confirm the foregoing Remark: The *Pythagoreans* divided this Science into four Parts only, two of the abstract and two of the mixed Mathematicks; the two latter were Musick and Astronomy. Already the Observations of *Pythagoras* upon Sound, with those made from Time to Time on the celestial Phenomena, joined to some Hypotheses contrived to explain and calculate the Motions of the Stars, afforded an Opportunity of employing pure Mathematicks in Physical Enquiries. The Extent of those Sciences was not much greater in the *Platonick* School; their Division into Geometry, Stereometry, Arithmetick, Musick, and Astronomy (g), was injudicious, and comprehended no more than that of the *Pythagorean* School: in Effect, the two first Divisions are only a Subdivision of Geometry. However, pure Mathematicks were considerably improved by the *Platonicks*; but as those Philosophers were too much addicted to Contemplation, they were less successful in physical Enquiries. It does not appear that they established one Fact that

(f) *De Augmento Scient. lib. 2. cap. 6.* (g) *Plat. Dial. 7 de Rep. Theophrastus in Loca Math. Platonis.*

could serve as a leading Principle to a new Science, if we except perhaps the rectilinear Propagation of Light, and the Equality of the Angles of Incidence and Reflection.

However that may be, it seems Opticks and Mechanicks were not ranked among the mathematical Sciences 'till a long Time after; and that about the Time of *Aristotle*, when they had at length traced out some of the Laws of the Propagation of Light, of Vision, and the Equilibrium of Bodies, the Questions of this Philosopher concerning Mechanicks, some of his Problems, the Treatise of Opticks attributed to *Euclid*, seem to have been the first Rudiments of these Sciences. The general System of Mathematicks was then composed of six Parts, Geometry and Arithmetick, Musick and Astronomy, Opticks and Mechanicks; no other Branches were known to the Ancients.

The Moderns, by cultivating natural Philosophy with Success, have reduced a great Number of other Subjects to Geometry, scarcely known to the Ancients: Opticks, among them, consisted only of a very simple Theory of the Illumination of Bodies; Catoptricks, or the Science of reflected Light; and a few Principles of Perspective. The Science of Vision or direct Opticks, they were ignorant of; Dioptricks likewise as yet were unknown. It is not much above a Century and a half ago since the Principle on which they are entirely established has been discovered, as also that which serves as a Foundation to direct Opticks; those two Branches of a Science so far extended at present, owe even their first Outlines to modern Discoveries.

It is also but lately since Mechanicks emerged from that feeble State in which they were transmitted to us by the Ancients. Confined then to the Science of Equilibrium, they only included what we now call Staticks and Hydrostaticks, in which the Equilibrium of Bodies is only considered. At present, Mechanicks are the Science of Motion in general; and what that Science in former Ages totally consisted in, is now only a small Part of it. Is Motion impeded by a contrary Resistance, which, without destroying the Tendency, annihilates the Effect and produces an Equilibrium? This is the Mechanicks of the Ancients. Do we consider the actual Motion in Bodies, the Phenomena resulting from their Concourse and Collisions, the Paths they describe, and the Velocities with which they move when acted upon by a Combination of different Powers, the Resistance of Fluids to Bodies that move in them, &c. these are the Subject of Dynamicks. Thus all the Parts of Mathematicks, without changing their Names, comprize, at this Day, Objects more vast and extensive; and each of them have sent forth a Number of Scions, which cultivated with Care by the Moderns, soon surpassed the Stock from whence they took their Rise.

A principle Part of our Plan being to unfold the whole System of Mathematicks, and to give a clear Idea of its different Branches, in order to exhibit the Progress of the human Mind in this considerable Part of natural Knowledge, it will not appear unnecessary to trace its metaphysical Generation.

Bodies are endued with several Properties, as Extension, Impenetrability, &c. but of all these Properties, that which seems to hold the first Rank, and that without which the others could not exist, and which is equally observed by those the least accustomed to reflect as by the most sagacious, is Extension: it does not require much Skill in Metaphysicks to form this Idea, to distinguish its different Species; tho' physically inseparable. The least informed know well how to distinguish in a Globe of any Size, Matter or Colour, what constitutes it a Globe, and not a Cube, or a Pyramid. Should you speak of the Extent of a Plane, the Mind naturally lays aside the Idea of Depth, and only annexes to it the Idea of Length and Breadth.

Is the Distance of two Objects considered, the Mind attends only to the Length; it even goes further, and excludes every Idea of Extension from the two Terms of this Distance. Such is the Origin of mathematical Points, Lines, and Surfaces, the Subject of so many groundless Objections made by Persons, who, ignorant of Metaphysicks or Partisans of Scepticism, have strove to raise Doubts concerning the Solidity of Mathematicks.

Bodies therefore considered only with Relation to their Extension, will be the Limit to which the Mind can arrive, when by a natural Impulse it analyses the Objects of its Research. Thus, Extension and the Figure that bounds it, will be necessarily the first Considerations that Men will attend to, when they examine the Nature of Bodies that surround them. They will begin by comparing them under those two Points of View, which alone can be the Subject of Consideration, in consequence of the Abstraction that lays aside all the other Qualities that may serve as a Foundation to any Comparison: such is the metaphysical Origin of Geometry.

The Idea of Multitude or Number, is not less natural to Man than that of Extension. Surrounded by distinct Objects, more or less Numerous, this Idea is every Instant suggested to us by our Senses; besides, at the same Time that the Mind gets the Idea of Space, conceives it divided into Portions of different Forms, and compares them with one another, it acquires the Idea of Number; hence arises the Division of Quantity into discrete and continued. Quantity considered as divided into Parts, more or less in Number, is the Object of *Arithmetick*; considered as extended and inclosed by Boundaries, is the Object of *Geometry*, some Divisions of which we shall proceed to point out.

Among the different Dimensions of Bodies, there are some more simple

than others: straight Lines are more simple than Curves, and of those the Circle is the least complex. In like manner, plain Surfaces, bounded by straight or circular Lines, Solids bounded by those Surfaces, are the most simple of their Kind. These Subjects, therefore, of Consideration, should have paved the Way to more difficult Researches; they are the Object of *elementary Geometry*. *Transcendental Geometry* is the Part of this Science by far the most extensive, which treats of curvilinear Figures of a more elevated and abstruse Nature, such as the Conic Sections, and an infinite Number of others, to the Theory of which the former serves as an Introduction. Figures may be considered as Spaces endued with certain Properties; or the Spaces may be analyzed and resolved, if we may so express ourselves, into the infinitely small Elements of which they are composed; whence arises the Division of transcendental Geometry into *Finite* and *Infiniteesimal*. The Speculations of the Ancients and Moderns on the Theory of Curves, furnish an Example of the former; their Researches respecting the Measure of Curves, Researches which commonly proceed only by considering the Law according to which their Elements increase or decrease, form the latter.

The Mind, after being some Time occupied in Researches purely geometrical, Researches so much the more pleasing, because always accompanied by a pure and luminous Evidence, is soon forced, either by Necessity or Curiosity, to return to the natural World. The Motions of Bodies, and their mutual Efforts, occasioned by their impenetrability, are the first Objects that claim his Attention, and thus give Rise to the most important and useful Branch of mixed Mathematicks, to wit, *Mechanicks*.

We conceive in Bodies considered as moveable, either a simple Tendency to Motion, a Tendency counteracted by contrary Efforts, or the Motion itself. From the first Consideration arises *Statics*, which is divided into *Statics*, properly so called, when it relates to Solids, and *Hydrostatics* when it relates to Fluids. If Bodies are considered in Motion, that Science is called *Dynamics*, which is divided in the same Manner as the former, into *Dynamics* and *Hydrodynamics*. From Dynamics a Variety of Theories arise, as the Laws of the Motion and Collision of Bodies, the Theory of central Forces, Balistic, Theory of Oscillations, &c. Many Sciences are only a particular Application of Dynamics, such as the Theory of the Direction and Motion of Water, *Navigation*, or the naval Science, in as much as it is the Art of conducting a Ship by the Aid of mechanick Powers which put it in Motion, as the Oars, the Sails impelled by the Wind, the Rudder, &c.

Next to the Knowledge of these Sciences requisite to supply our Wants, Astronomy is the most pleasing. The Motions of the heavenly Bodies are so regular, that they must have ever drawn the Attention of Men capable of Reflection. The human Mind was soon prompted to investigate the

Cause and the different Relations of the heavenly Motions. I call, with *Kepler (h)*, *Spherical Astronomy*, that whose Object is the Phenomena arising from that seeming true Supposition, that the Earth is in the Centre of a Sphere, in whose Surface the Stars are placed; this is the first Branch of Astronomy: the second is *Theoretical Astronomy*, wherein it is proposed to trace the different Relations of Position, Distance, and Velocity, of the heavenly Bodies, that is to say, to discover the real Form of the Universe. From Astronomy arise some subordinate Sciences, such as *mathematical Geography*, wherein is determined the Figure of the Earth, and the Position of the principal Places by Observation. *Navigation*, or the Art of conducting a Ship across the Seas by the Observation of the Stars alone. *Gnomonicks*, or the Art of dividing Time, and marking its Divisions by Help of the heavenly Bodies, but particularly by the Shadow projected by Bodies exposed to the Sun. *Chronology*, or that Part of the Science concerning Time, which consists in regulating the Method of counting it, by making the civil Period coincide as near as possible with those of the Sun and Moon.

The Phenomena of the propagation of Light, that is to say, of the Motion whereby it is transmitted from the luminous to illuminated Bodies, or from those to our Eyes, have given Rise to *Opticks*. The first Observation upon the Rays of Light, is, that they are propagated in straight Lines as long as they remain in the same Medium. After this Manner we generally perceive Objects, and the Sensation we thence receive is variously modified according to the Circumstances of their Distances, their Positions, &c. those Considerations form what we call *direct Opticks*. It would be natural to rank under this Head, *Perspective*, which is no more than the Art of representing on a Surface those Degradations of Form and Magnitude observed in Objects that surround us, and all its Rules are solely founded on the Principle of the rectilinear propagation of Light.

But Light is only propagated in straight Lines, when its Motion is not impeded by any Obstacle; if it meets an opaque Body in its Passage, it is reflected, and if the Surface of the Body be smooth, the Rays of Light move on, making the Angle of Reflection, equal to that of Incidence; if the Body it meets is transparent, and more or less dense than the first Medium, passing into this Body, it moves in a Direction more or less Oblique than before, which is called Refraction. From the former Observation, the various phenomena of Mirrors are deduced, from the latter, those of Glasses and Instruments for remedying the Defects of Sight. The two Sciences that are taken up with these Objects are called *Catoptricks* and *Dioptricks*.

Acausticks are nearly with regard to Sound, what *Opticks* are with regard to Light, but is not near so rich as the former in certain and incon-

(h) *Epitome Astron. Copern.* p. 14.

testable Discoveries, the Reason of which appears from the Difficulty of reducing its Principle to the Simplicity of a Supposition purely mathematical. This Principle is that of the Vibrations of the elastic Particles of the Air, which, it is obvious, is complicated with several physical Difficulties.

We may refer to this general Division, Musick, that enchanting Art of ravishing the Ear by the Harmony and Succession of Sounds; it is founded on a Principle partly discovered formerly by *Pythagoras*, partly in our Days by *Rameau*. Not that we pretend, that by the Aid of mathematical Rules alone agreeable Musick may be composed. No without Doubt, Harmony mathematically exact, may not be very pleasing: To Genius, to Taste, it belongs to select the various Tones best suited to the Subject proposed; and the Musicians who have treated this Art mathematically, have pretended no more, than to assign the Reasons of certain Phenomena observed either in Melody or Harmony.

To avoid Prolixity, we shall only point out the other Parts of Mathematicks, the Consideration of the Relations of the Gravity, Elasticity, Density of the Air, and other Fluids endued with these Properties, has been called by some Moderns, *Pneumaticks*. Calculation applied for determining the Probability of Events, has produced the *Art of Conjecturing*, of which the *Doctrine of Chances* is one of the principal Branches. Pure Geometry, applied to the Art of cutting Stones in such a Form, as from their Union certain Works of Architecture may result, compose what is called Stone-cutting; in general, the Symmetry necessary to be observed, whether in rearing Edifices for Defence as well as Ornament, is altogether owing to the mathematical Sciences.

v.

We have already observed more than once, that all the Parts of mixed Mathematicks are intimately connected with abstract Mathematicks, of which they are only particular Applications. This is an Observation proper to be insisted upon, for the Advantage of those who, desirous of acquiring a solid and extensive Knowledge of these Sciences, might mistake the Road to attain it, or would be desirous to know it: With this View, we shall endeavour to point out clearly their Connection and mutual Dependence.

Every Question in mixed Mathematicks is reducible to a Problem of pure Geometry, by stripping it of some physical Circumstances immaterial to its Solution, as will appear by the following Example. In Gnomonicks, as it is well known, it is required to find the Position of the Shadow projected at the different Hours of the Day, by a Style parallel to the Axis of the World, on a Surface whose Position is given. A slight Knowledge of the Sphere is sufficient to shew, that the Hours are determined by the Position of the Sun in the twelve horary Circles that divide his diurnal Revolution into 24 equal Parts, and that these Circles intersect each other in the same Line; it is further observable, that the Style fixed in a proper Position, that

is to say, parallel to this Axis, sensibly coincides with it, and would so in effect if we were at the Centre of the Earth; and our Distance from it, when compared with that of the Sun, is so inconsiderable, that this Supposition may be allowed. Lastly, it is manifest that the Shadow of the solid Axis fixed in the common Intersection of all the horary Planes, is in the same Plane with the Sun and this Axis, the Shadow therefore projected by this Axis, is only the horary Plane produced; hence the Problem for determining the Position of this Shadow, is reduced to the following: *A certain Number of Planes that cut each other in the same Line and in the same Angles every Way, being given; to find their Intersection with a Surface whose Form and Position are also given.*

Now it is easy to perceive, that this is merely a geometrical Problem; and whilst he who is unacquainted with, or indifferently skilled in Geometry, takes great Pains to learn the practical Rules of Gnomonicks and their Reasons, the intelligent Geometrician finds in Himself those Resources; he solves the Question, and invents and frames practical Methods of Solution.

The same may be said of Perspective; this Branch of Opticks consists in a Problem very easy to a Geometrician: it is required to determine on a Plane, whose Position is given, the Intersection of the different Lines conceived drawn from the Eye to the Outlines of the Figure, which is supposed to be placed behind the Plane. Little skill in Geometry suffices to resolve this Problem in its full Extent, whilst he that has made no Progress in it, is stopped every Instant, meets continually Difficulties which he is unable to solve; and we shall not hesitate to pronounce, that Geometry is the universal Key of Mathematicks: He alone can aspire to penetrate into those Sciences, who is Master of the first; any other will ever remain confined in a narrow Sphere, and in a State of Mediocrity.

VI.

Mathematicks were always held in great Esteem by the most celebrated Philosophers of Antiquity. We find, in Effect, that all those who were eminent for their Learning and the Purity of their Morals, cultivated those Sciences: I say, all those eminent for their Learning and the Purity of their Morals, being well aware that a sophist *Pratogaras*, a voluptuous *Aristippus*, an epicurean *Zenon of Sidon*, and some others of the same Stamp, have endeavoured to decry them; but the most respectable Characters have rendered them the Justice they deserve, such as *Thales*, *Pythagoras*, *Democritus*, *Anaxagoras*, and all the Philosophers of the *Ionian* and *Italick* Schools; in fine, *Plato*, *Xenocrates*, *Aristotle*, &c. It is well known that the former were indefatigable in promoting those Sciences in *Greece*; that *Plato* was one of the most eminent Geometricians of his Time, and that his Works are full of honourable Testimonies in Favour of Mathematicks. *Xenocrates*, one of his Successors, entertained no less high an Opinion of them; wit-

ness his Answer quoted above (i). The Principle of the Peripatetick School in his metaphysical Works frequently makes use of Examples taken from Geometry; which clearly evince that he considered the geometrical Method as the fittest to be employed in the Investigation of Truth: besides, it is well known that he wrote upon several mathematical Subjects. We find among the Antients only *Socrates*, whose Opinion can with any Shew of Reason be opposed to this general Suffrage in Favour of Mathematicks. This Sage, we must own, disapproved of too great a Curiosity to penetrate into those Sciences. When we know, says he, as much Geometry as is necessary to measure our Land, as much Astronomy as is requisite to point out the Hours and regulate Time, to guide us in our Journies, either by Sea or Land, we should not affect to know more (k).

We shall make some Remarks on those Expressions of *Socrates*, in order to obviate the Consequences that some might be apt to draw from them.

First then, does not this Philosopher make large Concessions, and even more than in Appearance he intended, by allowing us to cultivate Mathematicks as far as the Exigencies of Society may require? If the Circumstances of the Times in which he lived, rendered the Use of Mathematicks very confined, it is not so now; we no longer navigate thro' a narrow Sea, as they did at that Time; never more secure from the Dangers of Navigation, than when out of Sight of the Coasts, we steer thro' the Ocean, having, during a considerable Time, no other Intercourse but with the Stars: the Knowledge of the position of all those celestial Bodies is therefore necessary. It is requisite that the Geography of our Earth should be correct; this only can be effected by perfecting and increasing the astronomical Methods. If such Pains are now taken to improve the Theory of the Moon, and so great an Apparatus of Observations and Calculations employed for this Purpose, let it not be thought this is done merely to gratify Curiosity; tho' even that might be easily justified: It is with a View to procure to Navigators a certain and perfect Method of discovering, at all Times, the Place of their Situation. Thus we see a profound Knowledge in Astronomy becomes necessary, even according to the Judgment of *Socrates*. We chose Astronomy for an Example, because the Utility of this Branch of Mathematicks being less generally known, it might perhaps be considered as a vain and useless Science. What great Advantages do not accrue to Mankind from cultivating Mechanicks, Opticks, &c.

But we are principally to consider the Motives that induced *Socrates* to hold Mathematicks in so little Esteem. This Philosopher devoting himself entirely to the Study of Morality, was of Opinion (so difficult it is to observe a just Medium) that the sole Study of Man should be that which serves to make him better and more virtuous. We grant, it is the first and most

(i) Art. 1.

(k) Diog. in *Socrat. Xenoph. b. ix. de dic. & fac. Socr.*

essential Study; that, without moral Virtue, the most eminent Qualities deserve little Regard: yet must we not also allow, that it is too rigid to confine the human Mind to that Study alone. If it be necessary to furnish some Aliment to a Curiosity, too natural to Man, that to gratify it, should be looked upon as criminal, what can suit it better than the Study of Mathematicks. These Sciences, in effect, incapable of misguiding the Heart, whilst at the same Time they enlighten the Understanding, answer best this Purpose. *Socrates*, tho' his extreme Severity inspired him with little Esteem for them, however acknowledged they were highly useful in some Respects. If we believe *Plato*, he regarded them as very fit to strengthen the Faculties of the Mind: "Have you not observed, said he (1), that those who naturally count well, are endued with an Understanding capable of making a rapid Progress in all Arts; and that those who are slow and dull, become, after being exercised in Arithmetick, more quick and ready of Apprehension." And elsewhere (m), he seems to acknowledge the Utility of the Mathematicks in all Arts, which greatly softens his harsh Judgement, or, at least, renders it inconclusive: For it is incontestible, that Branches of Knowledge useful to Society, should be the Occupation of some Men endued with Talents and Genius sufficient to improve them; and that it were to be wished, that all could contribute by their Labours. In fine, it must be allowed, that a Study calculated to render the Understanding more ready to conceive, more capable of exercising the Faculties of Reasoning and Reflection, ought to form a considerable Part of the Education of all those who are intended for a Way of Life that requires the Exertion of those Faculties. The Testimony of *Socrates*, therefore, is no way unfavourable to Mathematicks.

We might collect in all Ages a Train of Suffrages no less honourable to these Sciences, than those of the Philosophers of Antiquity. If there were found even in those Days of Darkness that so long reigned in the West, some worthy of a more enlightened Age, who soared far above their Contemporaries, we observe that they cultivated Mathematicks. Such were the illustrious *Boetius*, *Cassiodorus* in the sixth Century; the venerable *Beda* and *Alcuin* his Disciple, Preceptor to Charles the Fifth in the eighth Century; *Ferbert* in the tenth; *Albertus Magnus*, *Roger Bacon*, and some others in the thirteenth Century: Those personages so much the more to be admired because they were able to make their Way through the ignorance and barbarism of their Age, those personages I say held Mathematicks in Esteem, and cultivated them with Ardour, witness *Roger Bacon*, in whose Writings we find the Seeds of so many important Discoveries; witness *Ferbert*, who, thirsting after the Knowledge of those Sciences, fled from his Convent to seek among the Arabians the Assistance which he could not find among Christians.

(1) In *Phedro* & in lib. VII. de *Repub.*

(m) In *Phil.*

Let us now pass to the Moderns ; we shall find that the most eminent Philosophers who have flourished since the Revival of Letters cultivated the Mathematical Sciences ; such was the illustrious Chancellor *Bacon*, that great Genius, who at a Time that Learning only began to dawn, traced the Road to be pursued for its Improvement. Mathematicks appeared to him indispensably necessary for the Restoration and Advancement of Physicks, the Study of which he so warmly recommends. Who does not know that *Gallileo*, *Torrilicelli*, *Descartes*, *Pascal*, &c. held the first Rank among the Mathematicians of their Time, and that they have enriched Physicks with several useful Discoveries. *Boyle*, the principal Restorer of experimental Philosophy, has often regretted (a) his not having cultivated Mathematicks ; however, it cannot be said that he was totally unacquainted with them, as sufficiently appears by his Works. But he was sensible that a more profound Knowledge of those Sciences would have been of signal Service to him. The same Genius to whom we are indebted for the noblest Improvements in Geometry, the great *Newton*, is Author of the most sublime physical Discoveries. It was reserved for the first of Mathematicians to analyze Light, and to discover and irrefragably demonstrate the Constitution of the World, and the Laws according to which it is maintained and preserved. The most illustrious Metaphysicians have likewise added their Suffrages to those we have already collected. *Mallebranche* thought he could not propose a better Example of the Manner of proceeding in the Investigation of Truth, than the geometrical Method (o). I shall finish by the Testimony of *Locke* (p), *I have mentioned*, says he, *Mathematicks as a Way to settle in the Mind an Habit of reasoning closely and in Train, not that I think it necessary that all Men should be deep Mathematicians, but that having got the Way of Reasoning, which that Study necessarily brings the Mind to, they might be able to transfer it to other Parts of Knowledge as they shall have Occasion.*

To avoid Prolixity, I omit several other Passages of *Locke* favourable to this Science. If the Authority of great Men is of any Weight, what Names might we oppose to the Enemies of Mathematicks, and to those Writers who from Time to Time have attacked them, so little versed in those Sciences, that from their first setting out they fall into the most gross Mistakes. The famous *Bayle* (q), who from a Propensity to Scepticism, was induced to say, that even Mathematicks had a weak Side, acknowledged however, that none but an able Mathematician could oppose those Sciences with Success, but we confidently affirm, that this Attack would in no Respect be disadvantageous to Mathematicks ; and that nothing would

(a) In *Consid. circa utilit. Phil. experim.* Exercit. vi.

(o) *Rech. de la Verité*, liv. 6. chap. 5. & *passim*.

(p) Of the Conduct of the Under, § 6, 7, &c.

(q) *Critical Dictionary*, art. of *Zenon de Sidon*.

and that nothing would contribute more to make the Enemies of Mathematicks retract, than a profound Study of the Truths it contains.

The Annals of Philosophy and of the human Mind furnish a Number of Traits honorable to Mathematicks. The physical Discoveries we are now in Possession of have for the most Part been made by Mathematicians, as we have shewn by the Examples of *Descartes*, *Paschal*, *Gallileo*, *Newton*, &c. on the contrary, if some useful Doctrines have met with Opposition, it arose principally from Persons unacquainted with mathematical Learning. The mechanical Discoveries of *Gallileo*, the Gravity of the Air were opposed only by Men who shewed they were destitute of this solid Branch of Knowledge. Who are those in the present Age who attack the mechanical and optical Discoveries of *Newton* but Men for the most Part ignorant of this Science? If we now take a View of the literary Societies, where a Number of Mathematicians commonly take the Lead, we shall find the same Opinions in Physicks adopted a long Time before they have made their Way into the Universities, where in general Mathematicks are much neglected, they do not gain Admittance there 'till very late, and even then rather under the Denomination of popular Opinions than after a rational Discussion. The Physicks of *Descartes* were discussed in the Academies at the very Beginning of their Institution, whilst *Aristotle* was implicitly received for more than forty Years after in the most learned Universities. These illustrious Societies rejected the Opinion of the French Philosopher respecting the Collision of Bodies, the ebbing and flowing of the Sea, Colours, &c. They rejected his Vortices in Proportion as undoubted Experiments and new physical Phenomena demonstrated their inconsistency.

VII.

After such respectable Testimonies, so well attested Facts that depose in Favour of Mathematicks, it perhaps would be unnecessary to attend to the vain Declamations of its Enemies (r); however, as there are some ca-

(r) In this Note we shall take Notice only of such as have endeavoured to ridicule them, or who have condemned them thro' Motives that do not deserve a serious Answer. A Sun Dial being shewn to *Epicurus*, to prove the Utility of Mathematicks, *Admirable Invention*, said he, *not to miss the Hour of dining*.

Verdier Vauprivas (in his Biblioth.) thinks *Euclid* void of common Sense: This is pardonable in a Man whose Head was crammed rather with the Titles of Books, than with real Knowledge.

Hobbes, tho' praise-worthy in other Respects, when convicted of the gross Errors he had committed in attempting to square the Circle, maintained Mathematicks to be an illusory Science. The Particulars of this Contest may be seen in the Philosophical Transactions.

I omit mentioning a Multitude of other Authors who have declaimed against the pretended Vanity and Uncertainty of the Sciences, the greater Part excite the Laughter of the Mathematicians, by the Manner they treat Mathematicks; deserving no other Answer, but an Exhortation to make themselves acquainted with their first Principles, before they attempt to speak of them. Of this Kind are several Pieces inserted in the Literary Journals. There are others who have regarded Mathematicks as dangerous: The Understanding of *Picus de la Mirandula* appears to have been greatly on the Decline, when he asserted that Mathematicks were incompatible with Theology, because they accustom the Mind to demonstrative Reasoning. *Peter Poiret*, in a Book intitled *De Vera falsa et superficiali Eruditione* 1694 *Leips.* treats it as

pable of misleading those who are not thoroughly acquainted with the Nature of those Sciences, it will not be improper to discuss them and expose their Weakness.

Two Sects among the Ancients were the declared Enemies of Mathematicks, the *Pyrrhonics* and the *Epicureans*. We shall first examine the Motives of the Aversion of the former for this Science. This Sect, as it is well known, only studied to raise Doubts against all Branches of human Knowledge; it was but reasonable therefore to expect that their first Attacks would be levelled against Mathematicks. *Sextus Empiricus* has transmitted to us the Arguments of his Sect in his famous Book *against the Mathematicians*, under which Denomination are comprised all those who make Profession of any Kind of Learning whatsoever, whom he attacks, one after the other, and the Mathematicians strictly so called, in the III, IV, V, and VI Books.

To answer these Objections, it would be almost sufficient to observe, how ridiculous is a Doctrine that pretends, that there is no Demonstration, no Means of attaining the least Certainty, that the Evidence of Reason is of less Weight than that of our Senses so liable to Deception; that, in fine, attempts to annihilate all Science founded upon Reason. We do not propose here to refute in Form this Method of philosophising, or to assert the contested Prerogatives of human Reason; there is Nobody that is capable of the least Reflection, who would not be able to answer those vain Subtilties. What Man in his Senses would not laugh to hear *Empiricus* undertaking to prove against the Geometricians, that there is neither Body nor Extension; against the Arithmeticians, that even Number does not exist; against the Musicians, that there are no sounds? Paradoxes so ridiculous in their Nature, that barely to mention them, is to refute them.

The Objections started by the *Pyrrhonics* against Mathematicks that deserve any Attention, are those that regard the Nature of the Objects they are employed about, and particularly Geometry; to those a general Answer has been given by Men of Eminence. The Objects of Mathematicks, say they, are so Metaphysical, that it is no Way surprising they are liable to Difficulties; but it is a Rule observed in the Research of Truth, that Objections, though they were unsurmountable, cannot invalidate a Doctrine supported by Demonstration; and this is the Case with Mathematicks; the Doubts that are raised against them arising from our imperfect Knowledge of the Nature of Bodies, Extension and Motion, cannot affect Conclusions deduced from evident Principles.

a Study that rather weakens than strengthens the intellectual Powers; he disapproves of it particularly because it diverts the Mind from the Contemplation of the Divinity. This devout Philosopher perhaps did not stand in need

of being refuted; he has been however refuted by the Count d'Herbstein, in a Dissertation intitled *Mathemata ad. umbratiles P. Poireti impetus propugnata*, 1709, in 8vo.

We shall not, however, confine ourselves to this Kind of Defence; and we shall discuss some of those so much boasted Objections of the *Scepticks*, or of the Enemies of Mathematicks.

The Objects of Mathematicks, say they, have no Reality, and cannot exist; Lines without Breadth, Surfaces without Depth, a mathematical Point, that is, without Length, Breadth, or Thickness, are mere Chimeras: The same may be said of the Figures whose Properties are demonstrated in Geometry; there cannot be a perfect Circle, a perfect Sphere, &c. whence they conclude, that the Objects of this Science are purely chimerical. They enforce this Objection with several Arguments: If, say they, from the Center of a Circle, Lines be drawn to every Point of the Circumference, they will fill up the whole Area of this Circle, and consequently the Circumference of every Circle concentric to the former, being cut by those Rays in as many Points will be equal to it, because it will contain the same Number of Points. If a perfect Sphere be supposed placed upon a perfect Plane, their Contact will be a Point without Extent, a true mathematical Point; but when this Sphere rous upon the Plane, it will describe a Line by the continual Application of its Surface to the Plane, and in this Manner will be generated a Line composed of Points without Extension, that is, Extension formed of Parts that have no Extension, which is absurd. Hence it appears, that the Supposition of a perfect Circle, of a perfect Sphere, &c. involves palpable Contradictions. Again, say they, if thro' every Point of the Ray of a Circle, concentric Circles be described, they will all touch, and will fill up the Area, new Absurdity, which consists in this, that Surfaces may be made up of Lines, or the Geometricians will be obliged to allow, that Lines have Breadth, which is sufficient to overturn all their Demonstrations. It will be unnecessary to produce more Objections of this Nature; because, for the most Part, they are only the same Idea represented in a different Manner, and that the Solution of some of them may serve as an Answer to all the rest.

To solve these Difficulties, it would be almost sufficient to observe, that the Mathematicians never pretended that there were Bodies extended in Length and Breadth without having Solidity. That there are others that have only Length without having any other Dimension, they only resolve Extension into its Parts, attending to some of them, and abstracting from all the rest. All Bodies have Length, Breadth, and Thickness; but we may consider the Length and Breadth, without attending to the Thickness: Hence arises the Idea of a Surface, and this Idea resolved again by a new Degree of Abstraction, produces the Idea of Length, thus the Surface is the Term of the Bulk of the Body, and consequently has no Thickness; the Line is the Term of a bounded Surface, and the Point the Term of a Line.

It follows from hence, that Bodies, Surfaces, Lines, are not made up of Surfaces, Lines, Points, for the Term of an Extension cannot be considered as one of its constituent Parts, whence the Hypothesis on which the first and last Objection is founded, is not admissible. Whatever Number of Lines are drawn from the Center of a Circle to its Circumference, or from the Vertex of a Triangle to its Base, they will never compose a Surface, they will only be the Terms of the Divisions of this Surface into Parts, as the Points of the Circumference are only the Terms of the Portions of this Circumference, for it is those Portions that compose it, and not their Extremities: When, therefore, it is pretended that there are as many Points in a little Line as in a great one, nothing else can be meant by it, than that one can be divided into as many Parts as the other, consequently there will be the same Number of Terms of Divisions in each; but we can conclude nothing with Respect to their Size, which depends on that of the Portions into which they have been divided. The pretended Absurdity that the last Objection seemed to point out, is no less groundless: all those concentric Circumferences do not fill up the Surface of the Circle, they only divide it into circular Zones, of which they are the Boundaries.

It is of no Consequence whether a perfect Sphere or a perfect Plane exists or not, those Figures are only the intellectual Limits of material Magnitudes considered by the Geometricians. What they demonstrate with regard to those Limits, is more sensibly true with respect to material Bodies, as they approach the nearer to them: Allowing therefore that the Truths of Geometry are only Hypothetical, that is, for Example, if a perfect Globe and Cylinder existed, they would be to each other in such a Ratio, they would be very far from being ill grounded; it would be necessary to demonstrate, that a perfect Sphere is the two Thirds of its circumscribed Cylinder, to discover that the same Ratio subsists sensibly between material Bodies that approach to those Figures, as far as we are able to judge by our Senses.

With respect to mixed Mathematicks their Certitude depends partly upon Geometry partly upon the Truth of the Hypothesis assumed for their Basis. Hence by pleading the Cause of that Science we have defended mixed Mathematicks, at least as far as regards the Consequences deduced from the Fact they presuppose. As to this Fact or Principle, since it is founded upon Observation or incontestible Experiments, it would be entering Scepticism further than the Scepticks themselves to disallow it; for those Philosophers did not contest the Truth of Fact and Experiments. *Empiricus* who refused to acknowledge the Truth of the Axioms of Geometry allowed that Part of judicial Astronomy which consists in foretelling the Vicissitudes of the Seasons, because he believed it was founded upon astronomical Observations.

The Invectives of *Aristippus* against Mathematicks, the contemptuous Opinion that *Epicurus* and his Followers affected to entertain of them can have but little Weight with those who are acquainted with those Characters. It is no Way surprising to find a Science that requires a close Application of the Mind condemned by a voluptuary such as the former; the Pleasures they afford, Pleasures purely mental, are quite different from those in which they made our supreme Felicity consist. (s) With regard to *Epicurus*, to whom it would be unjust to impute so sensual a System of Morality, other Motives induced him to reject the mathematical Sciences; it was because his Opinions were incompatible with the Truths they contain. In Effect, what Mathematician could he have persuaded that the real Magnitude of the Sun is the same with the apparent, or even less; that the Eclipses of the Sun and Moon, the setting of the Stars, are occasioned by a total Extinction of their Light; that they are newly lighted up at their Rising, &c. Such was the physical System of *Epicurus*, a System well worthy of one who slighted Mathematicks. Hence it is that *Cicero* ridicules him in several Places, among others (t), where he says, he can be easily induced to believe, without *Epicurus* swearing it, when he asserts, that he never received any Instructions from a Master, but that he would have done much better to have had one, and to have learned Geometry rather than to have decried it: Finally he adds, that this salutary Advice would have saved him from much Ridicule (u).

It is observable that the greater Part of the Scholastick Philosophers opposed the Study of Mathematicks thro' the same Motives, as likewise in our Days some pretended Philosophers, who espouse these Systems, by Means of which every Thing is explained in general and nothing particularly and with Precision. Those Sciences expose the Weakness of the Physicks of the former, and Geometry destroys the physical Romances of the latter.

I shall say nothing concerning those Edicts issued by the Emperors against the Mathematicians. It is well known, that the Astrologers, were distinguished by that Name, who flocked to *Rome* during several Ages, even down to the Time of *St. Augustine*, who wrote a Homily on the Re-

(s) Diog. Laert. in *Aristippo*.

(t) *De finib. Bon. et Mal.* lib. 1. § 7.

(Acad. *Quæst.*) It appears by another Passage of *Cicero*, that *Epicurus* had gained over a certain *Polynæus*, a good Mathematician, who afterwards maintained, that Geometry was a mere Tissue of Fallacies. It is very possible that this *Polynæus* might have passed for an able Mathematician, though very little versed in them; it might also happen, that an able Mathematician might have been misled. To this Example we may add that of

Chevalier de Mere, who, in a Letter to *Pascal*, set up for a first-rate Mathematician, (*Bayle* Dict. Art. *Zenon de Sidon* Let. du *Cheval. de Mere*, Num. 19.) treats as false the Demonstrations of *Pascal* and of Geometry. Those Arguments prove nothing against Mathematicks: either more Examples should be produced of eminent Mathematicians that decried this Science after having founded it, or Objections equal in Force and Evidence to the Principles on which Geometry is founded.

conciliation of one of those pretended Mathematicians with the Church; (x) but the Men of Understanding, the Philosophers, the Emperors themselves who proscribed the Mathematicians through the Empire, knew how to distinguish the real Professors of this Science from the Impostors who usurped their Name, conferred Marks of distinction on the former, whilst they decreed Punishments against the latter. There is a Decree of the Emperors *Theodosius* and *Valentinian*, (y) conferring the honorable Titles of *Spectabiles* and *Clarissimi* on the Professors of Geometry. The Emperors *Diocletian* and *Maximien* declared by a Rescript that the Cultivation of Mathematicks was an Object of public Concern; *Artem Geometriæ discere atque exercere publice interest.*

VIII.

It remains now to answer Objections of another Nature; these do not regard the Certainty of Mathematicks, but the Rank they should hold among the human Sciences. It is common in these Days for Men of Literature, upon every Occasion, to undervalue these Sciences, and to extenuate the Merit of those who excel in them. According to these Criticks, Mathematicks flourished along with the Schoolmen in the Ages the most destitute of Taste, Science, and Delicacy. The greatest Mathematicians, say they, elsewhere have been always the most aged or the most laborious. We can clearly discern the Motive that induced them to speak in this Manner. It is evidently in order to exclude all Genius from Mathematicks, and reduce them to the Level of the scholastic Puerilities. According to *Scaliger*, and others who have retailed his Opinions, Genius is not requisite to succeed in those Sciences, and consequently those who devote themselves to the Study of them need not expect a distinguished Place in the Annals of Literature.

These Reproaches or Invectives rather, will not surprise those who are acquainted with the human Heart; it arises from that Failing to which the greater Part of Mankind are addicted, of over-rating their own Pursuits, and contemning those of others. With regard to Mathematicks, there is a further Reason for this Conduct. Those Sciences being of a rude and difficult Access, and requiring much Pains and Study to become thoroughly acquainted with their fundamental Principles, the greater Part of those who endeavour to decry them with so much Malignity, are animated with a Sort of Despight for their having wanted Talents to pursue such Enquiries. It was, for Example, Vanity mixed with Envy, that excited *Scaliger* to speak with so much Contempt of the Mathematicks. *Joseph Scaliger*, full of that Self-sufficiency which made him fall into so many Errors, was desirous of acquiring a Reputation even among Mathematicians; far from entertaining at that Time so contemptible an Opinion of them, he attempt-

(x) In *Psalm*. l. LXI. p. 32. Ed. *Frob*
1556.

(y) L. 2. *Cod. de excusat Artif.*

ed the Solution of all those Problems that hitherto had baffled all their Skill; such as the Quadrature of the Circle, the Trisection of an Angle, the Duplication of the Cube, &c. he at length disclosed (z) his pretended Discoveries with much Pomp. He proposed also a new Method of reforming the Calendar, which he opposed to that of *Gregory XIII*; but all those Novelties, far from being applauded by the Mathematicians, met with that Reception which a Tissue of gross Paralogisms, proposed with the greatest Assurance, deserved: An universal Cry was raised against *Scaliger*, and *Clavius* among others convicted him of the Mistakes he had committed. From that moment, all those who cultivated Mathematicks with Success, were only dull heavy men; and the Jesuit Geometrician, his principal Adversary, was loaded with Abuse in Proportion to the Affront he had received. They deserve no other Answer than this short History.

Those who applaud such groundless Imputations, shew themselves either very ignorant of Facts, or have very little Candour. Were then *Pythagoras*, *Plato*, and all the eminent Mathematicians among the Ancients, were *Descartes* and *Newton* among the Moderns, dull and heavy Men? It would be the Excess of Injustice, to treat thus the celebrated Mathematicians of our Times. Whoever can taste the preliminary Discourse of the *Encyclopedy*, a Discourse wherein the Talents of an able Writer appear conspicuous; whoever, I say, can perceive the Beauty of this Discourse, will not hesitate to rank the Author among the first Men that grace the Republick of Letters. This is, however, the Production of one of our first-rate Mathematicians, who, with the same Pen that he calculated the Action of Fluids, the Irregularities of the Moon's Motion, wrote this truly sublime Piece. There is another whose Name has been rendered famous by one of the greatest Operations that was ever attempted, by various mathematical and physical Discoveries, and who had the Talent of adorning the driest philosophical Subjects. It would be easy to cite a Number of others, in whom profound Meditation has not dulled the Vivacity of Imagination. If there are found Mathematicians of a different Cast, they are either not of distinguished Abilities, or it may be a Defect contracted by Solitude, so apt to extinguish all Brilliancy and Liveliness of Imagination. Great Geniuses of every Species have experienced this Fate in different Ages; but particularly in those Days when Men of Learning were confined solely to their Books, and never passed the Limits of the Science to which they devoted themselves. If some Mathematicians were then entirely Strangers to polite Literature, how few were the Professors of Belles Letters who were acquainted with the first Elements of the Sphere? I say, some Mathematicians, for it would be easy to prove, by a Multitude of Instances, that most of them were versed in several Branches of polite Literature: But if

(z) *Cyclametrica*.

there were even more that lived in a Kind of literary Barbarism, this was common to them with many others; however, they are much reformed as to this Point, in this Age. It would be easy to find actually, Men of Letters, particularly among the Poets, who know not the Reason why the Days are longer in Summer than in Winter. Should a Phenomenon, so regularly and frequently occurring, less excite the Admiration and Curiosity of the human Mind, than the sublime Beauties of Poetry and Eloquence?

Those who speak of Mathematicians with such Contempt, doubtless have mistook the Compilers of voluminous Works for the ablest Mathematicians; and as Authors do not compile large Volumes in the Prime of Life, or without an immense Deal of Pains, they conclude from thence, that the oldest and most laborious were the most eminent Mathematicians. This Mistake is only pardonable in a Stranger to mathematical Learning: Had these Criticks been less so, they would have thought otherwise. Mathematicians have always discovered more Genius in some Pages of *Vieta*, *Kepler*, *Copernicus*, *Tycho Brahe*, than in the voluminous Writings of *Clavius*, *Renaldini*, *Guarini*, &c. *Descartes*, while yet at the Flower of his Age, instructed all the Mathematicians of his Time, by publishing his Geometry, written in a very concise Manner, The vast Improvements Geometry has received within this last Century, are almost entirely owing to young Mathematicians: *De Fermat* was as young as *Descartes* when he contended with him, and laid the Foundations of the *Infiniteimal Calculus*; *Wallis* was very young at the Time he grafted his Discoveries upon those of *Descartes* and *Fermat*; *Newton* had scarce attained his 23d Year, when he was the first Mathematician in *Europe*, since at that Age he had discovered several of his sublime analytical Methods, and among others the Principles of the direct and inverse Method of Fluxions; a few Years after, he analyzed Light, and at the Age of 28, published his profound Theory of Opticks; his immortal Treatise *Principia Mathematica Philosophiæ Naturalis* is partly the Production of his Youth, he had then laid the Foundation of that immense and admirable Structure; the Lives of many ordinary Men would scarce be sufficient to collect and digest the infinite Number of Materials he has employed, and which he drew from Geometry and the most subtle Mechanics; however, he had scarce completed the Half of his Career, when at the repeated Solicitations of the Learned, he published this Work. *Lebnitz*, proposing Cartels to the Geometricians, or answering them, was but very young: This great Man's profound Knowledge in Antiquities, in History, in Politicks, and Jurisprudence, his Taste for the most refined Metaphysics, are universally known; had it not been for these various Studies which equally shared his Attention during his Life, probably his youthful Days would have been distinguished, as well as those of *Newton*, by important Discoveries. What shall I say of the illustrious Brothers, *James* and *John Bernoulli*, who, following in the same Path *Newton* and *Lebnitz*, were,

after them, the ablest and youngest Mathematicians in *Europe*? In fine, we may venture to affirm, that there has not been a Mathematician of Reputation who has not distinguished himself in his Youth by some Work of Genius.

We may conclude from those Traits, of which it would be easy to augment the Number, that the first Accusation of the Enemies of Mathematicks is destitute of Foundation; nor is the second more equitable. Those Mathematicians that flourished in the Ages of Ignorance and Barbarism, were not such as those Criticks would represent them; independant of their Number being very small, while the Seminaries of Learning swarmed with Schoolmen; independant that the greatest Part of even those opposed the false Taste that prevailed in the Schools, could the most eminent among them enter into Competition with those Geniuses that *Greece* in her flourishing Days produced, with those that appeared in *Europe* since the Revival of Letters. Their Knowledge, confined to the elementary Parts of those Sciences, to understand *Euclid's* Elements perfectly was considered as an extraordinary Effort of Genius; the Geometricians of a more distinguished Character, such as *Archimedes*, *Apollonius*, &c. were scarce known to them. But let us admit for a Moment, that these dark Ages produced some eminent Mathematicians, why should the Fecundity of Nature, that from Time to Time produces great Geniuses, should it be suspended? Those Men were so much the more praise-worthy, because they were able to make their Way in spite of the Ignorance and Prejudice that prevailed in those Times; and nothing is more honorable to Mathematicks, than that the great Geniuses in all Ages were versed in those Sciences: We might conclude from thence, that no Study is more proper to give that Strength and Vigour to the Mind, which enables it to triumph over the Obstacles of Prejudice and Ignorance. Besides, might it not be asked, in what Age did *Homer*, *Hesiod*, &c. live? Was it not when *Greece* was plunged in Barbarism? Was it not in an ignorant Age that *Dante*, *Petrarch*, shone forth in *Italy*? How many Poets of Merit, how many Men of sound Literature, flourished in the 16th Century, so little productive of eminent Mathematicians every where but in *Italy*, where Arts and Sciences were cultivated with unremitted Zeal? The Objection, therefore, of the Enemies of Mathematicks, proves nothing. Upon a more candid Examination it will appear, that for the most Part Men famous for polite Literature, and those eminent for Mathematicks, lived in the same Age. The eminent Mathematicians that *Italy* produced at the Revival of Letters in *Europe*, were cotemporary with *Ariosto* and *Tasso*. The same Age that produced in *France* a *Descartes*, a *Paschal*, a *Fermat*, and a *Marquis de L'Hopital*, produced also *Corneille*, *Moliere*, *Racine*; in *England*, *Wallis*, *Newton*, and *Halley*, were cotemporary with *Milton*, *Addison*, and *Pope*.

The Ancients, more equitable, seem to have been sensible of this Truth, when they assigned to one of their Muses the Employment of presiding over the Study of the Heavens. As this Study, by the Nobleness of its Subject, claims the most distinguished Rank, they had it chiefly in View when they created that allegorical Being; but the Attributes (a) they gave it, appertain to Mathematicks in general: In effect, the Compass and Square are the Symbols of Geometry, and evince that they had more extensive Views than one would at first be apt to imagine; besides, it is only by the mutual Aids they afford, that we can attain to the sublime Knowledge of the Constitution and Laws of the Universe; they are therefore of the Number of those Sciences over which this Divinity presides. The Muse *Urania*, therefore, not only guides the Astronomer thro' the Heavens, but also inspires the Geometrician and the Mechanician, who also have a Seat on *Parnassus*; it being but just, that those who explore the Mysteries of Nature with so much Sagacity, should ascend with those who paint Nature with so many Charms.

I might here avail myself of the Testimony of *Cicero*, Not of *Cicero* exalting his own Profession, by shewing how few had attained Perfection in it; but of *Cicero* the Philosopher, weighing the Sciences in the Balance of Reason. What Encomiums does he not give to Physicks and Mathematicks (b): *Quid dulcius otio Litterato; iis dico litteris quibus infinitatem rerum ac Naturæ et in hoc ipso Mundo, Cælum, Maria, terras cognoscimus.* He makes Wisdom partly to consist in contemplating and unravelling those Mysteries: He exclaims (c), What Riches, what Crowns, can be preferred to the Pleasures tasted by a *Pythagoras*, a *Democritus*, an *Anaxagoras*, in contemplating the surprising Spectacle of the Universe! In another Place (d), he even ventures to call the Genius of *Archimedes*, divine, for having been able to imitate, in a frail Machine, the stupenduous Structure of the Universe. The Sagacity of Astronomers appeared to him so great, that from thence he deduces one of his principal Proofs of the Existence of a Soul, Portion, or Image, of the Divine Being.

IX.

It now remains to shew the Utility arising from the Study of Mathematicks (e). I shall confine myself here to the Advantages resulting from that

(a) See the Frontispiece.

(b) *Tuscul. Quæst.* lib. v. vers. fin.

(c) Several Authors, thro' a mistaken Zeal for Mathematicks, have extolled their Utility in a very puerile Manner. We think it necessary to observe this, lest such injudicious Pretensions, if not taken Notice of, might expose those truly valuable Sciences to ridicule. *F. Merfennus* (*Harm. Univ.* tom 2. lib. viii. *Syn. Mat. Pref.* § 13.) does not scruple to exhort the Orators to adorn their Discourses

(c) *Ibid*, lib. v. vers. med.

(d) *Ibid*, lib. i. vers. med.

with Passages and Texts taken from Mathematicks; in his Opinion, the Conic Sections afford excellent Materials for Similes for the Use of the Pulpit.

Others have made ridiculous Applications of the mathematical Truths to Questions in Theology, Metaphysics, and Morality. The *Pythagoreans* formerly shewed the Example, by the Analogies of Figures and Numbers they

Science, in supplying our Wants and administering to our Pleasures; how this Study strengthens and improves the Faculties of the Mind, we have already hinted. It were to be wished, as Mr. Locke says, that all those

pretended to find in all Nature; but the Moderns have far outdone the *Pythagoreans* in those Conceits, particularly *J. Caramuel de Lobkowitz*, Author of the Book entitled *Mathesis Andax, Rationalis, Naturalis, Supernaturalis*, &c. Lov. 1644, 4to. All the Subtilties of Metaphysics, all the Mysteries of revealed Religion, are explained by mathematical Reasoning, the Application of which is truly ridiculous. He examines whether God could create Angels whose Degree of Perfection is incommensurable, if the Motion of the Earth be possible admitting the Rapt of St. Paul, what Species of Triangle form the Trinity, &c. *Caramuel* had Imitators in one Michael Berns, Author of a Treatise wrote in High Dutch; in Gaspard Schmidt, who also found all the Mysteries and Precepts of Religion in Mathematicks. His Work, a mere Rhapsody, is entitled *Astrologia Cathetica*.

Vossius does not give any great Proofs of his Discernment in his Book *De Scientiis Mathematicis*, c. 7. when he discusses the Utility of the Mathematicks, he finds them fit for every Purpose; for Poetry, Grammar, Economics, Theology, &c. The Reasons he alleges are really curious: The Art of Combinations, says he, will teach the Poet that the Verse *Rex, Lex, Sol, Dux, Fons, Lux, Mons, Spes, Pax, Petra, Christus*, will admit of 3628800 Variations. The Grammarian will learn, that a Dictionary of the Size of Calepin, would scarce be sufficient to contain the different Words arising from the Combination of 16 Letters. The Economist will learn from Mathematicks, that a Pea in 12 Years would yield so plentiful a Crop, that sold at a moderate Rate, would amount to more than 1000000000000 Crowns. But the honest Vossius was ignorant of the first Rudiments of Trade, since a Commodity so abundant would be of no Value. The Divine, in fine, will find there Subject sufficient to calm the Apprehensions of those who might fear they would not find Room in Paradise: They would learn from Mathematicks, that tho' the World should subsist 12000 Years longer, and there should be twenty thousand Millions saved, the Empyrean Heavens are so vast, that God might assign to each of them a Space exceeding the Extent of several Kingdoms on the Surface of the Earth.

We shall also take Notice of several Books

published by injudicious Persons, with a View of shewing the Utility of Mathematicks for explaining the Holy Scripture. These are the Titles: *Andreas Arnoldi Mathesis Sacra*, 1676, 4to. *Altorf. Sam. Reyheri, Mathesis Mosaisca*, 1679. *Krist. Sturmii, Math. ad. S. Script. interp. Applicata*, Norib. 1710. *Wideburgi, Specimina Matheseos Biblica*. *J. Schmidt. Mathesis Biblica*, 8vo. 1736. There is no Doubt, that some Knowledge of Arithmetick and Geometry is necessary to explain certain Facts in Holy Writ; but it is quite ridiculous to rake together, as those Authors have done, a Multitude of trifling Questions, for the Sake of applying to them the first Principles of Arithmetick and Geometry, for such are for the most Part the Questions found in those Books; as the Calculation of the Sand, mentioned in *Genesis xiii, 6*; that of the Size of Goliath; the Weight of Absalom's Hair; and, of the Crown of the King of the Ammonites, &c. Vossius has taken Care to extract some of the most frivolous, to encourage Divines to study Mathematicks.

Among the Abuses of those Sciences, we may reckon the pretended Application of them to Metaphysics and Physick. There are some Authors who have imagined, that when they had digested their Conceits in the Form of Theorems, Problems, Corollaries, they had raised them to the Rank of mathematical Truths. Of late, several Works have appeared where the most contested Points in Metaphysics are treated after the Manner of the Geometricians, whose Authors, after having heaped up a Number of *quod erat Demonstrandum*, Scholia and Corollaries have believed their Opinions, have thereby acquired the Certainty of a geometrical Theorem. We shall here observe, that it is not from the Form of their Demonstrations, that Mathematicks derive their Certitude; they owe it chiefly to the Simplicity and Evidence of their Principles; to the clear and incontestible Connection of the Propositions deduced one from the other. *Sr. P. de Croza* seems to have intended to expose to Ridicule this improper Use of the Form of geometrical Reasoning, in a Treatise on the Spirituality and Immortality of the Soul: The Arguments he proposes are quite humorous; and all in the geometrical Style, which are a perpetual Satire on the Metaphysicians we have spoken of.

who are destined for a Way of Life that requires the Exertion of the intellectual Faculties, would apply themselves to the Study of Mathematicks, we should see fewer precipitate Conclusions, fewer false Arguments, boasted for Demonstrations; in fine, fewer Persons seduced by the specious Appearances of Truth. But we have already sufficiently insisted on this Article.

It must be allowed, that the Aid of Geometry and Arithmetick is absolutely requisite in Society, and in an infinite Variety of Cases, in Economics, Jurisprudence, &c. As the Property of each individual must be ascertained in Number, Measure, and Weight; in general a Knowledge of the Elements only of these Sciences suffices, but there are Circumstances in which the Aid of the most profound Knowledge of those Sciences is requisite: It is of Importance to a State, to private Companies, &c. that establish Annuities on Lives, Insurances, &c. that authorises certain Games of Chance, as Lotteries, to know the Advantages and Disadvantages, and to State a certain Equality. Upon those Questions, Mathematicians are always consulted; and the Inspection alone of the Books wrote on those Subjects, sufficiently evince that they are of a higher Investigation than those of common Arithmetick.

Physick furnish several Examples of the Abuse of Mathematicks. It is true, they may serve to explain some mechanical Effects observable in the human Body: Borelli's Treatise *De Motu Animalium*, is, in this Respect, very valuable; but, to pretend to apply Calculation to the combined Motions of Fluids and Solids, in a Machine the most complicated existing, is an Attempt, we may venture to declare, chimerical. See Maupertuis' xiv Letter on this Subject. It is entertaining, at least to Mathematicians, to see some Physiologists resolve, by a few Strokes of the Pen, Problems, whose Solutions, those who are most profoundly skilled in Mechanicks and Geometry, would not attempt. The Dissertation of Mr. Bernoulli, *De Motu Muscularum*, ought to be only considered as an ingenious Essay of his Skill on an hypothetical Problem, whose Solution is modified by a thousand Circumstances. Here follow the Titles of some Medico-mathematical Books: N. Stroem. *Ratioc. Mechanic. in Medicina usus Vindicatus*, L. Bat. 1707, 8vo. N. Gaukes *De Med. ad Math. Certitud. evehenda*, 1712, 8vo. Archibaldi Piscarrii, *Elementa Medicinæ Physico-Math.* Lond. 1717, 8vo. No Man appears to have abused Mathematicks more than this last Physician: He asserts what is truly ridiculous; that by their Assistance he had found out a Method of curing the Disorders incident to the Eyes; he even goes so far as to propose

this Problem: *A Distemper being given, to assign the Remedy for it.* In all Probability, Pitcairn resolved it badly; for notwithstanding his Solution, the Art of Healing is found every Day defective in the Treatment of the best known Distempers.

Questions sometimes occur in Jurisprudence, that require some Skill in Arithmetick, and some Knowledge in Geometry; this has given Rise to the following Works: N. Vogt, *Arith. Juridica*. N. Polackii, *Mathesis Forensis*. If those Works have been composed with an Intent to assist the Lawyers in discussing these Questions, the Motive is laudable; but if their Authors intended thereby to shew the Universality of Mathematicks, by applying them to a thousand frivolous Questions that have any Connection with Jurisprudence, we may class such Compositions with *Mathesis Biblica, Mosica, &c.*

A Mathematician (the Count d'Herbstein) has published a Dissertation on this Question: *An Studium Geometriæ, rempublicam administranti adminiculo an Obstaculo* (Prague). I know not how he resolves this Question: I conjecture, however, he concludes that Sovereigns should only chuse Mathematicians for their Ministers of State. We could not expect less from a Country and an Age, so productive of frivolous Compositions on the Utility of Mathematicks.

It is by Mechanicks and the ingenious Combination of its different Powers, that human Industry is enabled to remove and transport Weights so superior to our natural Strength; to make Water serve as a moving Power to several Machines, to raise it to the Tops of Mountains, and from thence diffuse it as Occasion may require. *Archimedes* defended his Country for a long Time by his mechanical Inventions; and almost all the Engines employed by the Ancients in War, were invented or perfected in those Ages when Mathematicks flourished in *Greece*; which proves that they were of great Use in perfecting that Part of the military Art.

The Advantages resulting from Astronomy, cannot be contested by those who will attend to the following Facts: This Science is the Soul of Geography, of Navigation, and of Chronology. It must be allowed, that it is of some Importance to Man to know the Form, the Extent, the exact Situations of the different Parts of the Globe he inhabits. How can this Knowledge be attained, but by the Assistance of Astronomy? It is easy to perceive how insufficient for this Purpose the most exact Journals of Travellers are, at least for determining the Situation of Places very remote from each other. Besides, in how few Cases can this Method be employed? And if it was the only one, we should not as yet have known the narrow Limits of the Places that environ us. By the Assistance of Astronomy, the most distant Countries, tho' separated by innavigable Seas, Deserts, and barbarous Nations, &c. hold a Sort of Correspondence by the Interposition of the Heavens alone.

Commerce, that Source of Wealth and Power to a Nation, is in a great Measure indebted to Mathematicks for the extensive Manner it is carried on at this Day. In effect, this Science has had a greater Share than is commonly imagined, in the Discovery of those Countries whence such Riches flow in upon us. When Infant *Don John of Portugal*, who principally promoted the Discovery of the *Indies*, proposed carrying this Scheme into Execution, he employed the most eminent Mathematicians to contrive Instruments for observing, and to invent proper Methods for keeping a Reckoning at Sea; by these Means, he induced Men to enter into his Views, and emboldened them to brave the Dangers of unknown Seas: Such was the Origin of Nautical Astronomy. This Prince, who was well skilled in Mathematicks himself, was the Inventor of the Charts employed in that Voyage; probably so great an Enterprize would have never been carried into Execution, without these Circumstances.

If we now are able to traverse the Ocean with so much Security and Skill, it is owing to Mathematicks which has furnished the Means. To *Mercator*, an Astronomer and Geographer of the *Low Countries*, we are indebted for the Invention of Charts by increasing Latitude, esteemed the best by intelligent Navigators. And, without Doubt, it is from Astronomers that this Art will receive its last Degree of Perfection, when the Motions

of the Moon will be sufficiently known as to enable them to determine its Place every Instant with Accuracy.

The Chronologers have always employed the celestial Phenomena as a Means of verifying the Dates of certain fundamental Epochas ; We find no Order in the Chronologies of ancient Kingdoms, arising from their ignorance of the celestial Periods. The Certainty of History, as far as it depends on the assigning to Events their proper Places, is entirely owing to Astronomy. A well regulated Calendar seems to be the most important Production of this Science. What Pains did not the ancient *Greeks*, *Persians*, the modern *Europeans*, take to give to their Calendar a permanent and perfect Form, which they attained only in Proportion as they cultivated Astronomy. I should not omit mentioning, that this Study has freed us from those Terrors so disgraceful to human Reason, that used to affright whole Nations ignorant of the Cause of certain unusual Phenomena. We recall to Mind, with Pity, the Story of that weak Prince, who, upon seeing an Eclipse of the Sun, ordered his Son's Hair to be cut off, as on a Day of publick Calamity. The Ignorance of *Nicias*, who commanded the *Athenian* Fleet in the War of *Sicily*, was the Cause of the signal Defeat they received on this Occasion : *Nicias*, terrified by an Eclipse, was afraid to set Sail when it was Time, in Order to raise the Siege of *Syracusa* : The next Day the Wind proved contrary, and prevented his Departure, and he was taken Prisoner with his whole Army. It is not long since, that the Appearance of a Comet inspired superstitious Terrors ; Astronomy alone could calm them, by unfolding the Causes of that alarming Phenomenon. The considerable Progress of those Sciences, has occasioned the Fall of judicial Astrology : That delusive Art, sprung from the Abuse of Astronomy whilst yet in its Infancy, has lost all Credit, except with a few weak Minds, since Astronomy has been so vastly improved. The different Branches of that Science, have thrown great Light on the general System of the World, an Object worthy of the Contemplation of rational Beings who enjoy that wonderful Spectacle ; such will doubtless be the Opinion of all those whose Eyes are not entirely turned to the Earth, and who recall to Mind those fine Verses of *Ovid* :

*Pronaque cum spectent Animalia cætera Terram,
Os Homini sublime dedit, Cælumque tueri
Jussit, et erectos ad Sydera tollere Vultus.*

x.

To enter into a Detail of the Advantages resulting from the Cultivation of the other Branches of mixed Mathematicks, would be affecting a useless Prolixity ; they are sufficiently obvious to dispense us from insisting any further on that Head, we shall therefore confine ourselves to some Observations on geometrical Speculations, the End and the Utility of which might be questioned. It must be allowed, that there are a great Number that are merely intellectual Curiosities, and of no apparent Utility ; but when we

consider that they are the only incontestible Truths which the human Mind can, by pure Reasoning, attain to, we shall cease to regard them as frivolous. In effect, Man being composed of two Parts, the one Intellectual, whose Nature is to reflect and to investigate the Properties of Objects; the other formed to perceive and enjoy those same Objects. It must be allowed, that if their sensible Properties should be studied with a View of supplying our Wants and administering to our Pleasures, those that are purely intellectual, are particularly suited to our intellectual Part. Besides, to what a narrow Compass would human Sciences be reduced, if all those were excluded, from cultivating which, no apparent Advantage accrue to Mankind, shortly Ignorance would prevail, and bring back all the Evils attending the rude and barbarous Ages.

We may further observe, that those purely theoretical Truths, whose Use is not obvious, may probably be productive of some which future Ages may discover, but particularly may pave the Way to more important Truths. What an Apparatus of Geometry does not some Questions in Mechanicks and Astronomy require? Among the latter, for Example, the Motion of the Moon, from the compleat Resolution of which we shall probably derive the inestimable Advantage of a perfect Navigation. The Fate of mixed Mathematicks is closely connected with that of abstract Mathematicks: Every Truth contained in the latter, is of Importance to the former.

I shall conclude by a Reflection. A Philosopher asked, what would be the Employment of Men, if they were exempt from Passions and from the Wants their Nature subject them to? Doubtless, the Enquiry after Truth and the Contemplation of the Phenomena of Nature, would be their only Pursuits thro' a Life equally tranquil and happy. Well, then, those Objects, so noble, the sole Occupation of perfect Beings, those Objects, I say, are those of the Mathematician. Abstract mathematical Enquiries, and their Application to the Study of Nature, enter therefore into the Plan of the Instructions given in the Drawing School established by the DUBLIN SOCIETY, pursuant to their Resolution of the fourth of February, 1768; to enable Youth to become Proficients in the different Branches of that Art, and to pursue with Success, geographical, nautical, mechanical, commercial, and military Studies.

XI.

Introduction.

WISE Regulations relative to the Education of Youth in *England, Scotland, and other Parts of Europe*. Fatal Consequences resulting to this Country from the Neglect of this important Object. How far the Drawing School established by the DUBLIN SOCIETY put on a proper Footing has supplied this Defect.

Plans of Instruction put in Execution in this School.

Plan of a Course of pure Mathematicks: Utility of this Science: Method of teaching it: Is divided into Arithmetick numeral, and specious; into

Geometry, Elementary, Transcendental, and Sublime. Conclusion. Authors who have furnished the Materials of this Course.

Instructions relative to young Noblemen and Gentlemen of Fortune.

Plan of the System of the Physical World. Utility of the Study of the System of the World. Advantages resulting from the Knowledge of the System of the World. Public Schools erected in *England, Scotland, &c.* for instructing young Nobleman and Gentlemen of Fortune, in what regards the System of the World. Method of teaching the Discoveries relative to the System of the World. Progress of the Discoveries relative to the System of the World. Principal Phenomena of the System of the World. Theory of the principal Planets. Theory of the Figure of the Earth. Theory of the Precession of the Equinoxes. Theory of the Tides. Theory of the Refraction of Light. Theory of the Moon. Theory of the Comets. Conclusion. Those Theories taken from the Principia, not as interpolated and anatomized by *Pemberton, Clark, M^r Laurin, &c.* but from the Original. The Demonstrations being compleated by supplying the Steps, which were designedly omitted by the illustrious Author. Enumeration of the Improvements which these Discoveries have received from the united Efforts of the first Mathematicians in *Europe*.

Plan of the Art of making Experiments, and that of employing them. Necessity there was of furnishing the School with a compleat Collection of the best executed Machines, adapted for experimental Enquiries, and of instructing Youth in the Management and Use of those Machines. I Class. Machines for making Experiments on the Gravity, Motion, and Equilibrium of solid Bodies. II Class. Machines for making Experiments on the Gravity, Motion, and Equilibrium of fluid Bodies. III Class. Machines for making Experiments on the Air. IV Class. Machines for making Experiments on Fire. V Class. Machines for making Experiments on Light and Colours. VI Class. Machines for making electrical and magnetic Experiments. VII Class. Machines for making Experiments and Observations in Cosmography. VIII Class. Machines for making Experiments and Observations in Meteorology. Conclusion. The Construction and Use of those Machines in experimental Enquiries proposed to be described.

Plan of the System of the Moral World. Origin of Civil Society. The different Forms of Government. Particulars in which all Forms of Government agree. Particular Circumstances which should modify the different Forms of Government. The Relations of which the different Forms of Government are susceptible. The Laws resulting from the Nature, Circumstances, and Relations of the different Forms of Government. Conclusion. That the moral Philosopher does not employ his Time in Speculations and Subtilties foreign to common Life.

Instructions relative to Engineers, Gentlemen of the Artillery, and in general to all Land Officers.

Plan of the Military Art. Necessity there was of erecting a Military

School. Studies there pursued. Mathematicks, Mechanicks, Dynamicks, Military Architecture, Balistic, Pneumatics, Hydraulicks, Hydraulick Architecture, Draughting, Attack and Defence, Geography, History, Tactics, Order of the Studies, practical Operations, public Examinations: Conclusion. Pointing out the Advantages the young Officer will reap from those Studies.

Instructions relative to those intended for Trade.

Plan of the Mercantile Arts. Dignity of the Trader. Disadvantages in Point of Education he laboured under. The Necessity there was of erecting a Mercantile School. Studies there pursued. Mathematicks, Drawing, Geography, History, Navigation, moral Philosophy, Book-keeping, Composition, practical Negotiations. Conclusion. Recapitulation of the Advantages which the young Trader, and the Public in general, will reap from this Institution

Instructions relative to Ship-builders, Sea Officers, and in general to all those concerned in the Business of the Sea.

Plan of the Naval Art. The Necessity there was of erecting a Marine School, where are taught naval Architecture, mechanical Navigation, the Art of Piloting, the different Branches of Drawing.

Instructions relative to Architects, Painters, Sculptors, Engravers, Clock Makers, &c. and in general to all Artists and Manufacturers.

Necessity there was of erecting a School of Mechanic Arts, where Artists receive the Instructions in Geometry, Perspective, Physicks, &c. which suit their respective Professions, and contribute to improve their Taste and their Talents.



[1]

E L E M E N T S

O F

N U M E R A L A R I T H M E T I C K .

C H A P. I.

Of Numbers, of the general Principles of numeral Arithmetick, and of the Operations performed upon simple Numbers.

COMPUTATION is either performed by Numbers, or by certain Signs and Symbols which have been contrived for this purpose and found convenient; whence the Science of computing is divided into numeral and specious Arithmetick.

I.

By Number is meant a Collection or Assemblage of Units, and by an Unit, an arbitrary Quantity, which is assumed to be a Measure of other Quantities of the same Kind. What is understood by number.

It has been found necessary for the convenience of Commerce, to employ different Kinds of Units. In Money the Units authorised by Law, are the Pound sterling, the Shilling, the Penny, and Farthing. In measuring Lengths and Distances, the Units employed are the Fathom, the Foot, the Inch and the Line, and several other Measures that have been introduced and confirmed by Custom. In measuring of Areas and Surfaces, the square Foot, the square Inch, the square Perch, the square Acre are taken for Units: and when there is Occasion of employing different Units in order to render the Computation more easy; it is usual to assume for Units different Measures which have all the same Length but different Breadths, from whence they take their different Denominations. In the Mensuration of Solids, the Units employ'd are the cubick Fathom, the cubick Foot, the cubick Inch, and different Solids which have all for Base a square Fathom but different Heights from which they take their different Denominations, not to mention a Number of other Measures such as the Gallon, the Bushel, &c. which vary in the different Parts of the World. In fine, every Species of Things have their particular Unit whose Value has been fixed and authorised by Custom or by Law. The different kinds of units agreed upon for the convenience of trade.

The Units we have spoke of, and all others to whatever Kind or Species of Things they relate, are called *concrete* or *applicate* Units, and a Collection or Assemblage of those Units is called a *concrete Number*. There is a Numbers are either concrete or abstract.

ELEMENTS OF

nother Unit which denotes no Kind of Things in particular, and is applicable to every Kind or Species of Things; this Unit is called *abstract* or *absolute* Unit and is expressed by the Word *One* or *once*; and a Collection or Assemblage of those Units is called an *abstract Number*.

II.

The formation of numbers.

By adding an Unit to another Unit, a Number is formed which is called *Two*; by adding to this Number a new Unit there results the Number which is called *Three*; and by continuing in this manner to add new Units to the Numbers already formed, there will result the following Numbers which are called *Four*, *Five*, *Six*, *Seven*, *Eight*, *Nine*, &c. As an Unit may be continually added to the Numbers that have been formed, let them be ever so great; it is manifest that the Series of Numbers has no Limits.

Inconvenience of expressing each number by a particular word or character.

If therefore each Number was to be expressed by a Word or particular Character, the Number of Words and Characters to be employed would be infinite, and the Life of Man would scarce be sufficient to learn to reckon up to Fifty Thousand, which is a small Number not only among the Numbers possible, but even among those which are in daily Use: but the Computists obliged to employ great Numbers have found out the Art of counting by means of a very few Words and Characters repeated several times; this Art is called *Numeration*.

III.

Characters employed to represent numbers.

Thow there is an Infinity of different Numbers, the Computists have found means of expressing them all with ten Characters differently combined and repeated, viz.

0 1 2 3 4 5 6 7 8 9

Cypher, One, Two, Three, Four, Five, Six, Seven, Eight, Nine.

It is easy to perceive that with those ten Characters we can reckon from Nothing to Nine inclusively, without having recourse to any new Artifice, but not further, if we have no other Unit but of the Species to be numbered. For Example, if the Units a Number consists of are Feet, we can reckon from Nothing to nine Feet, but we cannot reckon beyond 9 Feet, if we had no other Unit but the Foot.

The Computists, therefore, besides the Unit of the Species to be reckoned, which is called *principle Unit*, for Example, besides the Foot, which in the foregoing Example is the principle Unit, have imagined others which are called *collective Units*, which may be reckoned also from Nothing to Nine inclusively; and by help of those collective Units, all possible Numbers arising from the Repetition of the principle Unit may be expressed, in the manner we are going to explain.

IV.

How the Arithmetical characters are distinguished.

When the Number of principle Units do not exceed Nine, and consequently may be expressed by one Character, this character is called *Number of Units of the first Degree*, and the Place it stands in is called the first Place.

Of ten Units of the First degree is formed a collective Unit called *Ten* or an *Unit of the second Degree*, and may be reckoned from Nothing to Nine, by means of the Characters 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. posed to represent all possible numbers.

To distinguish the Character which represents a Number of those new *Units of the second Degree*, from that which represents a Number of *Units of the first Degree*; it is wrote down in the second Place to the left Hand of the Figure, which represents a Number of Units of the first Degree, and which stands in the first Place, this first Place being always filled up with a Character representing the Number of Units of the first Degree, as will appear by the following Examples.

1^o When the Number of Units to be reckoned consists only of *Tens* or of Units of the second Degree, and do not exceed *Nine*, the Character (0) which represents no Number, is set down in the first Place, and by denoting that there are no Units of the first Degree remove the Character which represents a Number of Units of the second Degree into the second Place, and determines it to represent a Number of this Kind of collective Units rather than any other.

Thus 10 represents *Ten*, 20 represents two Tens, or, *Twenty*, 30 three Tens, or, *Thirty*, 40 four Tens, or, *Forty*, 50 five Tens, or, *Fifty*, 60 six Tens, or, *Sixty*, 70 seven Tens, or, *Seventy*, 80 eight Tens, or, *Eighty*, 90 nine Tens, or, *Ninety*.

2^o Since by writing down some one of the ten Characters 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, in the second Place we can reckon a Number of Tens from Nothing to Nine, and by setting down some one of the same Characters in the first Place we can reckon a Number of *simple Units* or of the *first Degree* from Nothing to Nine; it is easy to perceive that we can by help of the two Characters which stand in the two first Places reckon from Nothing to Ninety nine. All numbers from nothing to ninety nine are represented by help of two characters.

The Characters which stand in the second Place, preserve the Names of the Numbers of Tens they represent, and those which stand in the first Place retain the Names of the Numbers of simple Units they represent. The following Numbers however are to be excepted, 11, 12, 13, 14, 15, 16, 17, 18, 19, which are read and wrote as follows, Eleven, Twelve, Thirteen, Fourteen, Fiveteen, Sixteen, Seventeen, Eighteen, Nineteen, and not Ten-one, Ten-two, Ten-three, Ten-four, Ten-five, Ten-six, Ten-seven, Ten-eight, Ten-nine, as the general Rule would require. Each character retains the name of the number it represents.

V.

As ten *Units of the first Degree* form an *Unit of the second Degree*, in like manner ten *Units of the second Degree*, form an *Unit of the third Degree*, which is called a *Hundred*, and the Character which represents a Number of those new Units is set down in the third Place to the left Hand. Those new Units of Hundreds may be reckoned from Nothing to Nine, with the same Characters 0, 1, 2, 3, 4, 5, 6, 7, 8, 9: and as we may All numbers from nothing to nine hundred and ninety nine are represented by

ELEMENTS OF

another Unit which denotes no Kind of Things in particular, and is applicable to every Kind or Species of Things; this Unit is called *abstract* or *absolute* Unit and is expressed by the Word *One* or *once*; and a Collection or Assemblage of those Units is called an *abstract Number*.

II.

The formation of numbers.

By adding an Unit to another Unit, a Number is formed which is called *Two*; by adding to this Number a new Unit there results the Number which is called *Three*; and by continuing in this manner to add new Units to the Numbers already formed, there will result the following Numbers which are called *Four*, *Five*, *Six*, *Seven*, *Eight*, *Nine*, &c. As an Unit may be continually added to the Numbers that have been formed, let them be ever so great; it is manifest that the Series of Numbers has no Limits.

Inconvenience of expressing each number by a particular word or character.

If therefore each Number was to be expressed by a Word or particular Character, the Number of Words and Characters to be employed would be infinite, and the Life of Man would scarce be sufficient to learn to reckon up to Fifty Thousand, which is a small Number not only among the Numbers possible, but even among those which are in daily Use: but the Computists obliged to employ great Numbers have found out the Art of counting by means of a very few Words and Characters repeated several times; this Art is called *Numeration*.

III.

Characters employed to represent numbers.

Thow there is an Infinity of different Numbers, the Computists have found means of expressing them all with ten Characters differently combined and repeated, viz.

0	1	2	3	4	5	6	7	8	9
Cypher	One	Two	Three	Four	Five	Six	Seven	Eight	Nine

It is easy to perceive that with those ten Characters we can reckon from Nothing to Nine inclusively, without having recourse to any new Artifice, but not further, if we have no other Unit but of the Species to be numbered. For Example, if the Units a Number consists of are Feet, we can reckon from Nothing to nine Feet, but we cannot reckon beyond 9 Feet, if we had no other Unit but the Foot.

The Computists, therefore, besides the Unit of the Species to be reckoned, which is called *principle Unit*, for Example, besides the Foot, which in the foregoing Example is the principle Unit, have imagined others which are called *collective Units*, which may be reckoned also from Nothing to Nine inclusively; and by help of those collective Units, all possible Numbers arising from the Repetition of the principle Unit may be expressed, in the manner we are going to explain.

IV.

How the Arithmetical characters are distinguished.

When the Number of principle Units do not exceed Nine, and consequently may be expressed by one Character, this character is called *Number of Units of the first Degree*, and the Place it stands in is called the first Place.

Of ten Units of the First degree is formed a collective Unit called *Ten* or an *Unit of the second Degree*, and may be reckoned from Nothing to Nine, by means of the Characters 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. posed to represent all possible numbers.

To distinguish the Character which represents a Number of those new *Units of the second Degree*, from that which represents a Number of *Units of the first Degree*; it is wrote down in the second Place to the left Hand of the Figure, which represents a Number of Units of the first Degree, and which stands in the first Place, this first Place being always filled up with a Character representing the Number of Units of the first Degree, as will appear by the following Examples.

1^o When the Number of Units to be reckoned consists only of *Tens* or of Units of the second Degree, and do not exceed *Nine*, the Character (0) which represents no Number, is set down in the first Place, and by denoting that there are no Units of the first Degree remove the Character which represents a Number of Units of the second Degree into the second Place, and determines it to represent a Number of this Kind of collective Units rather than any other.

Thus 10 represents *Ten*, 20 represents two Tens, or, *Twenty*, 30 three Tens, or, *Thirty*, 40 four Tens, or, *Forty*, 50 five Tens, or, *Fifty*, 60 six Tens, or, *Sixty*, 70 seven Tens, or, *Seventy*, 80 eight Tens, or, *Eighty*, 90 nine Tens, or, *Ninety*.

2^o Since by writing down some one of the ten Characters 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, in the second Place we can reckon a Number of Tens from Nothing to Nine, and by setting down some one of the same Characters in the first Place we can reckon a Number of *simple Units* or of the *first Degree* from Nothing to Nine; it is easy to perceive that we can by help of the two Characters which stand in the two first Places reckon from Nothing to Ninety nine. All numbers from nothing to ninety nine are represented by help of two characters.

The Characters which stand in the second Place, preserve the Names of the Numbers of Tens they represent, and those which stand in the first Place retain the Names of the Numbers of simple Units they represent. The following Numbers however are to be excepted, 11, 12, 13, 14, 15, 16, 17, 18, 19, which are read and wrote as follows, *Eleven, Twelve, Thirteen, Fourteen, Fiveteen, Sixteen, Seventeen, Eighteen, Nineteen*, and not *Ten-one, Ten-two, Ten-three, Ten-four, Ten-five, Ten-six, Ten-seven, Ten-eight, Ten-nine*, as the general Rule would require. Each character retains the name of the number it represents.

V.

As ten *Units of the first Degree* form an *Unit of the second Degree*, in like manner ten *Units of the second Degree*, form an *Unit of the third Degree*, which is called a *Hundred*, and the Character which represents a Number of those new Units is set down in the third Place to the left Hand. All numbers from nothing to nine hundred and ninety nine are represented by

Those new Units of Hundreds may be reckoned from Nothing to Nine, with the same Characters 0, 1, 2, 3, 4, 5, 6, 7, 8, 9: and as we may

help of three
characters.

reckon also with the same Characters the Tens from Nothing to Nine, and the simple Units from Nothing to Nine; it is manifest that we can by help of the three Characters which stand in the three first Places reckon from Nothing to *Nine Hundred and Ninety Nine*.

The different Numbers of Hundreds have no particular Names, they are specified by the name *Hundred*, preceded by the Word which expresses their Number; thus, 100 is named one Hundred, 200 two Hundred, 300 three Hundred, 400 four Hundred, 500 five Hundred, 600 six Hundred, 700 seven Hundred, 800 eight Hundred, 900 nine Hundred.

How a number represented by three characters is read.

The Reading of a Number represented by three Figures, is performed by pronouncing first the Number of Hundreds represented by the Figure in the third Place; and afterwards the Names of the Numbers represented by the two Figures which stand in the second and first Places; for Instance 364 is pronounced three Hundred and Sixty Four, 576 five Hundred and Seventy Six, 193 one Hundred and Ninety Three, 916 nine Hundred and Sixteen, 807 eight Hundred and Seven, 290 two Hundred and Ninety.

VI.

All numbers possible are represented by help of ten characters.

Of ten *Units of the third Degree* called *Hundreds*, is formed an Unit of the fourth Degree, which is called a *Thousand*. Those new collective Units may be reckoned as the other collective Units already mentioned, from Nothing to Nine, and the Figure which represents a Number of them is always set down in the fourth Place to the Left of the *Hundreds*.

By continuing in this Manner to form new collective Units each consisting of ten Units of the Degree that immediately precedes; there will result a Progression of Units decuple of each other, by means of which it is easy to perceive that all possible Numbers however great may be represented, employing only the ten Characters 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Every three places of Figures are called periods, and have the same names.

The three Figures which occupy the three first Places, have particular Names, that which stands in the first Place is called *simple Units*, that in the second, *Tens*, and that in the third *Hundreds*, particular Names might in like manner be given to the Units of higher Degrees, but as those new Names would burthen the Memory to no purpose, and it would be too difficult to pronounce a great Number of Words in a certain Order; the same Names of *Units*, *Tens*, *Hundreds* are given to the three following Figures which stand in the fourth, fifth and sixth Places, and to every three succeeding Figures: So that if the Figures which represent a great Number, be parted or distinguished into Periods of three Figures in each Period, beginning at the right Hand; each Period will contain a Number of Units in its first Place to the right Hand, a Number of Tens in its second Place; and a Number of Hundreds in its third Place, except the last Period which will not always consist of three Figures, but some times only of One or of Two; viz. a Number of Units, or a Number of Units with Tens.

This prodigious Number is thus pronounced : seventy one *Nonillions*, seven hundred and sixty four *Oillions*, five hundred and forty six *Sepillions*, eight hundred and nine *Sextillions*, two hundred and seventy *Quintillions*, eight hundred and fifty seven *Quadrillions*, one hundred and forty two *Trillions*, eight hundred and fifty seven *Billions*, one hundred and forty two *Millions*, eight hundred and fifty seven *Thousands*, one hundred and forty Two *Units* or *Grains of Sand*.

VIII.

Of decimal
parts.

Having sufficiently Explained how from the various Combinations and Repetitions of the ten Characters, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, all possible Numbers consisting of simple Units of every Denomination may be represented : we shall proceed to explain how by means of the same Characters all Numbers consisting of Units that are only the tenth the hundredth or thousandth Part, &c. of a simple Unit considered as the principle Unit are represented.

Since ten Units of any particular Order form an Unit of an higher Order, belonging to the next Place to the Left ; it is manifest that if we divide an Unit of any degree into ten equal Parts, those Parts will be ten Units of a lower Degree, belonging to the next Place to the right Hand. For Example, if we divide a *Hundred* which is an Unit of the third Degree, belonging to the third Place, into ten equal Parts ; there will result *Tens* or Units of the second Degree, belonging to the second Place : and if we divide again one of those new Units into ten equal Parts, there will result *simple Units*, or Units of the first Degree, which should stand in the first Place. Whence by dividing continually the Units into ten equal Parts ; there continually results new Units subdecuple or ten times less, and the Figures which represent a Number of them, stand in places more and more removed towards the right Hand, until we come to the simple Units which stand in the first Place ; but there is no reason why this Division may not be continued further on.

IX.

Numbers
set down
after the
units place
are decimal
parts.

The Figures standing in Places continually removed towards the right Hand, representing a Number of Units subdecuple of those of the preceding Figures : if a Comma be prefixed to the Figure representing the principle Units, to distinguish the Place of those Units, and if any Number of Figures be wrote after the Comma towards the right Hand ; as in this Number 576,347892 ; it is easy to perceive that the Figures 576 to the left Hand of the separating Point, represents a Number of principle Units ; and that the other Figures 347892 to the right Hand, reckon Units so much the more subdecuple of the principle Unit, as they are more removed to the right Hand from the separating point. For the Character 3 which stands next the Comma to the right Hand, or next the Character 6 which represents simple Units, will represent Units subdecuple or ten times less

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than those represented by the Character 6, whence the Units of this Number 3 will be the tenth Part of a simple Unit, and are called *Tenths*. In like manner the Character 4 which stands next to the Character 3 to the right Hand: represents a number of Units that are the tenth Parts of those of the Figure 3, and consequently will be the tenth Parts of a Tenth, or the hundredth Parts of the principle Unit, and for this reason are called *Hundredths*. In fine the Units represented by the following Characters 7892 to the right Hand, becoming continually subdecuple, will be the Thousandth, the ten Thousandth, the hundred Thousandth, the Millionth, Part, &c. of the principle Unit, and for this Reason are called *Thousandths*, *ten Thousandths*, *hundred Thousandths*, *Millionths*, as in the following

Example.

All the Figures which are to the right Hand of the simple Units, whose Place is distinguished by a Comma, are called *decimal Figures*, and their Units are called *decimal Parts* or *decimal Fractions*.

Decimal Figures									
5	7	6	3	4	7	8	9	2	
Hundredths	Tens	Units	Tenths	Hundredths	Thousandths	Ten Thousandths	Hundred Thousandths	Millionths	

X.

It is now easy to perceive how Numbers whose expression contain decimal Figures are read. Let for Example 576,347892 be proposed to be read.

1^o This Number may be considered as consisting of two Parts, the first part represented by the Figures 576 to the left Hand of the separating Point, reckon simple Units, and consequently may be wrote thus, 576 simple Units, as to the second Part represented by the Figures 347892 to the right Hand of the separating Point, the first Figure 2 expresses Millionths, and Ten of those Units forming an Unit of the following Figure 9 to the left Hand, this Figure 9 expresses Ninety Millionths, in like manner ten Units similar to those of the 9, that is ten Tens or one hundred Millionths form an Unit of the Figure 8 which follows to the left Hand, consequently this Figure 8 expresses eight hundred Millionths, and proceeding in this manner it will appear that the second Part of the proposed Number 347892 reckon Millionths, consequently may be wrote thus 347892 Millionths, wherefore the entire Number represented by (576,347892) may be read as follows, *five Hundred and Seventy six Units, and three Hundred and Forty seven Thousand eight Hundred and Ninety two Millionths*.

2. Since the Units of the proposed Number 576,347892 are continually decuples of one another, proceeding from the right Hand to the Left, and the Units of the lowest Degree are *Millionths*; it is easy to perceive that the entire Number may be considered as consisting of *Millionths* only; consequently may be read thus; *five Hundred and seventy six Millions three Hundred and forty seven Thousands eight Hundred and ninety two Millionths*.

First manner of expressing decimal numbers containing decimal figures.

Second manner of expressing numbers containing decimal figures.

ELEMENTS OF

In what a number containing decimal figures differs from one that has none.

From whence it appears that a Number that contains decimal Figures differs from one that has none, only because they consist of different Units. The Number which has no decimal Figures consists of absolute Units, or of Units of the Species to be reckoned. The Number which has decimal Figures consists of Units which are only Parts of absolute Unit or of the Unit of the Species to be reckoned; and it is the Place of the Comma which determines what Parts of the whole Unit the proposed Number consists of, for Example, if there are One, Two or Three or Six decimal Figures after the Comma, the Number will consist of Tenths or Hundreths or Thousandths or Millionths of the principle Unit.

Third manner of expressing numbers containing decimal figures.

3^o Lastly when Numbers contain decimal Figures, the Part to the left Hand of the separating Point may be first read, and afterwards the Values of all the decimal Figures expressed one after another, with the Names of their particular Units, for Example, the Number 576,347892 may be read thus, five Hundred and Seventy six *integer Units*, and three *Tenths*, four *Hundreths*, seven *Thousandths*, eight *ten Thousandths*, nine *hundred Thousandths*, and two *Millionths*.

X I.

How numbers consisting only of decimal parts are represented.

A Number very often has neither principle Units nor collective Units, but only decimal Figures: and as the Value of decimal Figures can only be known from the Places they stand in; the Place of the simple Units is distinguished by a Cypher to which a Comma is prefixed denoting that there are no simple Units, and that all the Figures to the right Hand are decimal Figures.

It likewise often happens, that a Number not only has no principle Units, but also has no *Tenths*, nor *Hundredths*, nor *Thousandths*, &c. and has decimal Figures of an inferior Denomination. In this Case in order to assign to each decimal Figure its proper Place, not only the Place of the simple Units is filled up with a Cypher to which a Comma is prefixed, but also Cyphers are wrote in the Places of the decimal Figures that are wanting; so that a Number which has neither Units nor Tenths, nor Hundreths, nor Thousandths, will have four Cyphers to the right Hand. For Example, (0,000289) will represent 289 Millionths.

X II.

Cyphers annexed to the right hand of the comma or separating point of a number do not alter its value.

Since the Comma prefixed to a Figure determines it to represent simple Units; we may annex as many Cyphers as we please to the right Hand of the Comma, without altering the Number to the right Hand of which those Cyphers have been annexed. For Example, the number 389 will not be altered by writing it thus (389,000) and will always represent *three Hundred and Eighty nine* simple Units; because the Cyphers annexed after the Comma, denote that there are no *Tenths*, no *Hundreths*, no *Thousandths*, nor any Kind of decimal Parts.

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By annexing Cyphers to the right Hand of the Comma of a Number, the Denomination of the principle Unit may be changed, or it may be reduced into smaller Units. For Example, if it was required to reduce the Units of the Number 389 into *Hundreths*; by writing it thus (389,00) it will represent 38900 *Hundreths*. If it be required to reduce the Units of this Number into *ten Thousandths* by writing it thus (389,0000) it will represent 3890000 *ten Thousandths*, and so of any other Transformation.

XII.

After having shewn how to read and write Numbers, we now proceed to explain the Operations performed upon them, which are reduced to the Four following. *Addition, Subtraction, Multiplication and Division*; all the other more complicated Operations being only different Combinations of those. We may perform those Operations either upon Numbers that have only one principle Unit, as the Pound sterling, the Foot, or any other Unit assumed at Will, or established by Custom, which are called simple Numbers; or upon Numbers which have several principle Units, such as 5 Pounds 6 Shillings and 8 Pence, 3 Fathoms 4 Feet 9 Inches, &c. which are called compound Numbers. We shall treat of the Addition, Subtraction, Multiplication and Division of simple Numbers in this Chapter.

The different operations performed on numbers are reduced to addition subtraction multiplication and division.

XIII.

Addition is an Operation by which a Number is found equal to the Sum of several Numbers.

As all the Units of the same Number should be of the same Species, it is easy to perceive that the Numbers to be added together in order to form but one, should consist of Units of the same Species, or that may be reduced to Units of the same Species. For Example, if it was required to add together 20 Pounds sterling, 50 Horses, and 60 Feet, three Numbers consisting of Units of different Species, and not reducible to the same Species of Units, we are not simply to add together 20, 50, and 60, their Units being different, but set them down separately with their particular Denominations. In like manner if it was required to add together 2 Fathoms, 5 Feet and 8 Inches, we are not simply to add together 2, 5, and 8, their Units being different. However as those different Units are reducible to the same Species, since 2 Fathoms are equal to 144 Inches, 5 Feet are equal to 60 Inches, and that a Number may be found equal to the three Numbers 144 Inches, 60 Inches, 8 Inches, we can add together 2 Fathoms, 5 Feet, 8 Inches, so that their Sum shall form but one Number, by reducing all the Units to the same Species.

The numbers to be added should consist of units of the same species.

XIV.

To perform this Operation more commodiously, the Computists place Proceſs of the Numbers to be added together under one another, in such a Manner

addition.

2 B

that the Figures of the same Degree stand in the same vertical Column; and draw a Line under all those Numbers, to separate them from that which is to represent their Sum.

They afterwards add together all the Figures of the lowest Degree, which are in the first Column to the right Hand. If the Sum of those Figures are less than 10, and consequently may be represented by one Figure; they set down this Figure in this first Column, under the Line; but if this Sum exceeds 9 and cannot be represented but by several Figures; they write the first Figure of Units of this Sum under the Line, and the other being of the same Species of those of the second Column, is retained to be added to them.

They perform the same Operation upon all the Columns until they come to the last which contains the Figures of the highest Degree; and adding the Figures of this last Column with that or those that have been retained from the preceding Column; they set down the Sum in this Column under the Line.

This Operation being performed; all the Figures under the Line represent the Sum of all the Numbers proposed to be added together.

xv.

Let the two Numbers 3456 and 4231 be given to be added together.

First example of the addition of integers,

Those two Numbers being disposed as in the Margin, to determine one after another the Figures of their Sum, I proceed thus

1^o I add together the two Figures 6 and 1 which form the first Column: and as their Sum is *Seven*, which may be represented by the one Figure 7: I set down 7 under the Line, in this first Column.

2^o I afterwards add the Figures 5 and 3 of the second Degree, which form the second Column: and as their Sum is *Eight*, which may be represented by the one Figure 8; I set down 8 under the Line, in this second Column.

3^o In like Manner I add together the Figures 4 and 2 of the third Column: and as their Sum may be represented by the one Figure 6; I set down 6 under the Line, in this third Column.

4^o Finally I add together the Figures 3 and 4 of the fourth and last Column: and their Sum being represented by the one Figure 7; I write 7 under the Line, in this fourth Column.

The Operation being finished; I find under the Line the four Figures 7687, which represents the Number seven Thousand six Hundred and eighty Seven, to which amounts the two Numbers 3456 and 4231 which were proposed to be added together.

1 st Column	2 nd Column	3 ^d Column	4 th Column
6	1		
5	3		
4	2		
3	4		
2	3		
1			
7	8	6	7

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XVI.

Let the four Numbers 5874, 9956, 459, 15 be given to be added together.

These Numbers being disposed as in the Margin, to determine one after another, the Figures of this Sum, I proceed as follows.

5874
9956

Second example of the addition of integers.

1^o I add together the Numbers represented by the Figures which form the first Column, saying: 4 and 6 is 10, and 9 is 19, and 5 is 24. As this Sum 24 is represented by two Figures, viz. 4 which represents 4 Units, and by 2 which represents 2 Tens, I write 4 in the first Column under the Line, and I retain the 2 or rather the two Tens to add them to the four Numbers of Tens which form the second Column.

459
15

2^o I next add the Numbers of the second Column, whose Figures consist of Units of the second Degree, saying: 2 I retained from the first Column and 7 is 9, and 5 is 14, and 5 is 19, and 1 is 20. As this Sum 20 Units of the second Degree, is represented by two Figures, viz. 2 which denotes that there are two Tens of the Units of the second Degree, which form 2 Units of the third Column or of the third Degree, and 0 which denotes that there are no Units of the second Degree that were to be added; I write 0 under the second Column, and I retain 2 to add it with the Figures of the third Column.

3^o In like manner I add together the Figures of the third Column which represent Numbers consisting of Units of the third Degree, saying: 2 I retained from the second Column and 8 is 10, and 9 is 19, and 4 is 23. As this Number 23 is represented by two Figures, viz. by 3 which denotes three Units of the Column which were added, and by 2 which represents two Tens of Units of the same Column, or two Units of the fourth Column, I set down 3 under the Line in the third Column, and I retain 2 to add it with the fourth Column which follows.

4^o Finally I add together the Figures of the fourth Column with what I retained, saying 2 that I retained and 5 is 7, and 9 is 16. This Sum consisting of Units similar to those of the fourth Column, and of an Unit of a higher Degree, I write 6 under the Line in the fourth Column. As to the Unit of a higher Degree, I would have retained it, if there was another Column of Figures to be added, but as there is not, I advance the 1 to the left Hand of the 5, that is I set down simply 16.

XVII.

Let the four Numbers (5874; 4631, 752; 6872, 44; 9797, 5) the three last of which contain decimal Figures, be given to be added together.

I set down all those Numbers one under another, so that the Figures of the simple Units are in the same Column, and all the other Figures of the same Denomination are likewise in the same Column one under another. Having disposed the Figures in this Manner,

Example of the addition of numbers containing decimal figures.

1°. I add the Figures of the lowest Degree, which are *Thousandths*: and as there are but 2 *Thousandths*, I set down 2 under the Line in the Column of *Thousandths*.

2°. I add together the Figures 5 and 4 of the following Column which is that of the *Hundredths*; and as their Sum 9 is represented by one Figure, I set down this Figure 9 in this Column under the Line.

3°. I add together the Figures of the Column of *Tenths*, saying 7 and 4 is 11, and 5 is 16; and as this Sum 16 consists of 6 Units of the Degree of this Column, and of a Ten which makes an Unit of the following Column; I set down 6 under this Column and I retain 1 to add it to the following Column.

4°. Adding the Unit I retained to the Figures of the Column of the *simple or principle* Units, I find 15, that is, 5 Units of this Column, and 1 Ten which makes an Unit of the following Column; whence I write 5 under this Column, and I retain 1 for the following Column.

5°. Adding in like Manner the Unit I retained to the Column of Tens, I find 27, that is, 7 Units of this Column, and 2 Tens which make two Units of the following Column: whence I write 7 under this Column and I retain 2 for the following Column.

I perform the same Operation upon the other Columns, and I find (27175,692) for the Sum of the four proposed Numbers.

XVIII.

Addition of decimal numbers performed by reducing them to units of the same denomination.

The Numbers (5874; 4631,752; 6872,44; 9797,5) which were given to be added together in the last Example, might be changed, or reduced so as to have principal Units of the same Denomination: and as one of those Numbers (4631,752) represents 4631 *simple* Units and 752 *Thousandths*, or 4631752 *Thousandths*; the other Numbers may be reduced so as to have *Thousandths* for their principal Units: then the Number 5874 which represents 5874 *simple* Units, will become 5874000 *Thousandths*, the Number (6872,44) will become 6872440 *Thousandths*, and the Number (9797,5) will become 9797500 *Thousandths*. All the proposed Numbers being reduced so as to consist of Units of the same Denomination, they may be added as Numbers that have no decimal Figures, and their Sum will be 27175692 *Thousandths*: to suppress afterwards the Word *Thousandths*, it suffices to set down the same Sum thus 27175,692 (Art. x).

XIX.

Quantities to be subtracted one from the other should be of the same species.

The Operation by which one Quantity is deducted or taken from another, is called *Subtraction*.

The Quantity therefore to be deducted must be contained in the Quantity from which it is to be deducted; and consequently those two Quantities must be of the same Species, or reducible to the same Species.

To perform this Operation more commodiously, the Computists set down the Number to be deducted under the Number from which it is to be deducted, placing the Units of the one under the Units of the other, the Tens under the Tens, the Hundreds under the Hundreds, &c. and the decimal Figures of the same Species under one another. The two Numbers being thus disposed, a Line is drawn to separate them from their difference sought.

Then the Subtraction is performed by Parts, each lower Figure is deducted from the Figure that stands over it, beginning at the right Hand, and proceeding from the Figures of the lowest Degree to those which are of a higher Degree. But in this Operation, there occur three Cases; either the lower Figure is less than the Figure that stands over it, or equal or Greater.

If the lower Figure is less than the Figure that stands over it, the Former may be easily deducted from the Latter, and there will be a Remainder which is set down under the Line.

If the lower Figure is equal to the Figure that stands over it, the former may also be deducted from the latter, and as there is no Remainder, a Cypher is set down under the Line for the remainder.

Finally if the lower Figure is greater than the Figure that stands over it, the Former cannot be deducted from the Latter without a Preparation which consists in borrowing an Unit from the Figure of an higher Degree and adding it to the Figure which is too Small: then the Figure from which an Unit is borrowed is accounted one Less than it is, and this Unit being carried to a Place one Degree lower, is accounted as Ten, consequently the Figure which is too Small being increased by Ten, the Figure immediately under it which cannot exceed 9 may be deducted from it.

XX.

Let it be required to deduct 324 from 466.

Having set down the Number to be deducted under the Number from which it is to be deducted, placing the Figures of the same Denomination under one another, and having drawn a Line as in the Margin, to determine the Figures of the Remainder one after another, I proceed as follows.

1°. Beginning at the place of Units, I say: if from 6 be taken 4 there will remain 2, which I set down under the Line in the Units Place.

2°. Then proceeding to the following Rank, I say: If from 6 be taken 2, there will remain 4 which I set down in the second Rank.

3°. Lastly, being come to the last Rank, in which are the Figures of the highest Degree, I say: if from 4 be taken 3, there will remain 1 which I write under the Line in the third Rank, and the Operation being finished, the Remainder of the Subtraction will therefore be 142.

First example of the subtraction of integers.

$$\begin{array}{r} 466 \\ 324 \\ \hline 142 \end{array}$$

XXI.

Let it be required to deduct 51 from 552.

Second example of the subtraction of integers.

Having placed the Numbers as in the Margin. 1° Beginning by the Figures of the lowest Degree, I say; if from 2 be taken 1, there

5 5 2

will remain 1 which I write under the Line.

5 1

2°. Proceeding to the following rank, I say: if from 5 be taken 5, there will nothing remain; whence I set down a Cypher under the Line to denote that the Figures representing Tens being Subtracted one from the other, there is no Remainder.

5 0 1

3°. Lastly proceeding to the third Rank in which there is no Figure to be deducted, I say: if from 5 be taken Nothing, there will remain 5; whence I set down 5 for the Remainder under the Line in this third Rank. The Operation being finished, the Remainder required will be 501.

XXII.

Let it be required to deduct 599 from 758.

Having disposed the Figures of the same Denomination under one another, as in the Margin, I proceed as follows.

Third example of the subtraction of integers.

1°. Beginning by the Figures of the lowest Order, because 9 cannot be deducted from 8, I take from the following Figure (5) of the upper Number an Unit, to add its Value to 8 which is too Small and as this Unit carried to the Place of 8 becomes 10, from 18 I deduct 9, and there remains 9, which I set down under the Line in the first Place.

7 5 8

5 9 9

1 5 9

2°. Proceeding to the following Rank in which 5 is reduced to 4, on account of the Unit that has been taken from it, and which is denoted by a Point set over it, because 9 cannot be deducted from 4, I take an Unit from the following Figure (7) of the upper Number to add its Value to the 4 from which 9 is to be deducted, and as this Unit of the third Degree carried to an inferior Rank becomes 10, from 14 I deduct 9, and there remains 5 which I set down under the Line.

3°. Proceeding to the Rank of Hundreds in which 7 is reduced to 6, on account of the Unit which has been taken from it, and which is denoted by a Point set over it; I say if from 6 be taken 5 there will remain 1 which I set down under the Line; whence the Remainder required will be 159.

XXIII.

It often happens that the Figure from which an Unit is to be borrowed is a Cypher, which representing nothing, can lend nothing; in this Case the Computists borrow from the Figure to the left Hand of the Cypher or Cyphers, if there are several; and leaving 9 above each of those Cyphers, reserve only Ten to add it to the Figure from which the one immediately under it is to be Subtracted.

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XXIV.

Let it be required to deduct 5486 from 7004.

Having disposed the Figures as in the Margin, I proceed as follows, because 6 cannot be taken from 4 and I cannot borrow from the Figure which is next to 4, to the left Hand, nor from the following, because they are two Cyphers; I borrow an Unit from the 7 which is next to those Cyphers. But as an Unit borrowed from the 7 is equivalent to 100 Tens, and only one Ten is required to add it to the 4 from which 6 is to be deducted, I leave above the Cyphers which are in the third and second Places the two Figures 99 which represent 99 Tens: and adding the Ten to 4 I deduct 6 from 14, and there remains 8 which I set down under the Line for the first Figure of the Remainder sought.

$$\begin{array}{r} 99 \\ 7004 \\ 5486 \\ \hline 1518 \end{array}$$

Fourth example of the subtraction of integers.

The upper Number being thus prepared, the 7 from which 1 has been borrowed is reduced to 6, and the two Cyphers over which the 9 have been left, are now to be considered as 9. To continue therefore the Subtraction, I deduct 8 from 9, there remains 1 which I write under the Line. I afterwards deduct 4 from 9, and there remains 5 which I set down likewise under the Line. Lastly I deduct 5 from 6, and there remains 1 which I also set down under the Line.

The Operation being finished the Remainder required will be 1518.

XXV.

When there are decimal Figures in the Number to be deducted or in the Number from which it is to be deducted, or in both; the Computists first dispose under one another the Figures of the same Degree; and when one of the two Numbers has not as many decimal Figures as the other, they annex Cyphers to the right Hand of that which has the Least, to make the decimal Figures equal in both; which will not change the Value of this Number, since the Comma preserves to each Figure the Place it had before the Cyphers were annexed.

How the subtraction of decimal numbers is performed.

The two Numbers being thus prepared and their Units of the lowest Degree being of the same Kind, the Subtraction is performed as if the Numbers had no decimal Figures.

XXVI.

Let it be required to deduct 8716,257 from 230009,3.

As those two Numbers have not the same Quantity of decimal Figures, the Second having two more than the first; I annex two Cyphers to the right Hand of the First, and the two proposed Numbers being transformed into those set down in the Margin: whose Units of the lowest Degree are *Thousandths*, I perform the Subtraction as on Numbers that have no decimal Figures.

$$\begin{array}{r} 230009,300 \\ 8716,257 \\ \hline 221293,043 \end{array}$$

Example of the subtraction of decimal numbers.

ELEMENTS OF

XXVII.

How the foregoing operation of subtraction may be contracted.

When there are Figures in the Number to be deducted greater than the corresponding ones in the Number from which it is to be deducted, and when placed under one another, to perform the Subtraction, Units are borrowed from the upper Figures of higher Degrees, to add their Values to the Figures which are too Small; instead of taking from the upper Figures the Units that have been borrowed before the Figures that are underneath are deducted from them, it is easy to perceive that the Unit borrowed from an upper Figure may be added to the Figure under it; and so from the upper Figure be deducted at once the Unit that was borrowed and the Figure under it.

XXVIII.

Let it be required to deduct 5486 from 7004.

An example of subtraction performed by the contracted method.

Having placed the Numbers as in the Margin, to find all the Figures of the Remainder, I proceed as follows

7	0	0	4
5	4	8	6
1	5	1	8

1°. Because 6 cannot be taken from 4, I add ten to 4 and deduct 6 from 14, setting down the Remainder 8 under the Line.

2°. The Ten added to the 4 should have been borrowed from the Cypher which follows the 4. Consequently as this Unit should be taken from this Cypher, as also the Figure 8 which stands under it, I add the Unit borrowed to the 8, to deduct their Sum 9 from the Cypher: but as this cannot be done, I borrow an Unit of a higher Degree, which being carried to the Place of the Cypher is equivalent to 10; and deduct 9 from 10, and set down the Remainder 1 under the Line.

3°. As the Unit borrowed should be deducted from the second Cypher, as also the 4 which stands under it, I add the Unit borrowed to the 4, to deduct their Sum 5 from the Cypher which is above it; but as this cannot be done; I borrow an Unit of a higher Degree, which being carried to the Place of the Cypher is equivalent to 10, and deduct 5 from 10, and set down the Remainder 5 under the Line.

4°. Lastly as the Unit borrowed should be taken from the 7, as also the 5 which is under it, I add this Unit to the 5, and deducting their Sum 6 from 7, I set down the Remainder 1 under the Line.

The Operation being finished, the difference required will be 1518.

XXIX.

Subtraction employed to prove addition.

To discover whether any Error has been committed in computing, a particular Operation is employed, which is called the Proof.

In Addition, as the Sum or total Amount should contain all the Numbers which are added together, if from the Sum be deducted all the Parts which were added, there should be no Remainder. Addition therefore may be proved, by deducting from the Sum all the Parts of the Numbers added together, the Operation being accurately performed, if after the Subtraction there is no Remainder.

The Order to be observed in this Operation, will appear in the following Example: in which the three Numbers 473, 567, 924 were added together as in the Margin, and their Sum found to be 1964.

Process of
this operati-
on explained
by an exam-
ple.

4	7	3
5	6	7
9	2	4
2	9	6
2	2	4

If this Sum 1964 is exact and contains all the Parts of the Numbers added together; it is manifest that if all the Hundreds all the Tens and all the Units of the Numbers added together, be deducted from it; there should be no Remainder.

The Subtraction might be performed beginning at the Units Place as usual; but as the Units, Tens and Hundreds would then be collected in the same Order they were added together to find their Sum, and consequently this Operation would become liable to the same Errors that might have been committed in performing the Addition. The Computists perform the Subtraction beginning by the Figures of the highest Degree.

To deduct, therefore 1^o the Hundreds of the Numbers added together, from the Sum of those Numbers, I add those Hundreds, saying: 4 and 5 is 9, and 9 is 18; and deducting those 18 Hundreds from the 19 Hundreds of the Sum, I set down the Remainder 1 under the 9, after having barred the two Figures 19.

2^o. To deduct the Tens, I add them together, saying: 7 and 6 is 13, and 2 is 15; and deducting those 15 Tens from the 16 Tens that remain in the Sum, I set down the Remainder 1 under the 6, after having barred the two Figures 16.

3^o. Lastly I add the Units, saying: 3 and 7 is 10, and 4 is 14; and deducting those 14 Units from the 14 Units which are still in the Sum, and there being no Remainder: I set down a Cypher under the 4, after having barred 14.

As there is no Remainder left after this Subtraction, I conclude, that the Addition of the three Numbers 473, 567, 924, has been accurately performed.

xxx.

In Subtraction as the Quantity deducted is contained in the Quantity from which it is deducted, and the Remainder is also contained in this Quantity; so that the Quantity deducted and the Remainder are all the Parts of the Quantity out of which Subtraction is made: if the Quantity Subtracted be added to the Remainder; their Sum will be equal to the Quantity out of which Subtraction has been made. Subtraction therefore may be proved by adding the Number subtracted to the Remainder, and their Sum, if the Operation has been accurately performed, will be equal to the Number out of which Subtraction has been made.

Addition
employed to
prove sub-
traction.

ELEMENTS OF

Let, for Example, 5486 be deducted from 7004, and the Remainder be found to be 1518, as in the Margin, adding 1518 to 5486, I find their Sum 7004 to be equal to the Number out of which Subtraction has been made, from whence I conclude that the Subtraction of 5486 from 7004 has been accurately performed.

$$\begin{array}{r}
 7004 \\
 5486 \\
 \hline
 1518 \\
 7004
 \end{array}$$

Process of this operation explained in an example.

XXXI.

Multiplication is an Operation by which a Quantity is repeated a certain Number of Times.

The particular denominations given to the numbers concerned in multiplication.

Two Numbers therefore are required to perform a Multiplication.
 1^o The Number to be repeated, which is called the *Multiplicand*.
 2^o The Number that denotes by the Number of its Units how often the *Multiplicand* is to be repeated, and which is called the *Multiplicator*.

The *Multiplicand* and the *Multiplicator* are also called the *Factors* of the Multiplication; and the Number which contains the *Multiplicand* as often as the *Multiplicator* contains Unity, is called the *Product*.

For Example, if it was required to multiply 8 by 4, those two Numbers 8 and 4 will be the two *Factors* of the Multiplication; the first (8) which is to be repeated will be the *Multiplicand*; the second (4) which denotes that the *Multiplicand* is to be repeated 4 Times, will be the *Multiplicator*; lastly the Number 32 found by repeating 8 four Times, will be the *Product*.

XXXII.

The sign $\times$ is employed to denote that two quantities are to be multiplied one by the other.

To denote that two Numbers are to be multiplied one by the other, the Computists place between them the Note $\times$ which signifies *Multiplied by*. Thus 8×4 denotes that 8 is to be multiplied by 4; and as the *Product* arising from this Multiplication is 32, it may be said that 8×4 is equal to 32.

If it was required to multiply again this *Product* 8×4 or 32 by a new *Multiplicator* 2; the Computists would write down $8 \times 4 \times 2$ or 32×2 , whose *Product* is 64; that is, they set down one after another all the *Factors* that enter into the *Product*, separating them by the Note $\times$.

Inconveniency of reducing multiplication to addition.

Multiplication may be performed by reducing it to Addition, for Example, the Multiplication of 32 by 4, might be performed by setting down the *Multiplicand* 32 four Times in the Order that Quantities are set down which are to be added; the Sum of this Addition containing 32 four Times, will be the *Product* of 32 multiplied by 4. But as this Method of performing Multiplication could only be applied when the *Multiplicator* consists of very few Units; being impracticable when the *Multiplicator* is a great Number. The Computists were under the Necessity of employing particular Rules to abridge the Operations.

XXXIII.

The *Product* therefore of a Multiplication will consist of Units of the same Species as those of the *Multiplicand*; for this *Product* resulting from the repeated Addition of the *Multiplicand*, can have no other Units than it.

From whence it follows, 1^o that if the Units of the lowest Order of the Multiplicand, are simple Units, or Tens, or Hundreds &c. and the Multiplier consists only of simple Units, the Units of the lowest Order of the Product will be also simple Units, or Tens, or Hundreds &c. whence the Product should have at the right Hand as many Places of Figures as there are at the right Hand of the Multiplicand. For Example, if 8 or 80 or 800 be multiplied by 4, which will be effected by repeating the significative Figure 8 of the Multiplicand four Times, the Product 32 will have to the right Hand as many Cyphers or Places as there are after the Figure 8 which has been multiplied, and will be 32 or 320 or 3200.

The product of two figures multiplied into one another should have as many places to the right hand as there are places to the right hand in the multiplicand and multiplier.

2^o If the Units of the Multiplier are collective Units, as Tens, or Hundreds or Thousands &c. each of those collective Units will denote that the Multiplicand is to be repeated ten Times, or a hundred Times, or a thousand Times; so that the Product will be ten Times, a hundred Times, or a thousand Times greater than if the Multiplicand was multiplied by a Number of simple Units; and consequently this Product will have also to the right Hand as many Places as there are to the right Hand of the Figure employed for Multiplier.

For Example, if 8 be multiplied by 4, or by 40, or by 400, or by 4000, &c. which will be effected by repeating 8 four Times; the Product 32 will have in all those Cases as many Cyphers or Places to the right Hand as there are to the right Hand of the Multiplier 4, and consequently will be 32, or 320, or 3200, or 32000.

From whence we may conclude that the Product of two Figures multiplied one by the other should have to the right Hand as many Cyphers or Places, as there are Cyphers or Places to the right Hand both of the Figure multiplied, and of the Figure employed as Multiplier.

For Example if 800 be multiplied by 4, or by 40, or by 400 or by 4000, &c; which will be effected by repeating the significative Figure 8 of the Multiplicand as often as there are Units in the significative Figure 4 of the Multiplier; the Product 32 will have to the right Hand the two Cyphers which are to the right Hand of 8, and all the Cyphers which are to the right Hand of the Figure 4, employed for Multiplier, and will consequently be 3200, or 32000, or 320000, or 3200000, &c.

XXXIV.

A Number let whatever be the Nature of its Units, may be repeated any proposed Number of Times; whence the Multiplicand of a Multiplication may be an abstract Number, or a concrete Number consisting of any particular Species of Units.

It is not so of the Multiplier which should denote how often the Multiplicand is to be repeated. Each of its Units whether Simple or Collective should only represent a Number of Times; whence it should necessarily be considered as an abstract or absolute Number, and never as a concrete Number.

The multiplier is always an abstract number.

However it is sometimes proposed to multiply a concrete Number by another concrete Number. For Example, if an oak Board cost 5 *Sbillings* and the Price of 20 Boards be required at that Rate, it is proposed to multiply 5 *Sbillings* by 20 *Boards*, but it is manifest that this Proposition is contrary to the Rules of Multiplication and that 5 *Sbillings* is not to be multiplied by the concrete Number 20 *Boards*, but by the absolute Number 20 *Times*. Since to obtain the Price of 20 *Boards* at the Rate of 5 *Sbillings* per Board, it suffices to repeat 5 *Sbillings* 20 *Times*.

XXXV.

In whatever order the factors of a multiplication are disposed the product will be always the same.

The Multiplicand remaining the same, if the Multiplier becomes double, or triple, or quadruple, &c. of what it was; the Product will also become double, or triple, or quadruple, &c. of what it was; because the new Product will contain the same Multiplicand Twice, Thrice, or four Times &c. oftner than the former Product contained it.

The Multiplier remaining the same, if the Multiplicand becomes double, or triple, or quadruple, &c. of what it was; the Product will also become double or triple or quadruple, &c. of what it was; because the new Product will contain a Multiplicand double, or triple, or quadruple, &c. as often as the first Product contained the simple Multiplicand; and it is manifest that a Whole should become double, triple, or quadruple, when its component Parts become double, or triple, or quadruple, &c. of what they were.

From whence it follows 1^o that the Product of two Factors, such as 7×5 , will be multiplied by 2, or by 3, or by 4 &c. by multiplying one of its Factors, either the Multiplicand 7, or the Multiplier 5, by the new Multiplier 2, or 3, or 4, &c. because by thus multiplying one of the two Factors, their Product is rendered double, triple, or quadruple, &c. of what it was.

It follows also from thence, that if any two Numbers, for Example, 3 and 7 are to be multiplied one by the other, there will always result the same Product whether we multiply 3 by 7, or 7 by 3, and consequently whether we write 3×7 or 7×3 .

For every Number may be considered as a Product of itself into Unity. Thus 1×7 is the same Thing as 7, and consequently the Product of 1×7 multiplied by 3, is the same Thing, as the Product of 7 multiplied by 3.

But to multiply 1×7 by 3 it suffices to triple the Multiplicand 1, which will give 3×7 ; and to denote that 7 is to be multiplied by 3, it is set down thus 7×3 .

Wherefore 3×7 is the same Thing as 7×3 ; that is, when two Numbers are to be multiplied one by the other, it is indifferent which of them is taken for Multiplicand or Multiplier.

From whence it follows 3^o, that if there are three Numbers such as 2, 3, 7 to be multiplied together, their Product will always be the same in

whatever Order they are multiplied, so that they may be set down the six different Ways following $2 \times 3 \times 7$, $3 \times 2 \times 7$, $3 \times 7 \times 2$, $7 \times 3 \times 2$, $7 \times 2 \times 3$, $2 \times 7 \times 3$. For since the two Products 3×7 , 7×3 , are equal; if they be multiplied by the same Number 2, the two resulting Products will be also equal: and as the two Products 3×7 , 7×3 , may be multiplied by 2, either by multiplying the Factor 3, or by multiplying the Factor 7, and that the new Factor 2 may be set down either before or after the Figure to be multiplied, there will result six different Arrangements of the three Factors 2, 3, 7. If there were a greater Number of Factors to be multiplied together, whatever way they are ranged, it is easy to perceive that the Product will be always the same.

XXXVI.

To find the Product of two Numbers represented each by one Figure, that is of two Numbers, each of which is less than 10. The Computists employ two different Methods. The first is a Table which Pythagoras is said to have invented, containing the Products of all the Numbers that may be expressed by one Figure.

This Table which consists of nine Rows of nine equal Squares, is constructed thus. In the first horizontal Row is set down the Series of natural Numbers from 1 to 9. In the first Square of the second horizontal Row is set down 2, and in the remaining Squares of this Row are set down the Numbers found by continually adding 2. In the first Square of the third horizontal Row is set down 3, and in the remaining Squares of this Row are set down the

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

First method of finding the product of two numbers less than 10 by the table of Pythagoras.

Numbers found by continually adding 3. In like Manner in the first Square of each of the following Rows is set down 4, 5, 6, 7, 8, 9, and in the remaining Squares of each Row, the Numbers found by continually adding the Number which is set down in the first Square of this Row.

To make Use of this Table in the Multiplication of Numbers less than 10, the Multiplicand is to be looked out for in the first horizontal Row, and then descending to the horizontal Row that begins by a Number equal

to the Multiplier, the Product required will be found. For Example, if it was required to multiply 8 by 7: the Figure 8 is to be looked out for in the first horizontal Row, and then descending to the horizontal Row, that begins by 7, the Product of 8 into 7 required, will be found to be 56.

XXXVII.

Second method for finding the product of two numbers less than 10.

The second Method employed by the Computists to multiply two Numbers less than 10, one by the other, consists in working with the Fingers; but for this Purpose it is necessary to know how to multiply two Numbers less than 6 one by the other. The Manner of performing this Operation is as follows.

The Computist shuts both his Hands, and attributing a Number to each Hand, he lifts up as many Fingers in each, as there are Units from the Number that is attributed to it, to Ten; then multiplying the Number of Fingers lifted up in one Hand, by the Number of Fingers lifted up in the other, he adds to the Product as many Tens as there are Fingers down in both Hands.

Let it be required, for Example, to multiply 8 by 7. attributing 8 to the right Hand, I lift up two Fingers, because there are two Units from 8 to 10, and there remain three Fingers down in this Hand; attributing 7 to the left Hand, I lift up three fingers, because there are three Units from 7 to 10, and there remain two Fingers down in this Hand. As there are three Fingers lifted up in one Hand, & two Fingers lifted up in the other, I multiply 3 by 2, which gives 6 Units for the Product; and because there are five Fingers down in all, viz Two in one Hand, and Three in the other, I add 5 Tens or 50 to the 6 Units already found, and there results 56 for the Product of 8 into 7.

XXXVIII.

Process of the multiplication of whole numbers when the multiplier is represented by one significative figure.

To multiply any Number which has no decimal Parts, by another which has none also, and that is represented by one Figure. The Computists set down the Multiplier under the Multiplicand, and drawing a Line to separate the two Factors of the Multiplication, from the Product which is set down under it; they multiply each Figure of the Multiplicand one after another, beginning with the Units Place, by the Multiplier. When each Product can be expressed by one Figure, it is set down under the Figure multiplied. But when some Products cannot be expressed but by two Figures, the First, which is of the lowest Degree is set down under the Figure multiplied, and the other is retained and added to the following Product. All the Figures of the Multiplicand being thus multiplied, and all the particular Products set down, all the Figures under the Line will represent the Product of the Multiplication required.

XXXIX.

Let it be required to multiply 964 by 4.

Having set down those Numbers as in the Margin, to find the Figures of this Product one after another.

1^o Beginning at the Units Place of the Multiplicand, I say: four times 4 is 16; and because this first Product is represented by two Figures the First of which (6) represents simple Units, and the second (1) a Ten or an Unit of the second Degree; I set down 6 in the Units Place under the Figure multiplied, and I retain 1 to add it to the following Product whose Units will be of the second Degree.

2^o Proceeding to the second Figure (6) of the Multiplicand which represents six Tens or six Units of the second Degree; I say: 4 times 6 is 24, and 1 that I retained from the foregoing Product makes 25, or rather 25 Tens or Units of the second Degree, and I set down the first Figure (5) in the second Place; retaining the second Figure (2) which represents two Tens of Tens, or two Units of the third Degree, to add it to the following product whose Units will be also of the third Degree.

3^o Proceeding to the next Figure of the Multiplicand, I say: 4 times 9 is 36, and 2 that I retained of the foregoing Product make 38, that is 38 Hundreds or Units of the third Degree, and I set down the first Figure (8) in the third Place. as to the second Figure (3) which represents three Units of the fourth Degree, I would have retained it to add it to the following Product, if there remained a Figure to be multiplied; but as the whole is multiplied, I set down this Figure 3 in the fourth Place to the left Hand of the Figure 8.

All the Figures of the Multiplicand 964 being Multiplied by 4, and the Figures of the particular Product being set down one after another, in the Manner we have explained; the Number 3856 under the Line is the Product required.

$$\begin{array}{r} 964 \\ \times 4 \\ \hline 3856 \end{array}$$

First example of the multiplication of integers.

XL.

Let it be required to multiply 964 by 60.

Having set down these Numbers as in the Margin. I observe that if the Number 964 was to be multiplied by 6, the Product would be 5784 but the Multiplier 60, being decuple of 6, the Product required should be decuple of the Product 5784 arising from the Multiplication by 6.

But according to the Rules of Numeration if each Figure of a Number be removed one Place towards the left Hand, which is effected by setting down a Cypher to the right Hand of this Number; it will be rendered decuple of what it was. Wherefore setting down a Cypher to the right Hand of the Number 5784, we will have 57840 for the Product required of 964 multiplied by 60.

$$\begin{array}{r} 964 \\ \times 60 \\ \hline 57840 \end{array}$$

Second example of the multiplication of integers.

ELEMENTS OF

XLI.

Let it be required to multiply 964 by 200.

Third example of the multiplication of integers.

Multiplying 964 by 2 Units, I find 1928 for the Product; but the Multiplicator 200 being centuple of 2: the Product required should be centuple of 1928 and consequently the simple Units of this Number should be Hundreds or Units of the third Degree, as those of the Multiplicator. 2 Hundreds.

$$\begin{array}{r} 964 \\ 200 \\ \hline 192800 \end{array}$$

Wherefore the Number 1928, should have two Cyphers or two Places to the right Hand, to express the Product of 964 by 200.

It will appear in like Manner that if the Multiplicator be represented by a Figure whose Units are of the fourth or fifth Degree, the Multiplicand is to be multiplied by this Figure, as if it represented only Units of the first Degree, and as many Cyphers are to be set down to the right Hand of the Product as there are Places to the right Hand of the Multiplicator.

XLII.

Process of the multiplication of integers when the multiplicator is represented by several significative figures.

When it is required to multiply a Number which has no decimal Parts, by another which has none also, and that is represented by several Figures. it is easy to perceive that the whole Multiplicand must be multiplied by each Figure of the Multiplicator; whence there will result, as many particular Products as there are significative Figures in the Multiplicator, and all those Products being added together: their Sum will be the Product required.

XLIII.

Let it be required to multiply 964 by 264.

First example of the multiplication of integers when they are both represented by several significative figures.

Since the Multiplicator 264 consists of four Units, six Tens, and two Hundreds, that is of 4, of 60, and of 200: the Multiplicand 964 must be multiplied by those three Numbers 4, 60, and 200. Having therefore set down the Multiplicator under the Multiplicand as in the Margin.

$$\begin{array}{r} 964 \\ 264 \\ \hline 3856 \\ 57840 \\ 192800 \\ \hline 254496 \end{array}$$

1° I multiply every Figure of the Multiplicand one after another by the first Figure 4, which represents 4 Units; setting down the Figures of the Product 3856 one after another according as I find them.

2° I multiply by the second Figure (6) of the Multiplicator, that is, by 60, and because the Units of the Product should be Tens, I set down a Cypher in the Units Place under the first Product, multiplying afterwards every Figure of the Multiplicand by 6, I find 57840 for the second Product.

3° I Multiply by the third Figure (2), that is, by 200, and because the Units of the Product should be Hundreds; I fill up with two Cyphers the Places of the Units and Tens; multiplying afterwards by 2 every Figure of the Multiplicand one after another, I find 192800 for the third Product.

Lastly I add together those three Products, and there results 254496 for the Product required of 964 multiplied by 264.

8452

XLIV.

Let it be required to multiply 43216 by 30050.

The Multiplicator consisting of two significative Figures only 5 and 3, the first denoting 50 or 5 Tens, and the second 30000 or 3 Tens of Thousands, there are no more than two particular Products to be found. Having set down the Multiplicator under the Multiplicand as in the Margin, and observing that the first Multiplication by 5 tens, will produce Tens; I set down a Cypher in the Units Place; and multiply the Multiplicand 43216 by 5; Saying: 5 times 6 is 30, writing down the first Character (0) in the second Place, and retaining 3. I multiply the other Figures of the Multiplicand by 5, and there results 2160800 for the first particular Product.

$$\begin{array}{r}
 43216 \\
 30050 \\
 \hline
 2160800 \\
 129648 \\
 \hline
 1298640800
 \end{array}$$

Second example of the Multiplication of integers represented by several significative figures.

The second Multiplication by 3 Tens of Thousands will produce Tens of Thousands, consequently will have four Places of Figures to the right Hand. I therefore set down four Cyphers in the four first Places, or leave them Void: and multiply the Multiplicand 43216 by 3; and there results 129648 for the Second particular Product.

Lastly I add together those two particular products, whose Sum 1298640800 will be the whole Product of 43216 multiplied by 30050.

XLV.

Let it be required to multiply 700800 by 40600.

1° Because the first significative Figure (6) of the Multiplicator has two Cyphers, or two Places to the right Hand of it, and consequently should produce Units of the third Degree, I set down two Cyphers in the two first Places; then saying: 6 times 0 is 0, I set down a Cypher in the third Place, and proceeding to the Multiplication of the second Figure of the Multiplicand, I say 6 times 0 is 0, setting down a Cypher also in the fourth Place. Then I say: 6 times 8 is 48, setting down 8 in the fifth Place, and retaining 4. I afterwards say: 6 times 0 is 0 and 4 I retained is 4, which I set down in the sixth Place; then I say: 6 times 0 is 0, and set down a Cypher in the seventh Place. Lastly I say 6 times 7 is 42, and set down 2 in the eighth Place, and 4 in the ninth Place, and there results 420480000 for the first particular Product.

$$\begin{array}{r}
 700800 \\
 40600 \\
 \hline
 420480000 \\
 2803200 \\
 \hline
 28452480000
 \end{array}$$

Third example of the multiplication of integers represented by several significative figures.

2° Because the second significative Figure 4 tens of Thousands of the Multiplicator will produce Units of the fifth Degree, I set down four Cyphers in the four first Places of the second Product, or leave them void, and then multiply 700800 by 4, and there results 2803200000 for the second particular Product.

Lastly I add together the two particular Products and their Sum 28452480000 will be the whole Product required.

XLVI.

Process of
the multi-
plication of
numbers
containing
decimal
parts.

If it be required to multiply a Number which contains decimal Parts, by another Number that also contains decimal Parts or that does not contain any. The Multiplicand is to be multiplied by the Multiplier, as if neither of them contained decimal Parts: and in the Product so many Places of decimal Figures to the right Hand are to be separated with a Comma, as there are decimal Parts both in the Multiplicand and Multiplier, as will appear by the following Examples.

XLVII.

Let it be required to multiply 74,964 by 264

First exam-
ple of the
multipli-
cation of num-
bers contain-
ing decimal
parts.

Since the Multiplicand (74,964) expresses 74964 *Thousandths*, the Units of the Product will be *thousandths*; and this Product will be 19790496 *Thousandths*; because the Units of the Product are always of the same Species as those of the Multiplicand, when the Multiplier contains no decimal Parts (Art xxxiii.)

$$\begin{array}{r} 74,964 \\ 264 \\ \hline 299,856 \\ 4497,84 \\ 14992,8 \\ \hline 19790,496 \end{array}$$

In order therefore to shew that the Units of the Product 19790496 are *Thousandths*, the same as those of the Multiplicand. I separate with a Comma three decimal Figures in the Product, or in general I separate with a Comma, as many decimal Figures in the Product, as there are in the Multiplier.

To multiply therefore a Number which contains decimal Figures by another that has none; the Multiplication is to be performed as if the Multiplicand had no decimal Parts, and afterwards as many decimal Figures are to be separated with a Comma in the Product, as there are in the Multiplier.

As the Multiplier has no decimal Parts, those in the Multiplicand are all that are both in the Multiplicand and Multiplier, whence the Rule of the foregoing Article is applicable to this first Example.

XLVIII.

Let it be required to multiply 74964 by 2,64

Second ex-
ample of
the multipli-
cation of
numbers
containing
decimal
parts.

Performing the Multiplication as if the Multiplier was a whole Number; I find for Product 19790496 *simple Units*. But the proposed Multiplier (2,64) which expresses 264 *Hundredths*, being only the hundredth Part of 264 *Units*: consequently the Product should be only the hundredth Part of the Product 19790496 arising from the Multiplication by 264; the *Units* therefore of this Product must be transformed into *Hundredths*; which is effected by separating with a Comma as many decimal Figures in this Product, as there are in the Multiplier.

$$\begin{array}{r} 74964 \\ 2,64 \\ \hline 2998,56 \\ 44978,4 \\ 149928 \\ \hline 197904,96 \end{array}$$

As the Multiplicand has no decimal Parts those in the Multiplier are all that are in both the Multiplicand and Multiplier; so that the general Rule is also applicable to this second Example.

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XLIX.

Let it be required to Multiply 74,964 by 2,64

Multiplying the Multiplicand (74,964) which contains three decimal Figures, by the Multiplier considered as a Number of *simple Units*; that is, Multiplying (74,964) by 264; I find (197,90496) for the Product.

But this Product is centuple of the One required; because it arises from the Multiplication by 264 which is centuple of the given Multiplier (2,64): whence this Product should be rendered a hundred Times less; consequently the separating Point must be still removed so many Places towards the left Hand as there are Places after the Comma in the Multiplier. And as there are already in the Product as many Places to the right hand of the Comma, as there are in the Multiplicand; there will necessarily be in the Product as many Places to the right hand of the Comma, that is, as many decimal Figures, as there are in both the Multiplicand and Multiplier. Consequently (197,90496) will be the Product of (74,964) into (2,64) agreeable to the general Rule. (Art. XLVI)

$$\begin{array}{r} 74,964 \\ \times 2,64 \\ \hline 2,99856 \\ 44,9784 \\ \hline 149,928 \\ 197,90496 \end{array}$$

Third example of the multiplication of numbers containing decimal parts.

L.

Let it be required to multiply 0,125 by 0,0625.

Having performed the Multiplication as if the two Factors had no decimal Figures, and added together all the particular Products; finding that there are not as many Figures in the Product as there are decimal Places in both the Multiplicand and Multiplier: there being seven decimal Places in both the Factors, and only five Figures in the Product; I set down two Cyphers to the left Hand of this Product in order to have seven decimal Figures, to which I prefix a Comma that they may be reputed as such: and setting down a Cypher to the left Hand of the Comma, to supply the Place of the Units; there results (0,0078125) for the Product required of (0,125) multiplied by (0,0625).

$$\begin{array}{r} 0,125 \\ \times 0,0625 \\ \hline 625 \\ 250 \\ 750 \\ \hline 0,0078125 \end{array}$$

Fourth example of the multiplication of numbers containing decimal parts.

LI.

To explain how the Operations in Multiplication are proved, let us suppose that 4872 was multiplied by 863, and the Product found to be 4204536. To prove it, the Figures 4,8,7,2 of the Multiplicand considered as representing *simple Units* are added together, and all the 9s contained in their Sum being taken away, the Remainder 3 is set down. In like manner, the Figures 8,6,3 of the Multiplier are added together, and the 9s in their Sum being taken away, the Remainder 8 is set down. Then the Remainder 3 of the Multiplicand, is multiplied by the Remainder 8 of the Multiplier; and the two Figures of the Product 24 being added together, if their Sum (6) not exceeding 9, is equal to the

How the operations in multiplication are proved explained by an example.

Remainder of the Sum of all the Figures 4, 2, 0, 4, 5, 3, 6 of the Product, after all the 9s are taken away; there is a Presumption that the Multiplication has been accurately performed, if not, it is a Proof that some Error has been committed in the Operation.

For it is easy to observe, that if the 9s contained in a Number expressed by a significative Figure followed by several Cyphers be taken away, the Remainder will be represented by a Figure equal to the significative Figure of this Number.

For Example, if all the 9s contained in the Numbers 4000, 800, 70 be taken away, the Remainders will be found to be 4, 8, 7.

So that if the 9s be taken away from any Number, such as 4872, which is equivalent to 4000 more 800 more 70 more 2, the Remainder will be 4 more 8 more 7 more 2; and if the 9s contained in those four Figures whose Sum is 21 be taken away; there will remain 2 more 1, or 3.

From whence it follows that if all the Figures of a Number be considered as representing simple Units, and from their Sum all the 9s are taken away; the Remainder will be equal to what remains when the 9s are taken away from the given Number.

Now it is easy to perceive that when two Numbers such as 4872 and 863 are proposed to be multiplied one by the other, that all the 9s may be cast out of their Product, by casting out of the Factors all the 9s (because the Multiplication of those 9s by any Number, can only produce 9s, which by the Supposition are to be taken away); and by multiplying their Remainders 3 and 8 one by the other, whose Product 24 will be reduced to 6, after the 9s are taken away.

LII.

To divide one Number by another, is to investigate a third Number which multiplied by the Second, gives a Product equal to the first Number.

The particular denominations given to the numbers concerned in division.

The Number which is proposed to be divided is called the *Dividend*; that Number by which the Dividend is to be divided is called the *Divisor*; and the Number which results from the Division is called the *Quotient*.

To denote that two Numbers are to be divided one by the other, the Dividend is placed above a small Line, and the Divisor under it. For Example, to denote that 32 Feet is to be divided by 8 Feet, it is set down thus

$$\begin{array}{r} 32 \text{ Feet} \\ 8 \text{ Feet} \end{array}$$

LIII.

From whence it follows that the Divisor or Quotient must be an abstract Number, and that one of them must consist of Units of the same Species of those of the Dividend, otherwise the Divisor and Quotient multiplied one by the other would not produce a Quantity equal to the Dividend; now according as the Divisor is an abstract Number, or a Number of the same Species with the Dividend we may form two distinct Notions of Division,

1^o when the Divisor is a Number of the same Species with the Dividend, the Quotient that will result will be necessarily an abstract Number, denoting how often the Divisor should be repeated to produce the Dividend. In this case *Division* may be said to be an Operation by which is discovered how often the Divisor is contained in the Dividend; and this how often that is found, expressed in Latin by the Word *Quoties*, is called the Quotient.

For Example if it be proposed to divide 32 Feet by 8 Feet, or to find a third Number by which 8 Feet being multiplied will produce the first Number 32 Feet; the Number 4 that will be found will be an absolute Number, denoting that 8 Feet is to be repeated 4 times to produce 32 Feet, and consequently that 8 Feet are contained 4 times in the Dividend 32 Feet, whence the Division of the Dividend 32 Feet by the Divisor 8 Feet, is reduced to find how often the Divisor 8 Feet, is contained in the Dividend 32 Feet; and the absolute Number 4 which is found, and denotes 4 times, will be properly speaking the Quotient of this Division.

2^o When the Divisor consists of absolute Units, it denotes how often the Quotient sought should be repeated, to produce the Dividend. So that to obtain this Quotient, the Dividend must be divided into as many equal Parts as the Divisor contains absolute Units. In this second Case, it may be said that *Division* is an Operation by which the Dividend is divided into as many equal Parts, as there are absolute Units in the Divisor, to obtain one of those Parts which will necessarily be of the same Species with the Dividend.

For Example, if it was proposed to divide 32 Feet by the absolute Number 4, or to find a Number which repeated four Times will produce 32 Feet; it is manifest that the Number which will be found will be the fourth Part of 32 Feet, and consequently will be a Number of Feet. So that the Division is reduced to divide 32 Feet into four equal Parts, to obtain one of those Parts which will be 8 Feet: but those 8 Feet when found will be improperly styled the Quotient, because not denoting 8 Times, it does not correspond to the Word *Quoties*, which signifies how often.

3^o when the Dividend and Divisor are both absolute Numbers, the Quotient will be also an absolute Number; and either of the foregoing definitions may be applied to Division.

For Example, if it be proposed to divide the absolute Number 32, by the absolute Number 8; according to the first Definition, the Division will be reduced to find how often the Divisor 8 is contained in the Dividend 32 which is of the same Species with it. And the Number 4 that will result and which will denote 4 Times will be properly speaking the Quotient of 32 divided by 8.

But the Divisor 8 being an absolute Number, may also denote that the eighth Part of the Dividend 32 is to be taken; agreeable to the second Definition.

Different notions that may be formed of division according as the divisor and dividend are of the same or of different species.

LIV.

It is easy to perceive that the Number of Units of the Quotient, will be the same whether the Dividend and Divisor consist of Units of the same Species, or whether the Divisor is an abstract Number, and that the Quotients will only differ because the Nature of their Units is different.

In the operation of division the number of units of the dividend and divisor is only considered and not their species.

For Example, if those two Divisions be proposed, 32 Feet to be divided by 8 Feet, and 32 Feet to be divided by 8; both one and the other Quotient will consist of 4 Units, and will differ only because the Units of the first will be absolute Units, and the Units of the second will be Feet.

The Computists therefore when a Division is proposed to be performed, consider only the Numbers of Units of the Dividend and Divisor, without attending to the Nature of those Units; and seek for Quotient an absolute Number which denotes how often the Number of Units of the Divisor is contained in the Number of Units of the Dividend, as if the Dividend and Divisor consisted of Units of the same Species. Determining afterwards the Species of those Units, by observing, that those Units should be abstract ones, when the Dividend and Divisor are really of the same Species, and that they are of the same Nature with those of the Dividend, when the Divisor is an absolute Number.

LV.

If the Dividend of a Division be multiplied by any Number, without altering its Divisor; the Quotient of the new Division will be equal to the Quotient of the former multiplied by the same Number.

For Example, if 32 is to be divided by 8, which is expressed thus $\frac{32}{8}$ the Quotient will be 4: and if the Dividend 32 be multiplied by 2 or 3, &c. without altering the Divisor 8; there will result those new divisions $\frac{64}{8}$, $\frac{96}{8}$, &c. their Quotients 8, 12, &c. being equal to the former Quotient 4 multiplied by 2 or by 3, &c. for it is manifest that a constant Divisor is contained twice oftener in a Dividend double, three-times oftener in a Dividend triple, &c.

And reciprocally if the Dividend of a Division be divided by any Number, without altering its Divisor, the Quotient of the new Division will be equal to the Quotient of the former, divided by the same Number.

For Example, if 96 is to be divided by 8, that is, $\frac{96}{8}$, whose Quotient is 12, and if the Dividend 96 be divided by 2 or by 3 &c; there will result those new Divisions $\frac{48}{8}$, $\frac{32}{8}$ &c. their Quotients 6, 4 &c. being equal to the former Quotient 12 divided by 2 or by 3 &c. for by dividing the Dividend by 2 or by 3 &c it is rendered twice or thrice &c. less than it was; so that the constant divisor 8 should be contained twice or thrice,

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&c. less than before; consequently the Quotient should be twice, thrice &c. less than the former Quotient, that is, equal to the former Quotient divided by 2 or 3, &c.

LVI.

If the Divisor of a Division be multiplied by any Number, without altering the Dividend; the Quotient of the new Division will be equal to that of the former divided by the same Number by which the Divisor was multiplied. For Example, if 96 is to be divided by 8, that is, $\frac{96}{8}$, whose Quotient is 12, and that without altering the Dividend 96, the Divisor 8 be multiplied by 2 or by 3, &c. there will result those new divisions $\frac{96}{16}$, $\frac{96}{24}$, &c. their Quotients 6, 4, &c. being equal to the former Quotient 12 divided by the Numbers 2, 3, &c. by which the Divisor was multiplied: for it is manifest that a Divisor become double or triple, &c. is contained twice or thrice less, &c. in the same Dividend: so that the Quotient should be twice or thrice &c. less than it would be if the Divisor had not been multiplied by 2 or by 3 &c.

And reciprocally if the Divisor of a Division be divided by any Number without altering the Dividend, the Quotient of the new Division will be equal to that of the former, multiplied by the Number which divided the Divisor.

For Example if 96 is to be Divided by 24, that is $\frac{96}{24}$ whose Quotient is 4; and that without altering the Dividend 96, the Divisor 24, is divided by 2 or by 3, &c. there will result those new Divisions $\frac{96}{12}$, $\frac{96}{8}$ &c. their Quotients 8, 12, &c. being equal to the former Quotient 4 multiplied by the Numbers 2, 3, &c. by which the Divisor was divided; because by dividing a Divisor by 2 or by 3, &c. it is rendered Twice or Thrice, &c. less; and it is manifest that a Divisor Twice or Thrice, &c. less, is contained Twice or Thrice &c. oftener in the same Dividend.

LVII.

From whence it follows 1^o that if the Dividend and Divisor of a Division be multiplied by the same Quantity; the Quotient of the Division of the new Dividend by the New Divisor, will be the same as the Quotient of the first Division; for Example if 32 is to be divided by 8, that is $\frac{32}{8}$ whose Quotient is 4, and the Dividend and Divisor be multiplied by 2 or by 3 or by a same Quantity: there will result those new Divisions $\frac{64}{16}$, $\frac{96}{24}$, &c. which give the same Quotient 4 as the former Division. For by multiplying

The multiplication or division of the dividend and divisor by the same number does not alter the quotient.

the Dividend and Divisor of a Division by a same Quantity, the Quotient of the first Division is at the same Time multiplied and divided by this same Quantity, (Art. LV. & LVI.) and its Value consequently is not changed.

2^o If the Dividend and Divisor of a Division is divided by a same Quantity, the Quotient of the new Division will be still the same as that of the former Division; because by this Operation the Quotient of the former Division is divided and multiplied by the same Quantity. (Art. LV, LVI.)

For Example, if 96 is to be divided by 24, that is $\frac{96}{24}$ whose Quotient is 4, and the Dividend 96 and Divisor 24 be divided by 2 or by 3 or by any same Quantity, there will result those new Divisions $\frac{48}{32}$, $\frac{32}{8}$, &c. which give the same Quotient 4 as the former Division $\frac{96}{24}$.

LVIII.

The quotient of a dividend divided by a composited divisor is equal to the quotient arising from the continual division by the component parts of the divisor.

When a Dividend is to be divided by a Divisor composed of several Factors multiplied together; instead of dividing the Dividend by the composited Divisor, it may be divided by a Factor of the Divisor, and the Quotient of this Division may be divided by a second Factor of the Divisor, and the new Quotient may be divided by a new Factor of the Divisor, and so on until the Division is made by all the Factors: the last Division being performed, the last Quotient will be the Quotient of the Division of the Dividend by the composited Divisor.

For Example, if 840 is to be divided by the compound Divisor $3 \times 5 \times 7$; which may be expressed thus $\frac{840}{3 \times 5 \times 7}$; instead of multiplying the three Factors 3, 5, 7 one by the other, and dividing the Dividend 840 by their Product 105; the Dividend 840 may be first divided by 3, and the Quotient 280 that results by 5, and the Quotient 56 arising from this last Division, by 7, which will give 8 for the last Quotient, the same that results by dividing 840 by the Product 105 of the three Factors 3, 5, 7.

For $\frac{840}{3 \times 5 \times 7}$ representing the Division of 840 by $3 \times 5 \times 7$, if the Dividend 840 and Divisor $3 \times 5 \times 7$ be divided by the same Number 3, the new Division $\frac{280}{5 \times 7}$ will give the same Quotient (Art. LVII.), and if the new Dividend 280 and Divisor 5×7 be again divided by the Number 5; the Division $\frac{56}{7}$ that remains to be performed will still give the same Quotient (Art. LVII.) wherefore by dividing 840 by 3, and the Quotient of this division by 5, & dividing again the new Quotient by 7, there will result the same Quotient as will arise by dividing 840 by the compound Divisor $3 \times 5 \times 7$.

LIX.

The Quotient arising from the Division of any Number less than 90, by another Number less than 10, and the Remainder of such a Division is found by Help of the Table of Pythagoras, after the following Manner.

The Divisor being looked out for at the Top of the Table; and then descending vertically under it until the Dividend is found, or a less Number that approaches the nearest to it; the Number opposite to it in the first Column to the left Hand, is the Quotient of the Division required.

Let it be proposed, for Example, to divide 72 by 8. I look out for 8 at the Top of the Table, and descending vertically to 72; opposite to which in the first Column, I find the Number 9, which is the Quotient required.

Let it be proposed to divide 78 by 9. I look out for 9 at the Top of the Table, and descending vertically under 9, and not finding 78; but 72 being less and approaching the nearest to it; opposite to which in the first Column, I find the Number 8 for the Quotient of 72 divided by 9; but as it is 78 and not 72 that is proposed to be divided by 9, there will be a Remainder 6 which cannot be divided.

Let it be proposed to divide 64 by 7. I look out for 7 at the Top of the Table, and descending vertically under 7, and not finding 64, but 63 approaching the nearest to it; opposite to which in the first Column, I find the Number 9 for the Quotient of 63 divided by 7; but as it is 64 and not 63 which is proposed to be divided by 7, there will be a Remainder 1, which cannot be divided.

LX.

To divide any Number by another Number represented by one Figure, the Computists divide all the Parts of the Dividend by the Divisor, beginning the Division by the Figures whose Units are of the highest Degree. Supposing, for Example, that the Dividend consists of three Figures, of *simple Units*, of *Tens* and of *Hundreds*; the Computists begin by dividing the Part of the Hundreds which can be divided into as many equal Parts as there are Units in the Divisor. Then reducing the Remainder of the *Hundreds* into *Tens*, and adding them to the *Tens* of the Dividend, they divide the Part of the *Tens* that can be divided into as many equal Parts as there are Units in the Divisor. Lastly reducing the Remainder of the *Tens* into *simple Units*, and adding those to the *simple Units* of the Dividend they divide the whole by the Divisor. So that there are as many particular Divisions to be performed, and consequently as many particular Quotients to be found, as there are different Parts of the Dividend to be divided.

Those Operations that have been announced in general, will be rendered obvious, by their application to the following Examples.

The division of any number less than 90 by another less than 10 performed by help of the table of pythagoras.

Process of the method of dividing a number by another represented by one figure.

Let it be required to divide 952 by 4.

First exam-
ple of the
division of
a number by
another re-
presented by
one figure.

Having set down the Divisor to the right Hand of the Dividend, separating them by a small vertical Line or Crotchet; and drawing a Line under the Divisor to separate it from the Quotient, under which is to be set down the Figures of the Quotient according as they are found; as in the Margin.

I begin by dividing the 9 *Hundreds* by 4; saying: the fourth Part of 9 *Hundreds* is 2 *Hundreds*; or more simply, the fourth Part of 9 is 2; which I set down in the Quotient in a Place which will be that of the *Hundreds*.

To determine the Part of the 9 *Hundreds*, I divided and to find the Remainder of the Division; I multiply the Divisor 4 by the Quotient 2 *Hundreds*, and set down the Product 8 *Hundreds* under the 9 *Hundreds*; then deducting 8 *Hundreds* from 9 *Hundreds*, the Remainder 1 *Hundred* will be the Part of the *Hundreds* which has not been divided by 4.

Setting down to the right Hand of the Remainder 1 of the *Hundreds*, the 5 *Tens* of the Dividend, there results 15 *Tens* to be divided by 4; I therefore take the fourth Part of those 15 *Tens*, which can only be 3 *Tens* with a Remainder.

I therefore set down 3 in the Quotient in the Place of the *Tens*, that is, to the right Hand of the 2 *Hundreds* which have been set down for the Quotient of the first Division.

To determine the Part of the 15 *Tens* that have not been divided. I multiply the Divisor 4 by the Quotient 3 *Tens*, and set down the Product 12 *Tens* under the 15 *Tens*; then deducting 12 *Tens* from 15 *Tens*, the Remainder 3 *Tens* will be the Part of the 15 *Tens* which has not been comprised in the Division.

Setting down to the right Hand of the Remainder 3 *Tens*, the 2 *Units* of the Dividend, there results 32 *simple Units* to be divided by the same Divisor 4. I therefore take the fourth Part of 32, which is 8, and set it down in the Quotient in the Place of *simple Units*, that is, to the right Hand of the 3 *Tens*.

To discover whether any Remainder is left after this last Division, I multiply the Divisor 4 by the Quotient 8, and setting down the Product 32 under the 32 *Units* that were to be divided, I deduct one from the other, and there being no Remainder left, and the division being ended, I conclude that the Divisor 4 is contained 238 Times exactly in 952.

$$\begin{array}{r}
 952 \left\{ \begin{array}{l} 4 \\ 238 \end{array} \right. \\
 \hline
 8 \\
 \hline
 15 \\
 12 \\
 \hline
 32 \\
 32 \\
 \hline
 00
 \end{array}$$

LXII.

Let it be required to divide 7264 by 9.

Having disposed the Dividend and Divisor as in the Margin; I divide one after another all the Parts of the Dividend, beginning by the *Thousands*, and proceeding afterwards to the *Hundreds*, and then to the *Tens*, and *Units*.

1° As the first Figure 7 cannot be divided by 9, I join it to the following Figure 2, and divide 72 *Hundreds* by 9, saying: the ninth Part of 72 *Hundreds* is 8 *Hundreds*, which I set down in the Quotient in a Place which will be that of the *Hundreds*.

$$\begin{array}{r} 7264 \left\{ \begin{array}{l} 9 \\ 807 \end{array} \right. \\ \underline{72} \\ 064 \\ \underline{63} \\ 1 \end{array}$$

To find the Remainder of this first Division, I multiply the Divisor 9 by the Quotient 8; and setting down the Product 72 under the 72 which was to be divided, I deduct one from the other; and as there is no Remainder, I set down a Cypher underneath to denote that the Division of 72 *Hundreds* by 9, gives exactly 8 *Hundreds* for Quotient.

Second example of the division of a number by another represented by one figure.

2° I bring down to the right Hand of the Cypher, the 6 *Tens* of the Dividend to divide them also by 9; and as that is not possible, I set down a Cypher in the Quotient, in the Place of the *Tens*, as well to denote that there are no *Tens* in the Quotient, as to conserve to the first Figure of the Quotient the Place of *Hundreds*: the 6 that has been set down and that could not be divided, will remain for the following Division.

3° I bring down the 4 *Units* of the Dividend to the right Hand of the 6 *Tens*, which makes 64 *Units* which I divide by 9; and as 64 contains 9 seven Times, I set down 7 in the Quotient in the *Units* Place: and the Division will be ended, with respect to the Quotient required; since all the Figures that it consists of have been found. To determine the Remainder of the Division, I multiply the Divisor 9 by the last Quotient 7; and setting down the Product 63 under 64, I deduct it from 64; and find 1 to be the Remainder of the Division.

Wherefore the Dividend 7264 being divided by the Divisor 9, gives 807 for the Quotient, with a Remainder 1 which has not been divided; so that 807 is the exact Quotient of 7263 divided by 9.

LXIII.

In General, when any Number is proposed to be divided by another, the Computists first determine the Number of Places of the Quotient, by pointing off so many Places from the right Hand of the Dividend, as are equal to, or not exceeding the Product of the Divisor into any of the Nine significative Figures; the remaining Places, in the Dividend more one, being equal to the Number of Places in the Quotient. To find afterwards the Figures corresponding to those Places, beginning by the Figure whose Units are of the highest Degree, they divide the first Figure of the Places

General method for dividing one number by another.

pointed off in the Dividend, which is called the first Member of the Division by the first Figure of the Divisor, when this Member and the Divisor consist of the same Number of Figures, or the two first Figures of this Member when it has one more; whereby is discovered the Number of Times the Divisor is contained in this Member, which is represented by the first Quotient Figure sought. They then subtract the Product of the Quotient into the Divisor from this Member, and to the Remainder affix the next Place in the Dividend, which will be the second Member of the division; with which they proceed as before, and if the Divisor is not once contained in any Member, they increase the Quotient with a Cypher before any new Place is taken down to the right Hand of this Member

LXIV.

Let it be required to divide 32361 by 469.

The foregoing method explained by an example.

Having set down the Divisor to the right Hand of the Dividend as in the Margin, I observe that as the Dividend contains Tens of Thousands, and the Divisor Hundreds, and that one Hundred is contained in Ten Thousand a Hundred Times, the Quotient may contain Hundreds, but no Units of a higher Degree. It is Question therefore to determine how many Hundred Times, how many Tens of Times, how many Units of Times, the Divisor is contained in the Dividend.

$$\begin{array}{r}
 32361 \left\{ \begin{array}{l} 469 \\ 69 \end{array} \right. \\
 2814 \\
 \hline
 4221 \\
 4221 \\
 \hline
 0000
 \end{array}$$

To determine how many Hundred Times the Dividend contains the Divisor, I point off so many Places from the right Hand of the Dividend as there are in the Divisor, that is, I Point off the Part 323 of the Dividend, which really is 32300; setting aside for a moment the two last Figures 61, and divide 32300 by 469, to discover how many Hundred Times 469 is contained in 32300; to effect which, I observe that it suffices to divide 323 by 469, considering the Quotient Figure not as representing simple Units, but Hundreds. But as 323 can not be divided by 469, I conclude that the Quotient will not contain Hundreds. It would have contained Hundreds if instead of 323 there had been 323, or in general a Number equal or greater than 469: for then the Quotient would contain at least 1 Unit of the third Degree, or 1 Hundred. The Units therefore of the highest Degree that the Quotient can contain will be Tens, but it will necessarily contain Tens, because the Dividend having two Figures more than the Divisor, is necessarily ten Times greater than the Divisor, in effect 469 repeated ten Times gives 4690 which has a Figure less than 32361.

I consider therefore how often 32360 contains 469, setting aside for a moment the Number 1, or what amounts to the same Thing, I consider how often 3236 contains 469, remembering that the Quotient Figure will represent a Number of Tens; but 3236 can not contain 469 oftener than the Number 32, (the two first Figures of the Dividend) contains the first Fi-

figure 4 of the Divisor : for 32 contains 4 eight Times, and if, for Example, 9 is set down in the Quotient instead of 8 ; multiplying 469 by 9 there would result a Number greater than 3236 for 4 times 9 giving 36, the two first Figures of the Number equal to 9 Times 469 would be greater than the two first Figures 32 of the Number 3236 ; so that it suffices to divide by the first Figure of the Divisor the first Figure of the Member of the Division, when this Member consists of as many Figures as the Divisor, or the two first Figures when it contains one more.

However we are not to conclude that this Operation will never give too much; but it is certain it will never give too little, and it is for this Reason that the Computists divide the first Figures of the Member of the Division by the first Figure of the Divisor. When the Division gives too great a Quotient, as in the present Case, where 8 is too great, and even 7, I diminish successively the Quotient until I find one that is not too great, which is 6 *Tens*, which I set down in the Quotient. I afterwards multiply the Divisor 469 by 6 *Tens*, subtracting the Product 2814, which is really 28140 from 3236 which is really 32360. To the Remainder 422 which is really 4220, I add the 1 I set aside, which will be the second Member of the Division, and divide 4221 by 469: proceeding as before, I find it is contained 9 Times, which I set down in the Quotient in the Units Place, that is, to the right Hand of 6, and multiplying the Dividend 469 by the Quotient 9, and subtracting the Product from this Member of the Division, and there being no Remainder, I conclude that 32361 is exactly divided by 469, and that 69 is the Quotient of this Division

LXV.

When as many Places has been pointed off in the Dividend to the right Hand, as there are in the Divisor, or one more if necessary, it is easy to perceive that the Quotient will consist of as many Figures more one, as will remain in the Dividend. For Example, let 523032 be proposed to be divided by 469, having pointed off 523 which consists of as many Places as 469, there will remain three Figures 032, I say the Quotient should contain three Figures more One, or Four: for it is manifest that 323000 is a Thousand Times greater than 523, which is greater than 469; and that 523032 is less than 469 repeated Ten Thousand times, because 4690000 consists of a Figure more, therefore the Quotient should contain Thousands, and not Ten Thousands, and therefore should consist of four Places, neither more nor less. If the Dividend was 1523032, pointing off 1523, which has a Figure more than the Divisor 469, it will appear in like Manner that the Quotient should consist of four Places, neither more nor less; whence it is that a Cypher is sometimes set down in the Quotient and sometimes several.

It remains to explain why in the Quotient, more than 9, is never set down at once, which will easily appear when it is observed that a Mem-

General rule for finding the number of places that a quotient of a division will consist of.

Why in each operation of

ELEMENTS OF

division a
number
greater than
9 is not set
down in the
quotient.

ber of the division is always less than ten Times the Divisor; for a Member of the Division consists of as many Figures as the Divisor, or of one more: In the first Case it is manifest that it is less than the Divisor repeated ten Times, in the second Case, if a Figure be taken from the Member of the division it will be less than the Divisor, therefore the Member of the division with this Figure restored is less than ten Times the Divisor.

LXVI.

Let it be required to divide 4571112 by 897.

Second example of the division of integers represented by several significative figures.

Having disposed the Dividend and Divisor as in the Margin, and observing that the three Figures of the Divisor, form a Number greater than the three first Figures of the Dividend to the left Hand; I Point off 4571 for the first Member of the division, and as there remain three Figures 112 in the Dividend, the Quotient will consist of four Places of Figures.

To determine the Numbers corresponding to those Places, beginning by that whose Units are of the highest Degree, I consider how often the Divisor 897 is contained in the first Member of the division 4571, and find 5 *Thousands* for the first Figure of the Quotient.

To find the second Figure of the Quotient, I multiply the Divisor 897 by the Quotient 5, and subtract the Product 4485 from 4571, and to the Remainder 86 I affix the 1 *Hundred* of the Dividend, setting a Point over it to denote that it has been brought down; which will be the second Member of the Division, and divide 861 by 897, and as that can not be done, I set down a Cypher in the Quotient, as well to denote that it contains no *Hundreds*, as to preserve to the first Quotient Figure found its Place.

To discover the third Figure of the Quotient, I affix to 861 the 1 *Ten* of the Dividend, (setting a point over it) which will be the third Member of the division, and consider how often 897 is contained in 8611, dividing 86 by 5, and I find 9 for the third Figure of the Quotient.

To determine the fourth and last Figure of the Quotient, I multiply the Divisor 897 by the Quotient 9, subtract the Product 8073 from 8611, and to the Remainder 538 I affix the Units of the Dividend (setting a Point over it) which will be the fourth Member of the Division, and consider how often 897 is contained in 5382, dividing 53 by 8, and I find 6 for the Fourth and last Figure of the Quotient, and multiplying the Divisor 897 by the Quotient 6, and subtracting the Product 5382 from this Member of the division, and there being no Remainder, I conclude that 4571112 is exactly divided by 897, and that 5096 is the Quotient of this Division.

$$\begin{array}{r}
 4571112 \quad \left\{ \begin{array}{l} 897 \\ 5096 \end{array} \right. \\
 \underline{4485} \\
 8611 \\
 \underline{8073} \\
 5382 \\
 \underline{5382} \\
 0000
 \end{array}$$

LXVII.

Let it be required to divide 239200 by 52

The Dividend and Divisor being disposed as in the Margin, the Divisor 52 not being contained in the two first Figures of the Dividend to the left Hand; I Point off the three Figures 239, and there being three Figures 200 remaining in the Dividend, I conclude that the Quotient will consist of four Places of Figures.

To determine the Numbers corresponding to those Places, beginning by that whose Units are of the highest Degree; I consider how often the Divisor 52 is contained in the first Member of the Division 239 *Thousands*, and find 4 *Thousands* for the Quotient with a Remainder 31 *Thousands*.

To determine the second Figure of the Quotient, I bring down the 2 *Hundreds* of the Dividend to the Remainder 31; and dividing 312 *Hundreds* by 52, find 6 *Hundreds* for the Quotient without a Remainder.

To continue the Division, according to the Rules already explained, the two Cyphers of the Dividend should be successively taken down; but as each of those affixed to the Remainder 0, being divided by 52 will give 0 for Quotient; I set down two Cyphers to the right Hand of the Quotient already found.

Whence the Quotient of the Division of 239200 by 52, will be 4600.

$$\begin{array}{r}
 239200 \quad \left\{ \begin{array}{l} 52 \\ \hline 4600 \end{array} \right. \\
 \underline{208} \\
 312 \\
 \underline{312} \\
 000
 \end{array}$$

Third example of the division of integers represented by several significative figures

LXVIII.

The general Rule for dividing Numbers containing decimal Parts, consists in reducing the Units of the Dividend, and those of the Divisor to the same Denomination; by setting down after the separating Point, or after the Decimals of the Term which has less decimal Figures than the other, as many Cyphers as are necessary that the Dividend and Divisor may have the same Number of Characters after their Commas. The Dividend and Divisor being thus prepared, they are to be divided one by the other, without attending to the Commas which may be suppressed, and which are conserved only that the Values of the Dividend and Divisor may not be changed.

Process of the division of numbers containing decimal parts.

For Example, if (172,8) was to be divided by (1,44) that is, 1728 *Tenths*, by 144 *Hundredths*; as the Divisor has a decimal Figure more than the Dividend, I set down a Cypher to the right Hand of the Dividend. This Dividend being transformed into (172,80) which denotes 17280 *Hundredths*, will consist of Units of the same Species, with those of the Divisor (1,44) which denotes 144 *Hundredths*.

The Dividend and Divisor being thus prepared and transformed into (172,80) and (1,44) will not have changed their Value. The Quotient therefore of (172,80) divided by (1,44) will be the same as that of (172,8) divided by (1,44).

ELEMENTS OF

Suppressing the separating Points both in the Dividend (172,80) and in the Divisor (1,44), all the Figures of the Dividend and Divisor will be advanced two Places, and thereby will be rendered centuple of what they were, that is, they will be multiplied both one and the other by 100. The Quotient therefore will be the same (Art. LVII.) as that of (172,80) divided by (1,44).

If therefore, to the right Hand of the Dividend and Divisor there be annexed Cyphers to make the Number of decimal Figures equal in both, and the new Dividend be divided by the new Divisor, after suppressing the separating Points, the Quotient sought will be obtained.

LXIX.

When the Units of the Dividend and Divisor are reduced to the same Denomination, by annexing to each the same Number of decimal Figures; and the Division is performed as if the Dividend and Divisor had no decimal Parts; the Quotient arising will consist of simple Units; but it seldom happens that the Divisor is contained in the Dividend a certain Number of Times without a Remainder: and in this Case, it may be found necessary to reduce into decimals the Part of the Units Time that the Divisor is contained in the Dividend. Those different Cases will be explained in the following Articles.

LXX.

Division of numbers containing decimal parts by others that have none.

To divide a Number which has decimal Parts, by a Divisor that has none; the Dividend is to be divided by the Divisor as if the Dividend had no decimal Parts: and when the Quotient is found; so many decimal Figures are to be separated with a Comma, as there are in the Dividend: as will appear by the following Example.

LXXI.

Let it be required to divide (79,58) by 23

Example.

I divide (79,58), by 23 as if it was proposed to divide 7958 by 23: and having found 346 for the Quotient; I separate with a Comma two decimal Figures in this Quotient; and there results (3,46) for the Quotient of (79,58) by 23. Which will easily appear, when it is observed that the Number (79,58) proposed to be divided, denotes 7958 *Hundreths*; and the Divisor 23 denotes that the twenty third Part of this Dividend 7958 *Hundreths* is to be taken. Now, since a Part of a Number whose Units are *Hundreths*, will also consist of *Hundreths*: the Quotient 346 found by dividing 7958 *Hundreths*, by 23, can consist of no other Units than *Hundreths*; consequently should express 346 *Hundreths* whence each Figure of this Quotient should be removed two Places towards the right Hand; which is effected, by separating with a Comma two decimal Figures.

$$\begin{array}{r}
 79,58 \left\{ \begin{array}{l} 23 \\ 3,46 \end{array} \right. \\
 \hline
 69 \\
 \hline
 105 \\
 92 \\
 \hline
 138 \\
 138 \\
 \hline
 000
 \end{array}$$

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LXXII.

To divide any Number by a Divisor that has no decimal Parts, and Method of carry on the Division until the Quotient shall not differ from the exact approxima- Quotient, by a decimal Unit of any proposed Order. Cyphers must be ting to the annexed to the Dividend, until the Place of the Decimals of the lowest Quotient of a Division Order that the Quotient should contain is filled up. And the Dividend when the thus prepared being divided by the Divisor, as if it did not contain any Divisor con- decimal Parts; so many Decimals must be separated by a Comma in the tains no de- Quotient, as there are in the Dividend prepared, as will appear by the cimal Parts. following Example.

LXXIII.

Let it be proposed to divide (103,2) by 33, and to find a Quotient that shall not differ from the exact Quotient by a thousandth Part of an Unit.

As the proposed Dividend contains in Decimals, only *Tenths*; I annex The forego- to it two Cyphers to fill up the Place of the *Hundredths*, and of the ing Method explained by an Example.

Thousandths; because the Error of the Quotient must be less than 1 *Thousandth*, the Division should be carried on to *Thousandths*. The Dividend being thus prepared, I divide (103,200) by 33; without attending to the separating Point, and I find 3127 for the Quotient which should represent *Thousandths*; as the Dividend is composed of *Thousandths*: whence this Quotient should be (3,127).

As it is not possible to put an Unit more in the Quotient 3,127, without rendering it too great; and that this Unit can only be 1 *Thousandth*; it is manifest that the Quotient (3,127) found, tho' not exact, yet does not differ from the real Quotient by a thousandth Part of an Unit.

$$\begin{array}{r} 103,200 \quad \left\{ \begin{array}{l} 33 \\ \hline 3,127 \end{array} \right. \\ \underline{99} \\ 42 \\ \underline{33} \\ 90 \\ \underline{66} \\ 240 \\ \underline{231} \\ 9 \end{array}$$

LXXIV.

To divide any Number by a Divisor greater than the Dividend, and Method of carry on the Division until the Quotient shall not differ from the exact approxima- Quotient by a decimal Unit of any proposed Order: If the Divisor ting to the contains no decimal Parts, Cyphers must be annexed to the Dividend, Quotient of a Division until the Place of the Decimals of the lowest Order that the Quotient when the Divisor is should contain, is filled up. But if the Divisor contains decimal Parts; greater than the Divi- its separating Point must be suppressed, and that of the Dividend must dend. be removed so many Places to the Right Hand as there were decimal Figures in the Divisor: then Cyphers must be annexed to the Dividend, that it may contain as many decimal Places as it is proposed the Quotient should contain. The Dividend and the Divisor being thus prepared, they must be divided one by the other, as if they contained no decimal Parts.

ELEMENTS OF

As the Quotient should contain as many decimal Places as the Dividend; and it may happen that there will not be as many Figures in this Quotient as there are Decimals in the Dividend, in this Case, Cyphers must be prefixed to it for the Defect.

LXXV.

Let it be proposed to divide the Number 2 by 189, and carry on the Division until the Quotient shall not differ from the exact Quotient, by the hundred-thousandth Part of an Unit.

First Example.

Having placed a Comma to the right Hand of the Number 2 to be divided; I set down five Cyphers to the right Hand of this Comma, that the Place of *hundred thousandths* may be filled up. I divide the Dividend (2,00000) thus prepared, by 189, as if this Dividend did not contain decimal Parts, and I find 1058 for the Quotient, which should represent *hundred thousandths*, as the Dividend is composed of *hundred thousandths*; whence its Figure 8 to the right Hand should stand in the Place of *Hundred Thousandths*, and consequently in the fifth Place to the right Hand of the separating Point. I therefore set down a Cypher to the left Hand of the Quotient 1058 prefixing a Comma to it, that the Quotient may consist of five decimal Figures as the Dividend, and there results (0,01058) for the Quotient.

$$\begin{array}{r}
 2,00000 \quad \left\{ \begin{array}{l} 189 \\ \hline 0,01058 \end{array} \right. \\
 \underline{189} \\
 1100 \\
 \underline{945} \\
 1550 \\
 \underline{1512} \\
 38
 \end{array}$$

LXXVI.

Let it be proposed to divide (0,025) by (1,89), and to carry on the Division until the Quotient shall not differ from the exact Quotient by the Millionth Part of an Unit.

Second example.

Having suppressed the separating Point of the Divisor, I remove the Comma of the Dividend two Places towards the right Hand, because there are only two decimal Figures in the Divisor. The Dividend and Divisor will thereby be multiplied by 100, (Art. ix) and the Value of the Quotient will not be altered (Art. LVII). The Division therefore will be reduced to that of (002,5 or 2,5) by 189. As the Division is to be carried on to *Millionths*, which are Decimals of the sixth Order, which stand in the sixth Place to the right Hand of the Comma; and that the Dividend (2,5) has already a decimal Figure, I annex five Cyphers to this Dividend, and divide 2,500000 by 189.

The Division being performed, there results 13227 for the Quotient: and as the Dividend

$$\begin{array}{r}
 2,500000 \quad \left\{ \begin{array}{l} 189 \\ \hline 0,013227 \end{array} \right. \\
 \underline{189} \\
 610 \\
 \underline{567} \\
 430 \\
 \underline{378} \\
 520 \\
 \underline{378} \\
 1420 \\
 \underline{1323} \\
 97
 \end{array}$$

consists of *Millionths*, this Quotient will also reckon *Millionths*; so that its last Figure 7 should stand in the sixth Place to the right Hand after the Comma. I therefore set down a Cypher to the left Hand of this Quotient, prefixing a Comma to it, and there results (0,013227) for the Quotient, of (2,500000) divided by 189, or for that of (0,025) divided by (1,89).

As the Dividend (2,500000) which denotes 2500000 *Millionths* is not exactly divisible by 189; after the Division there is a Remainder 97 *Millionths*, which cannot be divided by 189, unless it should be proposed that the Quotient should contain Decimals of an inferior Order to *Millionths*.

If the Division should be carried on to *Hundred Millionths*, which are Decimals of the eighth Order, two more Cyphers should be annexed to the Dividend, that is (2,50000000) should be divided by 189: and there would result for Quotient 1322751 *Hundred Millionths*, or (0,01322751) with a Remainder 61 *Hundred Millionths*.

In fine, if the Division was carried on indefinitely, there would result for Quotient (0,01 322751 322751 322751 &c.), the same Figures 322751 being continually repeated.

LXXVII.

When it is required to express in decimal Parts, the Quotient of a Division that cannot be obtained without leaving a Remainder, after a certain Number of Figures of this imperfect Quotient is found, the same Figure or Period of Figures will continually recur as in the foregoing Example, and in the following.

1^o If 1 be divided by 3, there will result (0,3333 &c.), that is, the Quotient will consist of three *Tenths*, three *Hundredths*, three *Thousandths*, and so on continually ad infinitum.

2^o If 1 be divided by 6, the Quotient will be found to be 0,16666 &c. that is, the first Figure of the Quotient will be 1 *Tenth* and all the other decimal Figures will be 6^s.

3^o If 1 be divided by 7, there will result (0,142857 142857 &c.) that is, after the six first decimal Figures 142857 of the Quotient are found, the same Figures will recur for the six following ones, and so on continually ad infinitum.

4^o If 1 be divided by 24, the Quotient will be found to be (0,041666 &c.) that is, the three first decimal Figures will be 041, which represent 41 *Thousandths*, and all the decimal Figures following, ad infinitum, will be 6^s.

When the same Figures recur in this Manner in a Quotient, two Periods only of those circulating Figures are set down, with an &c. annexed to them, to denote that those Periods will continually recur, ad infinitum.

A Quotient of a Division that cannot be obtained without leaving a Remainder is expressed by a decimal Series, consisting of equal periods of Figures succeeding each other ad infinitum

LXXVIII.

Property of
the Digit 9
which di-
vides Num-
bers less
than itself.

Every Number less than 9 will give for Quotient an infinite Series of Decimal Figures, the same as that of the Dividend: For every Number less than 9 cannot be divided by 9, until it is reduced into *Tenths*, and then will be equivalent to as many Tens of *Tenths*, as it consists of Units; now each ten *Tenths* will give 1 *Tenth* for Quotient with a Remainder 1 *Tenth*. Wherefore all the Tens of *Tenths* of which the Dividend is composed, will give as many *Tenths* for the Quotient and Remainder as there are Units in the Dividend; consequently the first decimal Figure of the Quotient, and the first decimal Figure remaining will be the same as that of the Dividend which is supposed less than 9.

Since the remaining Figure of the first Division is equal to that of the Dividend, and is to be divided by 9, it must be reduced into *Hundredths*, and consequently will give as many *Hundredths* for Quotient and Remainder as the Dividend contains Units, and so on ad infinitum. For Example, if 7 is proposed to be divided by 9, I reduce the Dividend into 7 Tens of *Tenths*. Now each ten *Tenths* being divided by 9 will give 1 *Tenth* for the Quotient, and there will remain 1 *Tenth*. Wherefore the 7 Tens of *Tenths* being divided by 9, will give 7 *Tenths* for the Quotient, and will also give 7 *Tenths* for the Remainder.

Since those 7 *Tenths* remaining after the first Division cannot be divided by 9, I reduce them into 7 Tens of *Hundredths*, and as each ten *Hundredths* divided by 9, will give 1 *Hundredth* for the Quotient with a Remainder 1 *Hundredth*, the 7 Tens of *Hundredths* will give 7 *Hundredths* for the Quotient, and 7 *Hundredths* for a Remainder, the same is to be said of the *Thousandths*, &c. that is, each decimal Figure of the Quotient and of the Remainder, will be the same as the Figure of the Dividend.

LXXIX.

Method of
multiplying
by Numbers
consisting of
equal Peri-
ods of Fi-
gures suc-
ceeding each
other.

Whence is derived an abridged Method of multiplying, when the Multiplier is a Number represented by the same Figure several Times repeated. Let it be proposed for Example, to multiply 3787 by 5555. I multiply 3787 by the repeating Figure 5, annexing as many Cyphers to the Right Hand of the Product, as the Multiplier consists of Places. I then divide the Number that results 189350000 by 9, neglecting the Remainder of the Division, and from the Quotient 21038888 I subtract the Number represented by as many Figures of this Quotient to the right Hand, as there are Places of Figures in the Multiplier, the Remainder will be the Product required.

For the particular Products of this Multiplication are 18935×1 ,

$$\begin{array}{r}
 3787 \\
 \times 50000 \\
 \hline
 9)189350000 \\
 \underline{21038888} \\
 2103 \\
 \underline{21036785}
 \end{array}$$

18935×10, 18935×100, 18935×1000, the Sum of which, or the Product required, is equal to 18935×1111, but $\frac{18935 \times 10000}{9}$ or 21038888, 8888 &c. is equal to 18935×1111 more 18935×,1111 &c. and $\frac{18935 \times 1}{9}$ or 2103,8888 &c. is equal to 1893×,1111 &c. wherefore deducting 2103,8888 &c. from 21038888,8888 &c. that is 18935×,1111 &c. from 18935×1111 more 18935×,1111 &c. the Remainder 21036785 will be equal to 18935×1111 the Product required of 3787 into 5555.

LXXX.

Every Number less than 99, divided by 99, will give for Quotient an infinite Series of decimal Periods of two Figures equal to those of the Dividend. For every Number less than 99, cannot be divided by 99, until it is reduced into as many Hundreds of *Hundreths* as it consists of Units; now each hundred of *Hundreths* being divided by 99, will give 1 *Hundreth* for Quotient, with a Remainder 1 *Hundreth*; wherefore all the Units of the Dividend will give for Quotient a Number of *Hundreths*, expressed by the same Figures as the Dividend, and there will remain the same Number of *Hundreths*, and so in like Manner of the other Figures of the Quotient.

Property peculiar to all Divisors less by an Unit than the Terms of the progression 10, 100, 1000, 10,000 which divide Numbers less than themselves.

For Example, if it was required to divide 42 by 99, I reduce the Dividend 42 into 42 Hundreds of *Hundreths*, and each hundred *Hundreths* divided by 99, giving 1 *Hundreth* for Quotient, with a Remainder 1 *Hundreth*, the 42 Hundreds of *Hundreths* will give 42 *Hundreths* for the Quotient, with a Remainder 42 *Hundreths*; so that the Number of *Hundreths* of the Quotient, and the Number of *Hundreths* of the Remainder will be expressed by the same Figures as the Dividend 42.

As the 42 *Hundreths* remaining, cannot be divided by 99, I reduce them into 42 Hundreds of *Hundreths* of *Hundreths*, that is into 42 Hundreds of *Ten-thousandths*, but each Hundred of *Ten-thousandths* being divided by 99, will give 1 *Ten-thousandth* for Quotient, with a Remainder 1 *Ten-thousandth*; whence 42 Hundreds of *Ten-thousandths* will give 42 *Ten-thousandths* for the Quotient, with a Remainder 42 *Ten-thousandths*; wherefore the Number of *Ten-thousandths* of the Quotient and the Number of *Ten-thousandths* of the Remainder, will be expressed by the same Figures as those of the Dividend 42, and it will appear in like Manner that all the other Figures of the Quotient will be equal two by two to those of the Dividend, which are supposed less than 99; whence dividing 42 by 29, there will result for Quotient (0,42 42 42 &c.). If the Number to be divided by 99, was represented by one Figure, for Example, if 5

or 05, was to be divided by 99, I transform 5 into 500 *Hundredths*, which I divide by 99, and there results for Quotient (0,05) that is 5 *Hundredths* with a Remainder (0,05), I transform this Remainder into 500 *Ten-thousandths*, which I divide by 99, and there results 5 *Ten-Thousandths* or (0,0005) with a Remainder 5 *Ten-Thousandths* or (0,0005); so that the Quotient will be composed of similar decimal Periods, each consisting of the two Characters 05 equal to those of the Dividend, that is, 5 or 05 being divided by 99, will give for Quotient (0,05 05 &c.). In fine, any Number divided by another Number greater than it, all whose Figures are 9^s, will give for Quotient an infinite Series of decimal Periods, consisting of *as many* Figures as the Divisor, and each of those Periods having the same significative Figures as the Dividend; so that if the Dividend has less Figures than the Divisor, there will be places in each Period filled up with Cyphers, set down to the left Hand of the significative Figure or Figures, equal to those of the Dividend.

LXXXI.

Method of finding from whence simple and compound repetends are derived.

Reciprocally the Sum of an infinite Series of decimal Periods, consisting of the same Figures, is equal to the Quotient of one Period divided by a number, consisting of as many 9^s as there are Figures in the Period.

For Example, the Series (0,333 &c.) each of whose Periods consists of one Figure 3, is equal to the Quotient of the Division of 3 by 9, or of 1 by 3,

The Series (0,23 23 23 &c.) each of whose Periods (23) consists of two Figures, is the Quotient of the Division of 23 by 99,

The Series (0,087 087 087, &c.) each of whose Periods (087) consists of three Figures, is the Quotient of the Division of 087 or of 87 by 999,

The Series 0,001 001 001 &c.) each of whose Periods (001) consists of three Figures, is the Quotient of the Division of 001 or of 1 by 999, and so on.

LXXXII.

Method of finding from whence simple and compound repetends preceded by Cyphers are derived.

According as the Comma or separating Point of any proposed decimal Number, is advanced one Place towards the left Hand, its Value is thereby rendered ten Times less. For Example, if in the Series (0,298 298 &c.) the Comma be advanced one Place to the left Hand, there will result (0,0298 298, &c.) whose Value is ten Times less than that of (0,298 298 &c.) if the Comma be again advanced one Place to the left Hand; there will result (0,00298 298 &c.) whose Value is ten Times less than that of (0,0298 298 &c.) or a hundred Times less than (0,298 298 &c.) and so on.

But the Series (0,298 298 &c.) composed of equal Periods, the first beginning immediately after the Comma, is the Quotient of the Division of 298 by 999.

Wherefore the Series (0,0298 298.298 &c.) whose Value is ten Times less than that of the Former, is the Quotient of the Division of 298 by 9990, that is equal to $\frac{298}{9990}$ and the Series (0,00298 298 &c.) whose Value is ten Times less than the Former, is the Quotient of the Division of 298 by 99900, or is equal to $\frac{298}{99900}$ and so on; that is, when a Series of decimal Periods does not immediately begin after the separating Point, it represents the Quotient of a Division, the Dividend of which, is equal to one Period; and the Divisor consists not only of as many 9's as there are Figures in the Period, but also of as many Cyphers, as there are Places between the Comma and the first significative Figure of the first Period.

LXXXIII.

Besides those decimal Series that consist only of equal Periods, there are others, that independant of an infinite Series of equal Periods, include a certain Number of decimal Figures, after which begin the equal Periods. Such are the following, 0,1666 &c. 0,08 333 &c. 0,004 629 629 &c.

Method of finding from whence simple and compound repetends preceded by a certain number of decimal figures are derived.

The first of those Series 0,1666 &c. consists of 0,1 and an infinity of Periods of 6's.

The second 0,08333 &c. consists of a Part 0,08 and an infinity of 3's.

The third 0,004 629 629, is composed of a Part 0,004 and an infinity of Periods 629, and so on.

To discover from whence those Quotients are derived; for Example, to discover from whence the Series 0,004 629 &c. is derived, I separate the infinite Series of Periods from the decimal Figures that precede them, in order to form two decimal Numbers of the proposed one, and there results 0,004 and 0,000629 629 &c. contained in the given Number 0,004 629 629 &c.

Now the first Part (0,004) which denotes 4 Thousandths, is equal to $\frac{4}{1000}$. The second Part 0,000629 629 &c. which consists only of

an infinite Series of equal Periods, is equal to $\frac{629}{999000}$; Wherefore the

Sum of the two Parts 0,004 and 0,000 629 629 &c. or the proposed decimal Series 0,004 629 629 &c. is equal to the Sum of the two Divisions

$\frac{4}{1000} + \frac{629}{999000}$. Multiplying the Dividend and Divisor of the first Di-

vision by 999, there results $\frac{3996}{999000}$ and $\frac{629}{999000}$ and their Sum

$\frac{4625}{999000}$ or $\frac{1}{216}$ (found by dividing its two Terms by the Dividend 4625) will be the Division from whence the proposed decimal Series 0,004 629 629 &c. is derived.

LXXXIV.

To abridge the foregoing Operation of Division, the Computists instead of setting down the Products of the Multiplication of the Divisor into the Figures of the Quotient, deduct from the Dividend the Figures of those Products according as they find them. Let for Example 7958 be proposed to be divided by 23, placing the Divisor at the right Hand of the Dividend, as in the Margin, and beginning the Division by the Figures of the highest Denomination, I first divide 79 *Hundreds* by the Divisor, and there results 3 *Hundreds* for the first Quotient Figure.

$$\begin{array}{r} 7958 \left\{ \begin{array}{l} 23 \\ \hline 346 \\ \hline 105 \\ \hline 138 \\ \hline 000 \end{array} \right. \end{array}$$

Method of
contracting
the Operati-
ons of Divi-
sion.

To obtain the Remainder of this Division, I multiply the Divisor 23 by the Quotient, and deduct the Figures of the Product according as I find them from the Dividend 79. Saying: 3 Times 3 is 9, which I deduct from the 9 of the Dividend, and there being no Remainder, I set down a Cypher under the 9, then saying: 3 Times 2 is 6, which I deduct from the Figure 7 of the Dividend, and set down the Remainder under the 7, whence from the Division of 79 *Hundreds* by 23, there results for Quotient 3 *Hundreds*, with a Remainder 10 *Hundreds*, which could not be divided by 23.

To find the second Figure of the Quotient I bring down the 5 *tens* of the Dividend to the right Hand of the 10 *Hundreds* that remain after the first Division, and divide 105 *Tens* by 23, setting down the Quotient 4 *Tens* at the right Hand of the 3 *Hundreds*. To obtain the Remainder of this second Division of Tens, I multiply 23 by 4, and deduct the Figures of the Product according as I find them, from the second Member of the Division 105, saying: 4 Times 3 is 12, which is to be deducted from 5, but as this cannot be done, I borrow 1 *Ten*, or an Unit of a superior Degree, and adding it to 5, I deduct 12 from 15, and set down the Remainder 3 under the 5, then saying: 4 Times 2 is 8 and 1 I borrowed, which is to be deducted, is 9, and deducting 9 from 10, I set down the Remainder 1 underneath, so that the Remainder of the second Division will be 13.

To find the third Figure of the Quotient, I bring down the 8 Units of the Dividend to the right Hand of the 13 *Tens* remaining of the forego-

ing Division, and there results 138 *Units* to be divided by 23; which gives 6 *Units* for the Quotient, which I set down to the right Hand of the two Figures 34 already found. To obtain the Remainder of the Division, I multiply the Divisor 23 by 6; and subtract from this Member of the Division 138, the Figures of the Product according as I find them. And there being no Remainder left, I set down a Cypher underneath.

The Division being ended, I find 346 for the exact Quotient of 7958 divided by 23.

LXXXV.

The foregoing Operation is performed by some Computists in a manner somewhat different, in each particular Division, the Divisor being set down under the Dividend, and the Remainder above it.

For Example, if it was proposed to divide 7958 by 23, having pointed off as many Figures to the left Hand of the Dividend, as will contain the Divisor: And as the two Figures 79 of the Dividend contains the Divisor 23, I set down 23 under 79, and consider how often it is contained in 79, or how often 2 is contained in 7; and as it is contained 3 Times, I set down 3 in the Quotient.

10
7958 (3
23

Spanish Method of performing Division explained by an Example.

To discover the Remainder of this Division, I multiply the Divisor 23 by the Quotient 3; and subtract the Figures of the Product according as I find them from the superior Figures, and set down the Remainders over those superior Figures. Saying: 3 Times 3 is 9, which I deduct from the superior Figure 9; and as there is no Remainder left, I set down a Cypher over this 9 which I barr, as also the Figure 3 multiplied. Then saying: 3 Times 2 is 6, which I subtract from the superior Figure 7, and set down the Remainder 1 over the 7, and barr this 7 and the Figure 2 multiplied.

By this first Operation, the 79 *Hundreds* of the Dividend will be divided by 23; the Quotient will be 3 *Hundreds*, and there will remain 10 *Hundreds* that could not be divided by 23.

To continue on the Division, I annex the 5 *Tens* of the Dividend to the 10 *Hundreds* that remain after the first Division; and set down anew the Divisor 23 under the Dividend 105 *Tens*, so that the 3 will stand under the 5, and the 2 under the Cypher; that is, I remove the Figures of the Divisor one Place towards the right Hand. I then consider how often 23 is contained in 105; and set down the Quotient 4 to the right Hand of the Figure 3 first found.

1
203
7088 (34
233

To obtain the Remainder of this second Division, I say: 4 Times 3 is 12; and as this particular Product cannot be deducted from the supe-

rior Figure 5, I borrow 1 *Ten* to add it to 5, and deduct 12 from 15; and set down the Remainder 3 over the 5, barring this 5 and the 3 multiplied. Then I say, 4 Times 2 is 8 and 1 I borrowed is 9, which I deduct from 10; setting down the Remainder 1 over the Cypher, after having barred 10 and the 2 multiplied.

By this second Operation, the 105 *Tens* will be divided by 23; the Quotient will be 4 *Tens*, and there will be a Remainder 13 *Tens* which could not be divided by 23.

To compleat the Division, I annex the 8 *Units* to the Remainder 13 *Tens*; and set down the Divisor 23 under the new Dividend 138: So that the 3 will stand under the 8, that is, I remove the Figures of the Divisor one Place towards the right Hand. And finding that 23 is contained 6 Times in 138, I set down 6 in the Quotient to the right Hand of 34, and multiplying the Divisor 23 by 6, I subtract the Figures of the Product according as I find them from those above them, as in the two foregoing Operations, and after the Subtraction, there being no Remainder left, I conclude that 346 will be the exact Quotient of 7958 divided by 23.

LXXXVI.

To render the foregoing Method of performing Division more commodious, the Computists have found Means of avoiding the Trouble of borrowing when the Product of the Divisor into the Quotient is subtracted from the Members of the Division. Let it be required, for Example, to divide 84162 by 98, as the Divisor 98 is not

French Method of performing Division explained by an Example.

contained in the two first Figures 84 to the left Hand of the Dividend, I point off the three Figures 841 for the first Member of the Division, and setting down the Divisor 98 under 41, I consider how often 98 is contained in 841 *Hundreds*, and set down the Quotient 8 *Hundreds*, in a Place which will be that of the *Hundreds*.

To obtain the Remainder of this first Division, I multiply successively the two Figures of 98 by 8, beginning by the Figure 9 of the highest Degree; and deduct the Figures of the Product, according as I find them from those immediately over them. Saying: 8 Times 9 is 72, which I deduct from 84, setting down the Remainder 12 over it, after having barred 84 and the Figure 9 multiplied. Then I say, 8 Times 8 is 64, which I deduct from 121, and set down the Remainder 57 over it, after having barred 121 and the Figure 8 multiplied.

By this first Operation, the 841 *Hundreds* will be divided by 98; the Quotient will be 8, and there will remain 57 *Hundreds* that could not be divided by 98.

2
203
7958 (346
2883
22

5
227
84162 (8
98

To continue on the Division, I remove the Divisor 98 one Place to the right Hand, setting it down under 76, in order to divide the 57 *Hundreds* remaining with the 6 *Tens*, that is, in order to divide 576 *Tens* by 98. The Divisor 98 being contained 5 Times in the second Member of the Division 576, I set down 5 in the Quotient at the right Hand of the 8 already set down, and multiply 98 by 5, beginning by the Figure of the highest Degree; and deduct the Figures of the Product according as I find them from the superior Ones. Saying: 5 Times 9 is 45, which I deduct from 57; and set down the Remainder 12 over 57, after having barred 57 and the Figure 9 multiplied. Then saying: 5 Times 8 is 40, which I deduct from 126, and set down the Remainder 86 over 126, or rather over 26, after having barred 126 and the Figure 8 multiplied.

$$\begin{array}{r} 18 \\ 82 \\ 2276 \\ 84262 \text{ (85)} \\ 988 \\ 9 \end{array}$$

By this second Operation, the Number 576 *Tens* will be divided by 98; the Quotient will be 5 *Tens*, and there will remain 86 *Tens* that could not be divided by 98.

To compleat the Division, I remove the Divisor still one Place towards the right Hand, in order to divide the 86 *Tens* remaining with the 2 Units, that is, in order to divide 862 Units by 98. Finding that 98 is contained 8 Times in the Member of the Division 862, I set down 8 in the Quotient to the right Hand of 85; and search for the Remainder of this Division, Saying: 8 Times 9 is 72, which I deduct from the superior Figures 86, and set down the Remainder 14 over these Figures, after having barred 86 and the Figure 9 multiplied. Then saying: 8 Times 8 is 64, which I deduct from 142; and set down the Remainder 78 Units over 42.

$$\begin{array}{r} 2 \\ 287 \\ 824 \\ 22768 \\ 84262 \text{ (858)} \\ 9888 \\ 99 \end{array}$$

Wherefore the Quotient arising from the Division of 84162 by 98 will be 858, and there will remain 78 Units, which cannot be divided by 98.

LXXXVII.

Since to divide a Number by another, is to find a third Number which multiplied by the second, will give a Product equal to the first Number, Proof of the Division may be proved, by considering it as a Multiplication of which Operations in Division made by casting out the Divisor is the Multiplicand, the Quotient the Multiplier, and the Dividend the Product, that is, by casting out the 9's out of the Divisor and Quotient, and multiplying the Remainders one by the other, after having added to the Product the Figures of the Remainder of the Division, when there is one; and casting out the 9's, if the Remainder is equal to what remains when the 9's are cast out of the Dividend; there is a Presumption that the Division has been accurately performed, otherwise it is certain that some Error has been committed in the Operation.

ELEMENTS OF

Example.

For Example, if 5478989 was divided by 375, and the Quotient found to be 14610 with a Remainder 239, I cast the 9's out of the Divisor and Quotient, and multiplying the Remainders 6 and 3 one by the other, and adding to their Product 18 the Remainder of the Division 239, and casting the 9's out of their Sum 257; the Remainder 5 being equal to what remains when the 9's are cast out of the Dividend 5478989, there is a Presumption that the Operation has been accurately performed.

LXXXVIII.

To prevent the Multiplicity of Errors and to discover them in the Course of the Division, the Computists verify each Figure of the Quotient according as they are found.

Method of
verifying
each figure
of the quo-
tient accord-
ing as they
are found.

For Example, if 254496 was to be divided by 264, beginning the Division, by the Figures of the highest Degree, I first divide 2544 *Hundreds* by 264, and find 9 *Hundreds* for the Quotient, I therefore set down 9 in the Quotient in a Place which will be that of the Hundreds; to discover whether the Figure 9 placed in the Quotient is exact, or if the Divisor 264 is contained 9 Times in the Member of the Division 2544; I multiply 264 by 9; setting down the Product 2376, under the Member of the Division 2544, and subtracting this Product 2376 from 2544 I find 168 for the Remainder.

$$\begin{array}{r}
 254496 \left\{ \begin{array}{l} 964 \\ \hline 264 \end{array} \right. \\
 \underline{2376} \\
 1689 \\
 \underline{1584} \\
 1056 \\
 \underline{1056} \\
 0000
 \end{array}$$

To discover whether this first Remainder 168 *Hundreds* is exact; there are two Things to be considered. 1°. We are to examine whether 2376 is the exact Product of 264 into 9; by taking the ninth Part of this Product: as this ninth Part should be equal to the Divisor 264. 2°. We are to examine whether the Subtraction has been accurately performed; by adding the Remainder 168 to the Product 2376: as this Sum should be equal to the Number 2544 from which it is subtracted.

To continue on the Division, I bring down the 9 *Tens* of the Dividend to the right Hand of the 168 *Hundreds*, and there Results 1689 *Tens* to be divided by 264; which gives 6 *Tens* for the Quotient. I then multiply 264 by 6, and deduct the Product 1584 from 1689; and there remains 105. I prove this second Division as the former. 1°. By taking the sixth Part of 1584, which is equal to the Divisor 264, and proves the Product 1584 to be exact. 2°. By adding 105 to 1584 whose Sum is equal to 1689, the second Member of the Division, and proves that the Remainder 105 which is less than the Divisor 264 is exact.

The Operations employed to obtain the 4 Units of the Quotient will be proved in like Manner.

C H A P. II.

Of Fractions and their various Reductions, &c. Of the Operations performed upon compound or applicate Numbers, and the Mensuration of Surfaces and Solids.

THE Numbers we have hitherto treated of are composed of several integer Units, which for this Reason are called *integer* or *whole Numbers*, but it often happens that the Unit that has been adopted, or established by Custom, is too great to be contained once or several Times exactly in the Magnitude which is proposed to be measured. In this Case the Computists form smaller Units that may measure exactly the proposed Magnitude, that is, which may be contained in it once, or a certain Number of Times, exactly.

I.

To obtain Units suitable to the Magnitude which is proposed to be measured; the Computists divide the principal Unit that has been adopted into several equal Parts, which in general are called *fractional Units*, and that receive their particular Denominations from the Number of Parts into which the principal Unit has been divided. For Example, if the principal Unit be divided into 2, or into 3, or into 4, or into 5 equal Parts, each Part is called 1 *Half*, or 1 *Third*, or 1 *Fourth*, or 1 *Fifth*; and those Parts are *fractional Units*. Origin of Fractions.

A fractional Unit or a Collection of several equal fractional Units, is called a *Fraction* or *broken Number*. There are therefore two Numbers required to represent a broken Number; viz. a Number to denote the Species of the fractional Unit; that is, to shew into how many equal Parts the principal Unit has been divided, and another Number to denote how many Times those new Units are repeated.

To distinguish those two Numbers, the Computists place one above the other, with a Line drawn betwixt them, placing under the Line the Number which denotes into how many equal Parts the principal Unit has been divided, and which consequently denotes the Species of the *fractional Unit*; and placing above the Line, the Number which denotes how often the *fractional Unit* is taken. Notation of Fractions.

For Example, $\frac{7}{8}$ is a Fraction or broken Number, the inferior Number (8) denotes that the principal Unit has been divided into 8 equal Parts, and that consequently each Part is the 1 *Eighth* of the principal Unit; and the superior Number (7) denotes that there are 7 of those new Units; so that the Fraction $\frac{7}{8}$ denotes 7 *Eighths* of the Unit or Quantity that has been taken for Unit.

As the *fractional Unit* receives its Denomination from the inferior Number of the Fraction, it is called the *Denominator*, and because the superior Number denotes the Number of those new Units, it is called the *Numerator*. Thus in the Fraction $\frac{7}{8}$ the inferior Number (8) which represents *Eighths*, is the *Denominator*, and the superior Number (7) which denotes that there are 7 of those Units called *Eighths*, is the *Numerator*.

The Numerator and Denominator of a Fraction are called the two *Terms* of this Fraction; the Numerator is called the *first Term*, and the Denominator is called the *second Term*.

II.

The
different
species of
fractions.

The Computists distinguish two Sorts of broken Numbers; *abstract broken Numbers*, and *concrete or applicate broken Numbers*. The broken Numbers, such as $\frac{1}{4}$, $\frac{2}{5}$, $\frac{5}{6}$, which denote 1 *Fourth*, 2 *Fifths*, 5 *Sixths*, that are not applied to number any Species of Things are called *abstract* or *absolute* broken Numbers. Broken Numbers that are applied to Number the Parts of any Thing, are called *concrete or applicate broken Numbers*. For Example, $\frac{1}{4}$ Pound, $\frac{2}{5}$ Foot, $\frac{5}{6}$ Hour, which denote *one fourth of a Pound*, *two Fifths of a Foot*, *five Sixths of an Hour*, are *concrete or applicate broken Numbers*. It is manifest that all broken Numbers, whether abstract or concrete, may be considered as Species of integer Numbers, having for principal Units their particular fractional Units.

III.

A Fraction may be also considered as the Quotient of a Division, of which the Numerator is the Dividend, and the Denominator the Divisor. For Example, the Fraction $\frac{7}{8}$ may be considered as the Quotient of 7 divided by 8. For to divide 7 by 8 is to take the eighth Part of 7, but to take the eighth Part of 7 the eighth Part of each of the Units which compose 7 must be taken: and as each Unit will give 1 Eighth, for its eighth Part; 7 Units will give 7 Eighths, that is, the Fraction $\frac{7}{8}$ for their eighth Part.

Whence the Fraction $\frac{7}{8}$ is the Quotient of 7 divided by 8.

How a
whole Num-
ber chang-
ed into a
fraction.

As a Number is not changed when multiplied and divided by the same Quantity; being rendered so much less by Division, as it is increased by Multiplication, it is manifest that an Integer may be changed into a Fraction; by multiplying it by any Number for to make a Numerator, and giving it this same Number for a Denominator.

Since a Fraction is equal to the Quotient of the Division of its Numerator by its Denominator ; it is manifest that it is equal to an integer Unit, when its Numerator is equal to its Denominator, for the Denominator will be contained once in its Numerator.

A fraction whose numerator and denominator are equal, is an unit.

IV.

Since by multiplying or by dividing the Dividend and Divisor of a Division, by a same Quantity, the Value of the Quotient is not changed if the Numerator and Denominator of a Fraction be multiplied or divided by the same Quantity ; the Value of the Fraction will not be changed.

If, for Example, the Fraction $\frac{4}{5}$ be given, and the Numerator 4 and the Denominator 5 be multiplied by 3 or by 4, there will result a new Fraction $\frac{12}{15}$ or $\frac{16}{20}$ of the same Value as the first $\frac{4}{5}$. And reciprocally, if the Numerator and Denominator of a Fraction $\frac{12}{15}$ be divided, by a same Number for Example, by 3, there will result a new Fraction $\frac{4}{5}$ which will be equal to the first $\frac{12}{15}$.

The value of a fraction is not changed by multiplying its numerator and denominator by the same quantity.

By dividing the Numerator and Denominator of a Fraction by a same Quantity it is rendered more simple, and so much the more so, as the Quantity by which it is divided is greater ; and when the two Terms of a Fraction are divided by their greatest common Divisor, the Fraction which results whose two Terms can no more be divided by a same Quantity, is said to be reduced to its lowest Terms.

V.

To reduce a Fraction to its lowest Terms, without altering its Value, the greater Term must be divided by the lesser ; and if there is no Remainder, the lesser Term will be manifestly the greatest common Divisor of the two Terms of the Fraction. If there is a Remainder, the lesser Term must be divided by this Remainder, and if there is no Remainder, the Remainder of the first Division will be the greatest common Divisor of the two Terms of the Fraction.

Method of reducing fractions to their lowest terms.

If this second Division leaves a Remainder, the first Remainder must be divided by the second, and the second by the third, and so on continually, dividing the last Divisor by its Remainder, till there is no Remainder left : And then the last Divisor will be the greatest common Divisor of the two Terms of the Fraction, and by dividing the two Terms by this last Divisor, the Fraction will be reduced to its lowest Terms. But if it should happen that the last Remainder should be Unit, then is the Fraction already expressed by its most simple Terms.

VI.

Let it be proposed, for Example, to reduce the Fraction $\frac{2016}{5796}$ to its lowest Terms.

The foregoing method of reducing fractions to their lowest terms, applied to an example.

1°. I divide the greatest Term 5796 by the lesser 2016. The Division being performed, there will remain 1764. 2°. I divide 2016 by the Remainder 1764; without attending to the Quotient, but only to the Remainder 252. 3°. I divide the first Remainder 1764, by the second Remainder 252; and as there is no Remainder left after the Division, I conclude that the last Divisor 252, will be the greatest common Divisor of the two Terms of the Fraction $\frac{2016}{5796}$ and dividing the two

Terms of this Fraction by 252, there results the new Fraction $\frac{8}{23}$, whose Terms cannot be rendered more simple, and which will have the same Value as the proposed Fraction $\frac{2016}{5796}$

VII.

Grounds of the foregoing method.

The ground of this Operation is as follows: 1° By dividing the Denominator by 2016 of the Fraction $\frac{2016}{5796}$, the Quotient will be found to be 2 with a Remainder 1764. Whence the Denominator 5796 is composed of two Parts 4032 and 1764, and the Fraction $\frac{2016}{5796}$ may be reduced to this Form $\frac{2016}{2 \times 2016 \text{ more } 1764}$, wherefore any Number that is the greatest common Divisor of the two Terms of the Fraction $\frac{2016}{5796}$, will be the greatest common Divisor of 2×2016 and of 1764, which are the two Parts of the Denominator, and consequently should be the greatest common Divisor of 2016 and of 1764.

2° By dividing 2016 by 1764 it will be found to be contained once with a Remainder 252, that is, 2016 is composed of two Parts, of 1764 and of 252. Whence the Number which will be the greatest Divisor of 2016 and of 1764 will be also a Divisor of 252. But 252 is the greatest Divisor of 252, and divides 1764 exactly: It will therefore also divide 2016 which is the Sum of 252 and of 1764, and will be also a Divisor of twice 2016, that is, of 2×2016 . The Number 252 being a Divisor of 2×2016 and of 1764, will be also a Divisor of their Sum 5796, and will consequently divide the Numerator and Denominator of the Fraction $\frac{2016}{5796}$. Moreover, 252 is the greatest common Divisor of 2016 and of 5796; since the common Divisor of those two Numbers should be a Divisor of 252.

VIII.

There is another Method of reducing a Fraction to its lowest Terms, Another more easy, and upon several Occasions, more commodious than the method of forego-^{ing}. 1°. If the Numerator and Denominator of a Fraction are even Numbers, they may be both divided by 2, till one of those two fractions to their lowest Terms becomes an odd Number. 2°. If the two Terms end by 5, they may be divided by 5, till one of them ceases to end by 5. 3°. When the two Terms of the Fraction are odd Numbers, and do not end by 5, Tryal may be made to divide them by 3, till one of them ceases to be divisible by 3; afterwards Tryal may be made to divide the two new Terms by 7, then by 11, afterwards by 13, 17, 19, 23; and so on by all the Numbers that have no other Divisors but themselves and Unity. In fine, when one of the Terms is no more divisible, or when the two Terms can be no more divided by a same Quantity, the Fraction will be reduced to its lowest Terms.

IX.

Let it be proposed, for Example, to reduce the Fraction $\frac{2016}{3780}$ to its lowest Terms.

I divide the two Terms which are even by 2, and there results $\frac{1008}{1890}$. I divide again by 2 the two new Terms, which are also even Numbers, and there results $\frac{504}{945}$. I then divide those two Terms by 3, and there results $\frac{168}{315}$. Dividing again by 3, there results $\frac{56}{105}$. Dividing afterwards by 7, I find $\frac{8}{15}$. And this Fraction $\frac{8}{15}$ will at Length be the Fraction reduced because its Numerator 8, cannot be divided but by 2, or by a Multiple of 2, and its Denominator 15 cannot be divided by 2. ^{The foregoing method applied to an example.}

X.

If the two Terms of the Fraction have Cyphers to their right Hand, it is manifest that an equal Number may be suppressed in those two Terms; because each Term will be divided by 10 each Time a Cypher is suppressed. ^{The foregoing method applied to another example.}

Thus to reduce the Fraction $\frac{3150}{11400}$ to its lowest Terms, I first divide the two Terms by 10, by suppressing in each of them the Cypher that occupies the Place of the simple Units, and there, results $\frac{315}{1140}$. Then because the new Numerator ends by 5, and is consequently divisible by 5, as also the new Denominator which ends by a Cypher, I divide by 5, and there results $\frac{63}{228}$. I afterwards divide by 3, and there results $\frac{21}{76}$.

And this Fraction $\frac{21}{76}$ will be reduced to its lowest Terms ; since the Numerator 21 is divisible only by 7 or by 3, and the Denominator 76 is divisible neither by one or the other.

XI.

Method of
reducing
any two
fractions to
the same de-
nominator.

To reduce two Fractions to the same Denominator without altering the Value of those Fractions ; having reduced them to their lowest Terms (if they are not so already) the Numerator and Denominator of the first Fraction must be multiplied by the Denominator of the Second, and the Numerator and Denominator of the second by the Denominator of the first, whence will result two other Fractions of the same Value with the two first ; (Art. iv.) having the same Denominator, since the Denominator of each of them will be the Product of the Denominators of the two first Fractions.

XII.

Let it be proposed, for Example, to reduce to the same Denomination the Fractions $\frac{2}{3}$ and $\frac{5}{7}$.

Application
of this me-
thod to an
example.

1^o. I multiply the two Terms 2 and 3 of the first Fraction $\frac{2}{3}$, by the Denominator 7 of the second, and there results a new Fraction $\frac{14}{21}$ equal to the first $\frac{2}{3}$. 2^o. I multiply the two Terms 5 and 7 of the second Fraction $\frac{5}{7}$ by the Denominator 3 of the first, and there results a new Fraction $\frac{15}{21}$ equal to the second $\frac{5}{7}$.

By this Means the two proposed Fractions $\frac{2}{3}$ and $\frac{5}{7}$ without having changed their Value will be reduced to the two Fractions $\frac{14}{21}$ and $\frac{15}{21}$, which have each for Denominator the Product of the Denominators 3 and 7 of the proposed Fractions.

XIII.

General
method of
reducing
any number
of fractions

To reduce any Number of Fractions to the same Denominator ; being reduced to their lowest Terms, 1^o. All the Denominators of the Fractions must be multiplied into one another, and the Product will be the Denominator that all the Fractions reduced to the same Denomination should have.

20. To obtain the Numerator of the first of the Fractions that are proposed to be reduced to the same Denomination, the Denominators of all the Fractions, except that of the first, must be multiplied into one another, and the Product that arises multiplied by the Numerator of the first will be the Numerator of this first Fraction reduced.

To obtain the Numerator of the second Fraction reduced, all the Denominators except that of the second, must be multiplied into one another, and the Product that results multiplied by the Numerator of the second will be the Numerator of this second Fraction reduced; and in like Manner the Numerators of the other Fractions will be found.

XIV.

Let it be proposed, for Example, to reduce to the same Denomination the four Fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$.

I multiply into one another all their Denominators 2, 3, 5, 7; and the Product 210, will be the new Denominator, which should be common to all the Fractions.

Application
of this Me-
thod to an
Example.

To obtain the Numerator of the first of the new Fractions; I multiply into one another all the Denominators, except the first (2), that is, I multiply together the three Denominators, 3, 5, 7; and the Product that arises 105 by the Numerator 1 of the first Fraction, and the Product 105, will be the Numerator of the first Fraction, which will become $\frac{105}{210}$.

To obtain the Numerator of the second new Fraction; I multiply into one another all the Denominators except the second (3), that is, I multiply together the three Denominators 2, 5, 7, which produce 70; I afterwards multiply 70 by the Numerator 2 of the second proposed Fraction, and the Product 140 will be the Numerator of the second new Fraction.

$\frac{140}{210}$. In like Manner I find the Numerators of the two other Fractions.

which will be $\frac{168}{210}$, $\frac{180}{210}$. Whence the four Fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$ reduced to the same Denomination will be $\frac{105}{210}$, $\frac{140}{210}$, $\frac{168}{210}$, $\frac{180}{210}$.

XV.

By the foregoing Method any Number of Fractions may be reduced to the same Denominator without altering their Value, but those Fractions will not always be reduced to as low Terms as they might be, and retain still a common Denominator.

Inconveni-
ency to
which this
foregoing
Method of
of reducing

1^o. If among the Fractions reduced to their lowest Terms, such as the following $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$; there are not several whose Denomina-

fractions to the same denomination is liable.

tors have a common Divisor: When those Fractions will be reduced to the same Denomination; the new Fractions $\frac{105}{210}, \frac{140}{210}, \frac{168}{210}, \frac{180}{210}$, that result cannot be reduced to lower Terms and retain still a common Denominator.

Method of remedying this inconvenience according to the two different cases that may occur.

2°. If among the Fractions which are supposed reduced to their lowest Terms, there are several whose Denominators have common Divisors: When all these Fractions are reduced to the same Denomination, they may be reduced to lower Terms, and retain still a common Denominator, by dividing their Numerators and the common Denominator by the common Divisor so many Times less one as there are Denominators to which these Divisors are common in the first Fractions.

For Example, if those four Fractions were given $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{7}$, reduced to their lowest Terms, among which are three whose Denominators 2, 4, 6, are divisible by 2: When all those Fractions are reduced to the same Denomination, and become $\frac{168}{336}, \frac{84}{336}, \frac{56}{336}, \frac{48}{336}$; the Numerator of each of them and the common Denominator may be divided twice successively by 2, which will reduce them to the following ones $\frac{42}{84}, \frac{21}{84}, \frac{14}{84}, \frac{12}{84}$.

XVI.

When the Divisors of the Denominators are composite Numbers, if their Factors are common to a greater Number of Denominators, they should be employed, as Divisors, preferably to those composite Divisors.

Example of the second case.

If those four Fractions $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}$, were proposed, which may be reduced to this Form $\frac{1}{2}, \frac{1}{2 \times 2}, \frac{1}{2 \times 3}, \frac{1}{2 \times 2 \times 3}$. Though the Denominators $2 \times 2, 2 \times 2 \times 3$ of two of them are divisible by the composite Divisor 2×2 ; I do not make Use of this composite Divisor, because the simple Divisor 2 is common to a greater Number of Denominators than the composite one 2×2 : And I observe,

1°. That the four Denominators 2, $2 \times 2, 2 \times 3, 2 \times 2 \times 3$ are divisible by 2, and may be reduced to 1, 2, 3, 2×3 .

2°. That of those four Terms reduced by Division, there are two, viz. 2 and 2×3 still divisible by 2, so that those four Terms are reducible to 1, 1, 3, 3.

3°. Lastly, that those four new Terms, still include two, viz. 3 and 3, which are divisible by 3.

Whence the four proposed Fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{12}$, when reduced to the same Denominator, and are transformed into $\frac{288}{576}$, $\frac{144}{576}$, $\frac{96}{576}$, $\frac{48}{576}$, may be reduced to lower Terms and still retain a common Denominator, by dividing their Numerators and Denominators, three Times successively by 2, or once only by 8, then once by 2, and lastly once by 3, which will reduce those Fractions to $\frac{6}{12}$, $\frac{3}{12}$, $\frac{2}{12}$, $\frac{1}{12}$, which cannot be reduced to lower Terms, and retain a common Denominator.

XVII.

The Operations performed upon Fractions will often produce other Fractions whose Numerators will be greater than their Denominators. Reduction of improper fractions. And since a Fraction is equal to a whole Unit when its two Terms are equal, those Fractions will contain as many integer Units as the Denominators are contained in their Numerators.

Whence to find the Number of integer Units contained in a Fraction; the Numerator must be divided by the Denominator, and the Quotient of this Division will be the Number of integer Units contained in the Fraction. And the Remainder of the Division, if there be any, will be the Numerator of a Fraction having the Divisor for Denominator.

For Example, $\frac{18}{4}$ being a proposed Fraction; I divide the Numerator 18 by the Denominator 4, and there results for Quotient 4 integer Units with a Remainder 2; and this Remainder being divided by the Denominator 4 will give the Fraction $\frac{2}{4}$ which is reduced to $\frac{1}{2}$; so that the Fraction $\frac{18}{4}$ will be changed into 4 Units and $\frac{1}{2}$. Example.

XVIII.

The foregoing Reductions of Fractions are necessary to prepare them for Addition and Subtraction, for we can really add together, or subtract one from the other only Quantities which consist of Units of the same Species. Whence Fractions cannot be added together or subtracted one from the other, but when their fractional Units are the same; and then they must have the same Denominator.

XIX.

10. If the Fractions which are proposed to be added have the same Denominator; their Sum will be obtained by forming a new Fraction of the same Denominator, and having for Numerator, the Sum of their Numerators.

For Example, if the Fractions $\frac{3}{7}$, $\frac{4}{7}$, $\frac{6}{7}$, which have the same Denominator 7, were to be added together; I consider them as concrete Numbers, whose principal Units are *Sevenths*; and I say 3 *Sevenths* and 4 *Sevenths* is 7 *Sevenths* which joined to 6 *Sevenths* make 13 *Sevenths*, which I set down thus $\frac{13}{7}$: That is, I add together the three Numerators 3, 4, 6, and subscribe under their Sum 13, the Denominator 7 common to all the Fractions added together; and there results $\frac{13}{7}$ for the Sum required.

20. If the Fractions to be added together have not the same Denominator: They must be reduced to the same Denomination, and those new Fractions added, as in the former Case.

For Example, if the Fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, were proposed to be added together; I transform them into $\frac{105}{210}$, $\frac{140}{210}$, $\frac{168}{210}$, $\frac{180}{210}$, which have the same Denominator. I then add together their Numerators 105, 140, 168, 180, and subscribing under their Sum 593 the Denominator 210, there will result the Fraction $\frac{593}{210}$ equal to the Sum of the Fractions $\frac{105}{210}$, $\frac{140}{210}$, $\frac{168}{210}$, $\frac{180}{210}$, which are equal to the four proposed ones.

XX.

From the Addition of several Fractions, there often results a Fraction whose Numerator is greater than the Denominator. Such a Fraction being greater than the principal Unit, should be reduced to the integers it contains, and to a Fraction that it may contain besides.

For Example, having found that the Sum of the three Fractions $\frac{3}{7}$, $\frac{4}{7}$, $\frac{6}{7}$ is $\frac{13}{7}$, whose Numerator 13 contains the Denominator 7 once with a Remainder 6, whence this Fraction $\frac{13}{7}$ may be divided into those two others $\frac{7}{7}$ and $\frac{6}{7}$ and consequently will be equivalent to 1 and $\frac{6}{7}$

XXI.

To subtract one Fraction from another, 1^o. If they have the same Denominator, it suffices to deduct the Number of fractional Units of the one from the Number of the fractional Units of the other, since their fractional Units are equal. Now the Numerators of the Fractions express the Number of fractional Units they contain; wherefore the Remainder of the Subtraction will be obtained, by deducting the Numerator of one from the Numerator of the other, and subscribing under the Remainder the Denominator common to the two Fractions; because the Units remaining after the Subtraction should be of the same Species as those of the Number from which the Subtraction has been made.

For Example, if it was proposed to deduct $\frac{2}{9}$ from $\frac{8}{9}$, I subtract 2 from 8, and under the Remainder 6, I subscribe the common Denominator 9; and there results for the Remainder of the Subtraction the Fraction $\frac{6}{9}$ which may be reduced to the Fraction $\frac{2}{3}$.

2^o. If the proposed Fractions have different Denominators, they must be reduced to the same Denomination, and then deducted one from the other as in the foregoing Case. For Example, if it was proposed to subtract $\frac{2}{3}$ from $\frac{6}{7}$; I first reduce those two Fractions to the same

Denomination, whereby they will be transformed into $\frac{14}{21}$ and $\frac{18}{21}$, then subtracting the Numerator 14 from the Numerator 18, I subscribe under the Remainder 4 the Denominator 21; and there results $\frac{4}{21}$ for the Remainder of the Subtraction.

XXII.

The Multiplications and Divisions of Fractions by Fractions, being Operations composed of the Multiplication and Division of Fractions by whole Numbers, before we proceed to treat of the Multiplication and Division of Fractions by Fractions, it will be convenient to explain the Multiplication and Division of Fractions by whole Numbers

XXIII.

To multiply a Fraction by a whole Number, for Example, by 2, or by 3, or by 4, &c. is to repeat it twice or three Times, or four Times, &c. or in general as often as the Multiplier contains Unity: consequently it is to make a Fraction twice or three Times or four Times, &c. greater than the proposed Fraction. Now this Operation may be performed two different Ways, either by operating on the Numerator only, or by operating on the Denominator only.

First method.

10. If it be proposed to operate only on the Numerator, the Numerator of the proposed Fraction must be multiplied by the whole Number which is to serve as Multiplier; and subscribing under this Product the Denominator of the proposed Fraction, there will result a new Fraction, which will be the Product required; as will appear by the following Example.

If it was proposed to multiply the Fraction $\frac{2}{9}$ by 4; I consider the

Example of this first Method.

Multiplicand $\frac{2}{9}$ as a concrete Number (2 Ninths) which is to be repeated 4 Times; whence I say 4 Times 2 Ninths is 8 Ninths, which I set down thus $\frac{8}{9}$: that is, I multiply the Numerator 2 by 4, and subscribe

under the Product 8 the same Denominator 9; which will give $\frac{8}{9}$

for the Product required.

Second method.

2^o. If it be proposed only to operate on the Denominator, this Denominator must be divided by the proposed Multiplier; and subscribing the Quotient under the Numerator of the Fraction to be multiplied; the new Fraction that will result from this Operation, will be the Product required, as will appear by the following Example,

Let it be proposed to multiply $\frac{5}{12}$ by 2: I divide the Denominator 12

Example of this second Method.

by 2, and there results a new Fraction $\frac{5}{6}$ for the Product arising from the Multiplication of $\frac{5}{12}$ by 2.

For the principal Unit being divided into twice more Parts in the Fraction $\frac{5}{12}$ than in the Fraction $\frac{5}{6}$, the fractional Units of the Fraction $\frac{5}{6}$ will be double of those of the Fraction $\frac{5}{12}$. And as those fractions consist of the same Number of Parts, it is manifest that $\frac{5}{6}$ will be double of $\frac{5}{12}$.

As it is always possible to multiply one Number by another, and that one Number cannot always be divided by another without a Remainder; it will always be possible to perform the Multiplication of a Fraction by a whole Number, by multiplying its Numerator by this whole Number, but not by dividing its Denominator by this whole Number.

XXIV.

To divide a Fraction by a whole Number, for Example, by 2, or by Division of 3, or by 4, &c. is to form a new Fraction which will be twice, or 3 Fractions Times, or 4 Times, &c. less than the Fraction proposed to be divided. by whole Numbers.

Now this Operation may be performed two different Ways; by operating on the Numerator only, or by operating only on the Denominator.

1^o If it be proposed to operate only on the Numerator: the Numerator of the proposed Fraction must be divided by the whole Number which is to serve as Divisor; and subscribing under the Quotient the Denominator of the proposed Fraction, there will result a new Fraction which will be the Quotient required. As will appear by the following Example.

First Method explained by an Example.

Let it be proposed to divide the Fraction $\frac{8}{9}$ by 4, I consider the Dividend $\frac{8}{9}$, as a concrete Number which represents 8 Nintbs; and to divide it by 4, I say the fourth Part of 8 Nintbs is manifestly 2 Nintbs; which I set down thus $\frac{2}{9}$. Wherefore the Fraction $\frac{2}{9}$ found by dividing the Numerator $\frac{8}{9}$ by 4, is manifestly the Quotient of the Fraction $\frac{8}{9}$ divided by the whole Number 4.

2^o. If it be proposed to operate only on the Denominator; the Denominator of the Fraction must be multiplied by the proposed Divisor: and subscribing the Product under the Numerator of the proposed Fraction; the new Fraction that results will be the Quotient required.

Second Method explained by an Example.

For Example, if it was proposed to divide the Fraction $\frac{6}{7}$ by 4; without meddling with its Numerator 6, I multiply only its Denominator 7 by 4; and there results the Fraction $\frac{6}{28}$, for the Quotient of the Fraction $\frac{6}{7}$ divided by 4.

For the principal Unit being divided into four Times more Parts in the Fraction $\frac{6}{28}$ than in the Fraction $\frac{6}{7}$, the fractional Units of the Fraction $\frac{6}{28}$ will be only the fourth Parts of those of the Fraction $\frac{6}{7}$: and as those two Fractions consist of the same Number of Parts, that, $\frac{6}{28}$ whose Parts are four Times less will be contained four Times in the other $\frac{6}{7}$, that is, as often as there are Units in the Divisor 4.

The second
Method not
always prac-
ticable.

As it is always possible to multiply the Denominator of a Fraction by a whole Number, and it often happens that its Numerator can not be divided exactly without a Remainder: it is manifest that a Fraction can always be divided by a whole Number, by multiplying its Denominator by this whole Number, but not by dividing its Numerator by this whole Number.

XXV.

Multiplica-
tion by
Fractions.

To multiply any Magnitude by a Fraction, is to multiply this Quantity by its Numerator, and afterwards divide it by its Denominator. For let it be proposed, for Example, to multiply any proposed Magnitude by $\frac{2}{3}$, if instead of multiplying it by the Fraction $\frac{2}{3}$, it had been multiplied by the Numerator 2; the Product arising would be triple of the one required, since the proposed Quantity has been multiplied by a Number triple of $\frac{2}{3}$ the given Multiplier: consequently the third Part of this Product must be taken, that is, it must be divided by the Denominator 3, in order to reduce it to its just Value.

Whence it follows that the Multiplication by a Fraction, whose Numerator is Unity, is really a Division by the Denominator of this Fraction. For Example, the Multiplications by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. are really Divisions by the Denominators 2, 3, 4 &c. of these Fractions.

For to multiply by $\frac{1}{2}$ or by $\frac{1}{3}$ or by $\frac{1}{4}$, we must first multiply by Unity, which will not alter the Quantity multiplied; and afterwards divide by 2, or by 3, or by 4.

XXVI.

Four differ-
ent Me-
thods of
multiplying
a Fraction
by a Frac-
tion.

The Multiplication of a Fraction by a Fraction may be performed four different Ways, three of which are confined to particular Cases, but the fourth is applicable to every Case.

1^o. A Fraction may be multiplied by a Fraction, by operating only on the Numerator of the Multiplicand, and consequently without altering the Denominator of this Fraction.

2^o. A Fraction may be multiplied by a Fraction, by operating only on the Denominator of the multiplicand, and consequently without altering the Numerator of this Fraction.

3^o. A Fraction may be multiplied by a Fraction, by operating on the two Terms of the Multiplicand, by Means of Division.

4^o. A Fraction may be multiplied by a Fraction by operating on the two Terms of the first, by Means of Multiplication.

We shall explain those different Methods of multiplying a Fraction by a Fraction, in the following Articles.

XXVII.

To multiply a Fraction by a Fraction, by operating only on the Numerator of the Fraction which is considered as the *Multiplicand*; the Numerator of the *Multiplicand* must be multiplied by the Numerator of the Multiplier; and this Product being divided by the Denominator of the Multiplier, and under the Quotient the Denominator of the *Multiplicand* being subscribed; there will result a Fraction equal to the Product required, as will appear by the following Example.

First Method explained by an Example.

Let it be required to multiply $\frac{9}{10}$ by $\frac{2}{3}$; the Fraction $\frac{9}{10}$ being considered as a concrete Number 9 *Tenths*, I first multiply it by the Numerator 2 of the Multiplier $\frac{2}{3}$, saying: twice 9 *Tenths* is 18 *Tenths* or $\frac{18}{10}$, but by multiplying by 2, I multiplied by a Number 3 Times too great; since it should have been only multiplied by $\frac{2}{3}$ which is the third Part of 2. Wherefore the Product 18 *Tenths* is also three Times too great; I therefore take the third Part of it, or divide it by 3; and there results 6 *Tenths* or $\frac{6}{10}$, for the real Product of $\frac{9}{10}$ multiplied by $\frac{2}{3}$.

XXVIII.

To multiply a Fraction by a Fraction by operating only on the Denominator of the Fraction which is considered as the *Multiplicand*; the Denominator of the *Multiplicand* must be divided by the Numerator of the Multiplier; and the Quotient multiplied by the Denominator of the Multiplier; and this Product being subscribed under the Numerator of the *Multiplicand*; there will result a Fraction equal to the Product of the two Fractions which were to be multiplied one by the other; as will appear by the following Example.

Second Method explained by an Example.

Let it be proposed to multiply the Fraction $\frac{9}{10}$ by $\frac{2}{3}$; by dividing the Denominator 10 of the first Fraction by the Numerator 2 of the second, the first Fraction $\frac{9}{10}$ will be multiplied by 2; (Art. XXIII.) consequently the Fraction that results $\frac{9}{5}$ will be triple of the one required, since the *Multiplicand* $\frac{9}{10}$, has been multiplied by 2 which is triple of the Fraction $\frac{2}{3}$ by which it should have been multiplied; this Pro-

duct therefore $\frac{9}{5}$ should be divided by 3 which is the Denominator of the Fraction $\frac{2}{3}$. But by multiplying the Denominator of the Product $\frac{9}{5}$ by 3, this Product will be divided by 3 (Art. xxiv.) Wherefore the Fraction that results $\frac{9}{15}$ will be the real Product of the Fraction $\frac{9}{10}$ multiplied by $\frac{2}{3}$.

XXIX.

Third
Method
explained
by an
Example.

To multiply a Fraction by a Fraction, by operating on the two Terms of the Fraction which is considered as the *Multiplicand*, by Means of Division. The Denominator of the first must be divided by the Numerator of the second, and the Numerator of the first by the Denominator of the second; and from those two Operations there will result a Fraction equal to the Product required.

Let it be proposed to multiply $\frac{9}{10}$ by $\frac{2}{3}$: I first divide the Denominator 10 of the first Fraction by the Numerator 2 of the Second, and by this Operation the Fraction $\frac{9}{10}$ will be multiplied by 2 (Art. xxiii.) but the Product $\frac{9}{5}$ will be triple of the one required; since the Number 2 by which I multiplied, is triple of $\frac{2}{3}$ by which I should have multiplied. This Product $\frac{9}{5}$ which I consider as a concrete Number 9 *Fifths*, should therefore be divided by 3; whence I say the third Part of 9 *Fifths* is 3 *Fifths* or $\frac{3}{5}$, and this Fraction $\frac{3}{5}$ will be the Product required.

XXX.

Fourth
Method
explained
by an
Example.

To multiply a Fraction by a Fraction, by operating on the two Terms of the Fraction considered as the *Multiplicand*, by Means of Multiplication, the Numerator of the first Fraction must be multiplied by that of second, and the Denominator of the first by that of the second: and the Fraction that results having for Numerator the Product of the two Numerators, and for Denominator the Product of the two Denominators, will be the Product required.

Let it be required to multiply $\frac{9}{10}$ by $\frac{2}{3}$: I consider the Multipli-

and $\frac{9}{10}$ as a concrete Number 9 *Tenths*, and multiply it first by the Numerator 2 of the Multiplier $\frac{2}{3}$, saying: twice 9 *Tenths* is 18 *Tenths* or $\frac{18}{10}$; this Product will be triple of the one required, because the Number 2 by which I multiplied, is triple of the Fraction $\frac{2}{3}$ by which I should have multiplied: the Product $\frac{18}{10}$ therefore must be divided by the Denominator 3 of the Multiplier $\frac{2}{3}$ to reduce it to its just Value: which is effected by multiplying the Denominator 10 by 3, (Art. xxiv.) which will give the Fraction $\frac{18}{30}$ for the Product required.

XXXI.

As a whole Number may be considered as a Fraction of which it is the Numerator, and Unity the Denominator: when a whole Number is proposed to be multiplied by a Fraction, this Operation may be reduced to the Multiplication of a Fraction by a Fraction. For Example, if it be proposed to multiply 3 by $\frac{4}{5}$; this Multiplication may be reduced to that of $\frac{3}{1}$ by $\frac{4}{5}$; and there will result for Product $\frac{12}{5}$ or 2 and $\frac{2}{5}$.

The Multiplication of a whole Number by a Fraction is the same as the Multiplication of a Fraction by a whole Number.

Whence the Multiplication of a whole Number by a Fraction, for Example, of 3 by $\frac{4}{5}$ is the same as the Multiplication of a Fraction $\frac{4}{5}$ by a whole Number 3. For $3 \times \frac{4}{5}$ or $\frac{3}{1} \times \frac{4}{5}$ is equal to $\frac{4}{5} \times \frac{3}{1}$ or $\frac{4}{5} \times 3$.

XXXII.

To divide by a Fraction, is to divide by its Numerator, and afterwards to multiply by its Denominator. For let any Quantity (for Example) be proposed to be divided by the Fraction $\frac{3}{4}$. Let it first be divided by the Numerator 3 of this Fraction: this Divisor being 4 Times too great, since it was proposed to divide only by the fourth Part of 3, will give a Quotient 4 Times too little; consequently this Quotient must be multiplied by the Denominator 4 of the same Fraction, to reduce it to its just Value.

In what consists the Division by a Fraction.

NUMERAL ARITHMETICK.

From whence it follows 1^o that the Division by a Fraction, for Example, by $\frac{3}{4}$, is reduced to a Multiplication by the converse Fraction

$\frac{4}{3}$, for the division by the Fraction $\frac{3}{4}$, is performed by dividing by 3 and afterwards multiplying by 4; now the Multiplication by the converse Fraction $\frac{4}{3}$ is performed by precisely the same Operations; that is, (Art. xxv.) by multiplying by 4 and dividing by 3.

The Division by a Fraction reduced to a Multiplication.

It follows 2^o. that the division by a Fraction whose Numerator is Unity, is really a Multiplication by the Denominator of this Fraction.

For to divide by the Fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. is to multiply by the converse Fractions $\frac{2}{1}$, $\frac{3}{1}$, $\frac{4}{1}$, &c. that is, by the whole Numbers 2, 3, 4, &c.

XXXIII.

The Division of a Fraction by a Fraction may be performed two different Ways. 1^o. Since the Division by a Fraction is reduced to a Multiplication by the converse Fraction; when a Fraction is proposed to be divided by a Fraction, for Example, if $\frac{9}{20}$ was proposed to be divided

First Method of dividing a Fraction by a Fraction.

by $\frac{3}{4}$, it suffices to invert the Terms of the Divisor $\frac{3}{4}$, to obtain its converse $\frac{4}{3}$; and afterwards multiply the Dividend $\frac{9}{20}$ by this converse $\frac{4}{3}$ of the Divisor, and the Product that arises $\frac{36}{60}$ or $\frac{9}{15}$, will be the Quotient of the Fraction $\frac{9}{20}$ divided by the Fraction $\frac{3}{4}$.

Second Method of dividing a Fraction by a Fraction.

2^o. If the Dividend and Divisor have the same Denominator or are reduced to the same Denominator; the Division may be performed by dividing the Numerator of the Dividend by the Numerator of the Divisor; or else by forming a new Fraction having for Numerator the Numerator of the first Fraction, and for Denominator the Numerator of the second Fraction, the Fraction resulting from this Operation being the Quotient of the Division of the Dividend by the Divisor.

Let 1^o the Fraction $\frac{8}{9}$ be proposed to be divided by the Fraction $\frac{4}{9}$; I consider these two Fractions as concrete Numbers 8 *Ninths*, 4 *Ninths* whose fractional Units are *Ninths*; whereby the Division of $\frac{8}{9}$ by $\frac{4}{9}$

will be reduced to find how often the Dividend 8 *Ninths*, contains the Divisor 4 *Ninths*, which is effected by dividing the Number 8 of the fractional Units of the Dividend by the Number 4 of the fractional Units of the Divisor, that is, by dividing the Numerator 8 of the Dividend $\frac{8}{9}$, by the Numerator 4 of the Divisor $\frac{4}{9}$; which will give the whole Number 2 for the Quotient required.

2°. If it had been proposed to divide the Fraction $\frac{4}{9}$ by $\frac{8}{9}$, or the concrete Number 4 *Ninths* by the concrete Number of the same Species 8 *Ninths*; the Number 4 of the Fractional Units of the Dividend must be divided by the Number 8 of the Fractional Units of the Divisor, which will give for Quotient the Fraction $\frac{4}{8}$ having for Numerator the Numerator of the Dividend $\frac{4}{9}$, and for Denominator the Numerator of the Divisor $\frac{8}{9}$.

XXXIV.

As the principal Unit is divided into several Fractional Units, so a Fraction considered as a collective Unit may be divided into several equal Parts, which are called *Fractional Units of a Fraction*; and a Quantity consisting of one or several Fractional Units is called a *Fraction of a Fraction*. For Example, if the Fraction $\frac{5}{6}$ considered as an Unit be divided into three equal Parts; each Part which is only one third of the Fraction $\frac{5}{6}$, will be a fractional Unit of a Fraction, and the Quantity consisting of one or several of those fractional Units is called a *Fraction of a Fraction*.

The different Subdivisions of Fractions and the Manner of expressing them.

To express a Fraction of a Fraction, two Fractions are required separated by the Word *of*. For Example, to represent the two Thirds of the Fraction $\frac{5}{6}$, it is set down thus $\frac{2}{3}$ of $\frac{5}{6}$, which expresses 2 Thirds of 3 Sixths. The Denominator 3 of the first Fraction denoting into how many Parts the second Fraction is divided; and the Numerator of the first Fraction denoting how many of those Parts of the Fraction are taken.

As a Fraction may be divided into several equal Parts, one or several of which form a *Fraction of a Fraction*, in like Manner a Fraction of a Fraction may be divided into several equal Parts, one or several of which form a *Fraction of a Fraction of a Fraction*; and so on ad Infinitum.

ELEMENTS OF

All those different Species of Fractions of Fractions are set down one after the other, being separated by the Article *of*. For Example, if the $\frac{2}{3}$ of the Fraction $\frac{5}{6}$ be taken, it is expressed thus, the two Thirds of five sixths, and set down thus $\frac{2}{3}$ of $\frac{5}{6}$, and if the three Fourths of this Fraction be taken, it is expressed thus, the three Fourths of two Thirds of five Sixths, and set down thus $\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{5}{6}$.

A Fraction of a Fraction is equal to the Product arising from the Multiplication of the two Fractions by which it is expressed.

XXXV.

To determine the Value of a Fraction of a Fraction, let us take for Example the Fraction of a Fraction $\frac{2}{3}$ of $\frac{5}{6}$, which represents twice the one Third of five Sixths. Now the one Third of $\frac{5}{6}$ is obtained by dividing this Fraction by 3: consequently twice the one Third of $\frac{5}{6}$ will be obtained by dividing it by 3, and multiplying the Quotient by 2;

A Fraction of a Fraction is always the same whatever Way the Fractions by which it is expressed are arranged.

that is, by multiplying one by the other, the two Denominators and the two Numerators of the Fractions $\frac{2}{3}$ and $\frac{5}{6}$; wherefore a Fraction of a Fraction is equal to the Product arising from the Multiplication of the two Fractions by which it is expressed.

Since a Fraction of a Fraction is the Product of the two Fractions by which it is expressed, and this Product will be always the same in whatever Order those Fractions are multiplied, it is manifest that $\frac{2}{3}$ of $\frac{5}{6}$

A Fraction of a Fraction of a Fraction, &c. is equal to a Fraction having for Numerator the Product of the Numerators of all the Fractions by which it is expressed, and for denominator the Product of their Denominators.

is equal to $\frac{5}{6}$ of $\frac{2}{3}$.

XXXVI.

Since a Fraction of a Fraction of a Fraction, E. g. $\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{5}{6}$, is equal to the Fraction of a Fraction $\frac{3}{4}$ of $\frac{2 \times 5}{3 \times 6}$, or $\frac{3}{4}$ of $\frac{10}{18}$, and

that $\frac{3}{4}$ of $\frac{10}{18}$ is equal to $\frac{3 \times 2 \times 5}{4 \times 3 \times 6}$ it follows that the Fraction of a

Fraction of a Fraction $\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{5}{6}$ is equal to the Fraction $\frac{3 \times 2 \times 5}{4 \times 3 \times 6}$:

and in general that a Fraction of a Fraction of a Fraction, &c. is equal to a Fraction having for Numerator the Product of the Numerators of all the Fractions by which it is expressed, and for Denominator the Product of their Denominators.

XXXVII.

If there was only one Kind of Unit for all Magnitudes of the same Species; those Magnitudes could not be expressed by Numbers with sufficient Accuracy, unless the Unit was very small; and in this Case the Numbers requisite to express the most ordinary Magnitudes would be excessive, and consequently very inconvenient in Trade.

Necessity of employing Units of different Denominations for measuring the same Kind of Magnitudes.

In Computations of Money, for Example, if the Farthing was always assumed for Unit, the Numbers representing the different Sums would be four Times greater than if the Penny was taken for Unit, and would be 48 Times greater than if the Shilling was taken for Unit, and 960 Times greater than if the Pound Sterling was taken for Unit.

If the Pound Sterling was assumed for Unit, and no other was employed; all the possible Sums could not be reckoned without omitting such as are less than the Pound Sterling, which though too considerable to be rejected; could not be attended to without dividing the Pound Sterling into different Parts, which would require a particular Knowledge of Fractions.

XXXVIII.

To remedy those two Inconveniencies, the Computists employ Units of different Degrees of Magnitude for measuring the same Species of Quantities

The different Kinds of Units employed in Money, Weights and Measures.

In Computations of Money; they commonly employ four Sorts of Units, viz. the *Pound Sterling*, the *Shilling*, *Penny* and *Farthing*, they also employ for Units the *Crown*, the *Guinea*, and all the Coins current in Trade.

In Computations of Weights; the Units employed are the *Pound*, the *Ounce*, the *Penny Weight*, the *Dram*, the *Scruple*, the *Grain*; and in heavy Weights, the *Stone*, the *hundred Weight*, the *Tun*, &c.

In Computations of Lineal Measure; the Units employed are the *Fathom*, the *Yard*, the *Foot*, the *Inch*, the *Line*, the *Ell*, the *Pole*, &c.

In Computations of Time; the Units employed are the *Year*, the *Month*, the *Day*, the *Hour*, the *Minute*, the *Second*, &c. the *Week*, which consists of 7 Days, is also employed for Unit, and an *Age* which consists of 100 Years.

In Computations of Angles and of Arcs of Circles; the Units employed are the *Degree*, the *Minute*, the *Second*, &c. the *Signs* each of which contain 30 Degrees are also employed for Units. In fine, for calculating each Species of Quantities, the Computists employ for Units some of the known Parts of the same Kind of Quantities.

XXXIX.

The Value of the different Units, above mentioned, and the Characters by which they are distinguished from each other are as follows.

Value of the different Units employed in Money, Weights

IN MONEY, *l.* denotes Pounds Sterling, *s.* Shillings, *d.* Pence, and *grs.* Farthings.

and Mea-
sures, and
the Charac-
ters by
which they
are distin-
guished
from each
other.

1 Pound Sterling is equal to 20 Shillings, 1 Shilling to 12 Pence, and 1 Penny to 4 Farthings.

IN WEIGHTS *lb.* denotes Pounds, *oz.* Ounces, *dwt.* Penny Weights, 3 Scruples, 3. Drams, *cwt.* hundred Weight.

IN TROY WEIGHT used for weighing Gold, Silver, Seeds, Liquors, Bread, Medicines, &c. 1 Pound is equal to 12 Ounces, 1 Ounce to 20 Penny Weight, and 1 Penny Weight to 24 Grains.

In compounding of Medicines, &c. 1 Pound is divided into 12 Ounces, 1 Ounce into 8 Drams, and 1 Dram into 3 Scruples, and 1 Scruple into 20 Grains.

IN AVOIR-DU-POISE WEIGHT, used for weighing coarse and heavy Goods, such as grocery Wares, Pitch, Tar, Rosin, Wax, Tallow, Flax, Hemp, &c. Copper, Tin, Iron, Lead, Steel, &c. 1 Tun is equal to 20 hundred Weight, 1 hundred Weight to 8 Stone, 1 Stone to 14 Pounds, and 1 Pound to 16 Ounces.

IN LINEAL MEASURE, *F.* denotes Fathoms, *Y.* Yards, *f.* Feet *I.* Inches, *L.* Lines, *E.* Ells.

1 Fathom is equal to 2 Yards, 1 Yard to 3 Feet, 1 Foot to 12 Inches, 1 Inch to 12 Lines, and 1 Line to 12 Points, 1 Ell is equal to $1\frac{1}{4}$ of a Yard.

XL.

In treating of the Addition of simple Numbers, we defined Addition to be an Operation by which a Number is found equal to the Sum of several other Numbers. We proved afterwards that the Numbers to be added should consist of Units of the same Species, and that the Units of the Sum resulting from their Addition are of the same Species. All which is likewise applicable to compound Numbers: for though the Units of all their Parts are not absolutely the same, they are reducible to similar Units; that is, a certain Number of Units of a lower Denomination, form an Unit of another Part of a higher Denomination.

XLI.

Process of
Addition of
mixed Num-
bers.

Compound or applicate Numbers that are proposed to be added together, must be set down one under the other; so that all the Parts whose Units are similar, may stand in the same Column; and all the Figures of the same Degree that are in decuple Proportion, may also stand under one another. Then a Line being drawn under them, to separate them from their Sum; all the Parts whose Units are of the lowest Denomination must be first added; and if their Sum contains one or several Units of the next superior Denomination; they must be retained to add them to those of the next Column; and what remains set down under the Column that was added. The same Operation being performed upon each Column, the Number under the Line will be the Sum required of the compound Quantities which were proposed to be added.

XLII.

To illustrate those Rules, we shall proceed to apply them to several Examples; in Money, assuming for Units the *Pound Sterling*, the *Shilling*, *Penny* and *Farthing*; in Weights, assuming for Units the *Pound*, the *Ounce*, *Penny Weight* and *Grain*; in lineal Mensuration, assuming for Units the *Yard*, the *Foot*, the *Inch*, and the *Line*. Those Examples well understood will be sufficient to shew how those Rules are to be applied, for finding the Sum of any other Kind of compound Numbers.

XLIII.

Let it be proposed to find the Sum of the four following mix'd Numbers
 387*l.* 12*s.* 8*d.* 3*qrs.* 759*l.* 19*s.* 11*d.* 2*qrs.* 896*l.* 17*s.* 10*d.* 1*qrs.*
 4563*l.* 19*s.* 9*d.* 3*qrs.*

	<i>l.</i>	<i>s.</i>	<i>d.</i>	<i>qrs.</i>	
Having set down those Numbers as in the Margin, 1 ^o . I begin by adding the Farthings,	387	12	8	3	
as being the Units of the lowest Denomination,	759	19	11	2	
saying: 3 Farthings and 2 <i>qrs.</i> is 5 <i>qrs.</i> and 1 <i>qrs.</i>	896	17	10	1	
is 6 <i>qrs.</i> and 3 <i>qrs.</i> is 9 <i>qrs.</i> and because 9 <i>qrs.</i> con-	4563	19	9	3	
tains twice 4 <i>qrs.</i> which is 2 <i>d.</i> and 1 <i>qrs.</i> over. I	6608	10	4	1	
set down 1 <i>qrs.</i> under the Line in the Column					
of Farthings, and retain 2 <i>d.</i> to add them to the following Column.					

Addition of
Pounds Shil-
lings and
Pence
explained by
an Example.

2^o. As the tens of Pence form no particular Species of Units, and that 12*d.* is equal to 1 Shilling; I add together not only all the Pence, but also the tens of Pence; to find the Sum of the Pence. Saying: 2 Pence I retained from the Column of Farthings and 8*d.* is 10*d.* and 11*d.* is 21*d.* and 10*d.* is 31*d.* and 9 is 40*d.* and as 40*d.* contains three Times 12 Pence or 3 Shillings and 4*d.* over; I set down 4 Pence in the Column of Pence, and I retain the 3 Shillings to add them to the following Column.

3^o. Since 2 tens of Shillings is equal to 1 Pound Sterling, and that there are Units of Shillings which may amount to some tens of Shillings; I add the Column of Shillings at twice, beginning by the Units place of the Shillings. And I find 30*s.* or 3 tens of Shillings, I therefore set down a Cypher in the Units place of Shillings, and retain 3 which I add to the tens of Shillings, whose Sum is 7; and as 2 tens of Shillings is equal to one Pound Sterling, the 7 tens of Shillings will be equal to 3*l.* 10*s.* I therefore set down 1 in the Column of tens of Shillings, and retain 3*l.* Sterling to add them to the Pounds.

4^o. Proceeding to the Column of simple Pounds, I say 3 Pounds I retained from the Column of tens of Shillings, and 7*l.* is 10*l.* and 9*l.* is 19*l.* and 6*l.* is 25*l.* and 3*l.* is 28*l.* which contains 2 tens of Pounds and 8*l.* over: I therefore set down 8*l.* in the Units Place of the Pounds, and I retain 2 tens to add them to those of the following Column. And performing the Remainder of the Operation as explained in Addition of simple Numbers, I find 6608*l.* 10*s.* 4*d.* 1*qrs.* for the Sum required.

ELEMENTS OF

XLIV.

Let it be proposed to find the Sum of the four following mixt Numbers
 472lb. 11oz. 19dwt. 22gr. 1438lb. 10oz. 14dwt. 21gr. 2568lb. 11oz.
 18dwt. 10gr. 348lb. 10oz. 17dwt. 9gr.

Addition of
Pounds,
Ounces,
Penny
Weights
and Grains
explained by
an Example.

Having set down the those Numbers as in the Margin, I begin by adding the Grains, as being the Units of the lowest Degree. Saying 22gr. and 21gr. is 43gr. and 10gr. is 53gr. and 9 is 62gr. which contains twice 24gr. or 2 Penny Weights and 14 Grains over. I therefore set down 14gr. in the Column of Grains, retaining the 2 Penny Weights to add them to the Column of Penny Weights.

lb.	oz.	dwt.	gr.
472	11	19	22
1438	10	14	21
2568	11	18	10
348	10	17	9
4829	9	10	14

Proceeding to the Column of Penny Weights, I find 68 dwt. to which adding the 2 Penny Weights retained from the Column of the Grains, there results 70dwt. which contains three Times 20 Penny Weights, or 3 Ounces with 10dwt. over. I therefore set down 10 dwt. in the Column of Penny Weights, and retain 3 for the Column of Ounces.

Proceeding to the Column of Ounces, and adding to it the 3 Ounces I retained, I find 45oz. which contains 3 Times 12 Ounces, or 3 Pounds with 9oz. over; I therefore set down 9oz. in the Column of Ounces, and retain the 3oz. to add them to the Column of Pounds.

Proceeding to the Column of simple Pounds, and adding to it the 3 Pounds that I retained from the Column of Ounces, I find 29lb. that is, 2 tens of Pounds and 9 Pounds over, which I set down in the Column of simple Pounds, and retain the 2 tens to add them to the Column of tens of Pounds, and performing the Remainder of the Operation as explained in Addition of simple Numbers, I find 4829lb. 9oz. 10dwt. 14gr. for the Sum required.

XLV.

Let it be proposed to find the Sum of the four following mixt Numbers;
 32Y. 2f. 6l. 10L. 144Y. 1f. 9l. 9L. 67Y. 2f. 10l. 8L. 78Y. 2f. 11l. 11L.

Addition of
Yards, Feet
Inches and
Lines ex-
plained by
an Example.

Having placed those Numbers as in the Margin, I begin by adding the Lines, as being the Units of the lowest Degree. Saying: 10L. and 9L. is 19L. and 8L. is 27L. and 11L. is 38L. which contains 3 Times 12 Lines, or 3 Inches with 2 Lines over, I therefore set down 2L. in the Column of Lines, and retain 3l. to add them to the Column of Inches.

Y.	f.	l.	L.
32	2	6	10
144	1	9	9
67	2	10	8
78	2	11	11
324	1	3	2

Proceeding to the Column of Inches, I say: 3l. that I retained from the Column of Lines, and 6l. is 9l. and 9l. is 18l. and 10l. is 28l. and 11l. is 39l. which contains 3 Times 12 Inches, or 3 Feet, with 3 Inches over, I therefore set down 3 Inches in the Column of Inches, and retain 3f. to add them to the Column of Feet.

Proceeding to the Column of Feet, I say: 3*f.* that I retained and 2*f.* is 5*f.* and 1*f.* is 6*f.* and 2*f.* is 8*f.* and 2*f.* is 10*f.* which contains 3 Times 3 Feet, or 3 Yards and 1*f.* over. I therefore set down 1*f.* in the Column of Feet, and retain 3 Yards to add them to the Column of Yards.

Proceeding to the Addition of the Column of Yards, I say 3 Yards that I retained and 2*Y.* is 5*Y.* and 4*Y.* is 9*Y.* and 7*Y.* is 16*Y.* and 8*Y.* is 24*Y.* which contains 2 tens of Yards, and 4 Yards over. I therefore set down 4 Yards in the Column of Yards, and retain 2 tens of Yards to add them to the Column of the tens of Yards that follow, completing the Addition, after the manner explained in the Addition of simple Numbers, I find 324*Y.* 1*f.* 3*l.* 2*L.* for the Sum required.

XLVI.

In treating of the Subtraction of simple Numbers, we defined Subtraction to be an Operation by which one Quantity is deducted from another, whence we concluded that the Quantity to be deducted should be contained in the Quantity from which the Subtraction is required to be made; and consequently that those two Quantities should be both of the same Species, or reducible to the same Species, it is the same with respect of applicate Numbers; one Quantity cannot be Subtracted from another if they are not both of the same Species, or reducible to the same Species. For Example, a Quantity made up of different Weights, cannot be deducted from another Quantity made up of Yards and Parts of a Yard; but Yards, or known Parts of a Yard, may be deducted from another Quantity consisting only of Yards or Parts of a Yard.

XLVII.

To subtract one compound Number from another compound or simple Number, the Parts of the Quantity to be subtracted must be disposed under the similar Parts of the Quantity from which the Subtraction is required to be made, observing to place the Units, Tens, Hundreds, &c. of one Species, under the Units, Tens, Hundreds, &c. of the same Species, then drawing a Line under those Numbers to separate them from the Remainder, each Part of the Quantity to be subtracted must be deducted from each corresponding Part of the other Quantity that stands over it, beginning by the Parts of the lowest Denomination, and proceeding to the next superior Denomination, and so on through the whole till all the Parts of the Quantity to be deducted are subtracted.

The process of Subtraction of mixed Numbers.

If the Number of Parts to be deducted, is less than the Number of the corresponding Parts in the other Quantity that stands over it, there will be no Difficulty in subtracting them, and the Remainder must be set down under the Line in the same Column with those Parts. But if in the inferior Number that is to be deducted, there are Parts whose Number is greater than that of the corresponding Parts of the upper Num-

ber, an Unit must be borrowed from the following Parts that are of the next superior Denomination, in the upper Number, and this Unit reduced into Parts of the same Species with those from which the Subtraction is required to be made, added to them, whereby they will be sufficiently increased, so as the Number of Parts may be deducted from them, that could not be deducted before. The same Operation must be performed for all the Species of Parts in the Quantity to be deducted, which are in greater Number than in the corresponding Parts of the Quantity from which the Subtraction is required to be made, and Care taken to deduct the Unit borrowed from the Number from which it has been borrowed.

XLVIII.

Let it be proposed to subtract 98l. 14s. 10d. 3grs. from 199l. 1s. 6d. 2grs.

Subtraction
of Pounds,
Shillings,
and Pence.

Having placed the Numbers as in the Margin, to obtain the Remainder of the Subtraction, I first subtract the Farthings, which are the Parts of the lowest Denomination, but as the 3 Farthings to be subtracted are not contained in the 2 corresponding Farthings of the upper Number, I borrow 1 Penny from the Number of Pence in the upper Number, and reducing this Penny into 4 Farthings, I add them to the 2 Farthings from which the Subtraction is required to be made, and deduct 3 Farthings from 6 Farthings, and set down 3 Farthings for the Remainder of the Farthings.

l.	s.	d.	grs.
199	1	6	2
98	14	10	3
100	6	7	3

First Example.

Proceeding to the Column of Pence, I have 10 Pence to deduct from 5 Pence, (because the Unit borrowed from 6 to carry it to the Column of Farthings has reduced this Figure to 5) I therefore borrow 1 Shilling from the upper Number of Shillings, and reducing this Shilling into 12 Pence, I deduct 10 Pence from 17 Pence, and set down 7 Pence for the Remainder of the Pence.

Then proceeding to the Column of Shillings, I have 14 Shillings to subtract from nothing, because, the Shilling in the upper Number was borrowed and employed in the Pence: I therefore borrow 1 Pound from the Pounds of the upper Number; and reducing this Pound into 20 Shillings I deduct 14 Shillings from 20 Shillings, and set down 6 Shillings for the Remainder of the Shillings.

Proceeding to the Units Place of the Pounds, I subtract 8 Pounds from 8 Pounds, (because the Unit which was borrowed from the Figure 9 to carry it to the Column of Shillings has reduced this Figure to 8) and there remains nothing. In fine, continuing the Subtraction as in simple Numbers, I find 100l. 6s. 7d. 3grs. for the Remainder required.

XLIX.

Let it be proposed to subtract 9l. 16s. 11d. from 40l. 0s. 8d.

Having placed those Numbers as in the Margin, and as the 11 Pence to be subtracted cannot be deducted from 8 Pence which stand over it, and that nothing can be borrowed from the Shillings, or from the Units of the Pounds, I borrow one ten of the Pounds, leaving 9 Pounds in the Rank of the Pounds, and 19 Shillings in the Rank of Shillings, and adding the Shilling remaining reduced into Pence to the 8 Pence, I deduct 11 Pence from 20 Pence, and set down 9 Pence for the Remainder of the Pence, and the Operation is there by reduced to deduct 9l. 16s. from 39l. 19s. I therefore deduct 16s. from 19s. and set down 3s. for the Remainder of the Shillings, then proceeding to the Pounds I subtract 9 Pounds from 39 Pounds, and set down 30 Pounds for the Remainder of the Pounds, and there results 30l. 3s. 9d. for the Remainder required.

l.	s.	d.
9	19	
40	0	8
9	16	11
30	3	9

Second Example.

Let it be proposed to subtract 8lb. 10oz. 7dwt. 22gr. from 16lb. 8oz. 7dwt. 17gr.

Having placed those Numbers as in the Margin, I begin the Subtraction by the Grains, which are the Weights of the lowest Denomination; and as 22 Grains cannot be deducted from 17gr. I borrow 1dwt. or 24gr. which adding to the 17gr. I deduct 22gr. from 41gr. and set down 19gr. for the Remainder of the Grains.

lb.	oz.	dwt.	gr.
16	8	7	17
8	10	7	22
7	9	19	19

Subtraction of Pounds, Ounces, Penny Weights and Grains.

Proceeding to the Column of Penny Weights, as 7 Penny Weights cannot be deducted from 6 Penny Weights, to which the Penny Weights of the upper Number are reduced, because 1dwt. has been borrowed from them, I borrow 1 Ounce or 20dwt. and adding them to the 6dwt. I deduct 7dwt. from 26dwt. and set down 19dwt. for the Remainder of the Penny Weights.

Then proceeding to the Column of Ounces, as 10oz. cannot be deducted from 7oz. to which the Ounces of the upper Number are reduced, because 1oz. has been borrowed from them, I borrow 1 Pound or 12oz. and adding them to the 7oz. I deduct 10oz. from 19oz. and set down 9oz. for the Remainder of the Ounces.

Lastly proceeding to the Column of the Pounds, I subtract 8 Pounds from 15 Pounds, because 1lb. has been borrowed from the 6 Pounds, and the Subtraction being finished, there results 7lb. 9oz. 19dwt. 19gr. for the Remainder required.

LI.

Let it be proposed to subtract 97Y. 2f. 11I. from 100Y. 2f. 0I.

Subtraction
of Yards,
Feet, and
Inches, ex-
plained by
an Example.

Having disposed those Numbers as in the Margin,
and as 11I. cannot be deducted from 0I. I borrow an
Unit from the Feet, and deduct 11I. from 12I, and set
down 1I. for the Remainder of the Inches.

Y.	f.	I.
100	2	0
97	2	11
2	2	1

Proceeding to the Column of Feet, as 2 Feet cannot
be deducted from 1 Foot, to which the Feet of the up-
per Number are reduced, because 1 Foot has been borrowed from them,
I borrow 1 Yard, or 3 Feet, and adding it to the 1 Foot, I deduct 2
Feet from 4 Feet, and set down 2 Feet for the Remainder of the Feet,
and completing the Operation according to the foregoing Rules, I find
2Y. 2f. 1I. for the Difference required.

LII.

A Question
for Practice
in Addition
and Subtrac-
tion of mixt
Numbers.

In the foregoing Articles we have explained all the little Difficulties
that occur in the Subtraction of mixt Numbers, and that might embar-
rass Beginners, either in passing from one Column to another, or in bor-
rowing, when the Number of Parts to be subtracted, is greater than
that from which the Subtraction is required to be made. For the
further Exercise of the young Practitioner the following Question is pro-
posed, in which Addition and Subtraction are equally concerned.

A Merchant in balancing his Books, finds he hath in ready Money,
156l. 17s. in Goods, 1749l. 19s. 6d. his Stock in a Company Trade was
499l. 19s. 6d. 2qrs. due to him in open Accounts 2977l. 19s. 7d. 3qrs.
In Consignments 479l. 19s. 7d. He owes to A. 1456l. 18s. 7d. 2qrs.
To B. 99l. 19s. 11d. To C. 497l. 17s. 10d. and to the Bank 490l.
What is his nett Stock? Answer, 3319l. 18s. 10d. 3qrs.

LIII.

Process of
the Multipli-
cation of
mixt Num-
bers.

The Multiplication of applicate Numbers, is performed by multiply-
ing the Multiplicand by all the Parts of the Multiplier; but as the
Multiplier when it is a mixt Number consists of Parts less than the
principal Unit, and that each of those principal Units denote that the
Multiplicand must be taken once, the Parts of the Multiplier which
are less than the principal Unit, will denote that Parts of the Multipli-
cand must be taken for the Product. To render the Rules of the Multi-
plication of applicate Numbers more intelligible, we shall proceed to ex-
plain them in their Application to several Examples.

Multiplication being an Operation by which the Mutiplicand is re-
peated a certain Number of Times, expressed by the Number of prin-
cipal Units in the Multiplier; the Multiplier should be considered
as an absolute Number, even when it is applied to reckon any particular
Species of Units, and the Units of the Product should consequently be
of the same Species as those of the Multiplicand.

LIV.

Let it be proposed to multiply 518l. 14s. 8d. by 74 $\frac{1}{4}$.

Having placed those Numbers as in the Margin. To Multiply regularly all the Parts of the proposed Multiplicand, by all the Parts of the Multiplier; I begin by multiplying the whole Multiplicand by 74, multiplying afterwards the same Multiplicand by the Fraction $\frac{1}{4}$. But the Multiplicand 518l. 14s. 8d. consisting of three Parts, I multiply those three Parts one after the other by 74.

1°. I multiply 518l. by 74, and set down the particular Products 2072 Units of Pounds and 3626 tens of Pounds, as in the Margin.

2°. To multiply 14s. by 74, I divide 14s. into Parts that may be contained each a certain Number of Times in the Pound, which is the principal Unit of the Multiplicand. Those Parts will be 10s. and 4s. which I multiply separately by 74.

To multiply 10s. by 74 I observe that 10s. is $\frac{1}{2}$ of a Pound, and that 1 Pound Multiplied by 74 will give 74l. whence I conclude that the $\frac{1}{2}$ of a Pound or 10s. should only give for Product the $\frac{1}{2}$ of 74l. that is 37l.

To multiply 4s. by 74, I observe that 4s. being $\frac{1}{3}$ of a Pound, and 1 Pound multiplied into 74 giving 74l. the $\frac{1}{3}$ of 1l. or 4s. multiplied by 74 should only give the $\frac{1}{3}$ of 74l. which is 14l. 16s. for the $\frac{1}{3}$ of 74l. is 14l. and there remains 4l. or 80s. the $\frac{1}{3}$ of which is 16s.

3°. To multiply 8 Pence by 74, I observe that 8 Pence being the $\frac{1}{30}$ of a Pound, and that consequently 8 Pence multiplied into 74 should only produce the $\frac{1}{30}$ of 74l. whence we may take for the Product the $\frac{1}{30}$ of the Multiplier 74, considered as a Number of Pounds.

But instead of taking the $\frac{1}{30}$ Part of 74l. it will be more commodious to make use of the Product 14l. 16s. found for 4s. since 8 Pence is the sixth Part of 4s. the Product of 8d. into 74 will be the sixth Part of the Product 14l. 16s. found by multiplying 4s. by 74; whence the Operation is reduced to take the sixth Part of 14l. 16s. which will be 2l. 9s. 4d. because the sixth Part of 14l. is 2l. and there remains 2l. or 40s. which adding to 16s. I take the sixth Part of 56s. which is 9s. and there remains 2s. or 24d. the sixth Part of which is 4 Pence.

Having multiplied the Multiplicand 518l. 14s. 8d. by 74 it remains to multiply it by the Fraction $\frac{1}{4}$, to effect which I observe that if 518l. 14s. 8d. was multiplied by 1, the Product would be the whole Multiplicand 518l. 14s. 8d. the Product therefore arising from the Multiplication of the same Number by $\frac{1}{4}$, will be the $\frac{1}{4}$ of 518l. 14s. 8d. which I find to be 129l. 13s. 8d. because the $\frac{1}{4}$ of 518l. is 129l. with a Remain-

2 L

	518l.	14s.	8d.
	74	$\frac{1}{4}$	
4	2072l.		
70	3626		
10s.	37		
4s.	14	16s.	
8d.	2	9	4d.
$\frac{1}{4}$	129	13	8
	38515l.	19s.	0d.

The Multi-
plication of
Pounds,
Shillings,
and Pence
explained by
an Examp.

der 2*l.* or 40*s.* which added to 14*s.* makes 54*s.* the $\frac{1}{4}$ of which is 13*s.* with a Remainder 2*s.* or 24*d.* which added to 8*d.* makes 32*d.* the $\frac{1}{4}$ of which is 8*d.*

All the Parts of the Multiplicand being thus multiplied by all the Parts of the Multiplier, and all the particular Products set down, I add them together and their Sum 38515*l.* 19*s.* will be the Product required of the mixt Number 518*l.* 14*s.* 8*d.* multiplied into $74\frac{1}{4}$

I.V.

The Rules to be observed in the Multiplication of Money sufficiently appear from the foregoing Example. The only Difficulty that can occur in this Operation, is to find what Parts of the Multiplier considered as a Number of Pounds are to be taken for the different Numbers of Shillings and Pence in the Multiplicand, which, for the Assistance of the young Practitioner, we shall proceed to explain.

Table for finding what Parts of the Multiplier should be taken for the different Numbers of Shillings, Pence, and Farthings in the Multiplicand.

When the Number of Shillings or Pence, or of Shillings and Pence together, are contained exactly a certain Number of Times in 20*s.* that is, in a Pound, the Shillings or Pence, or the Shillings and Pence together, are said to be *aliquot Parts* of a Pound. So that the aliquot Parts of a Pound is always a Fraction of a Pound having for Numerator Numbers of Unity, and for Denominator the Number of Times that the Shillings or Pence, or the Shillings and Pence together, are contained in a Pound.

10*s.* is the $\frac{1}{2}$ of a Pound, whence for 10*s.* the $\frac{1}{2}$ of the Multiplier, considered as a Number of Pounds is taken.

5*s.* Is the $\frac{1}{4}$ of a Pound, whence for 5*s.* the $\frac{1}{4}$ of the Multiplier is taken.

For 4*s.* which is the $\frac{1}{5}$ of a Pound, the $\frac{1}{5}$ of the Multiplier is taken.

For 2*s.* which is the $\frac{1}{10}$ of a Pound, the $\frac{1}{10}$ of the Multiplier is taken.

For 1*s.* which is the $\frac{1}{20}$ of a Pound, the $\frac{1}{20}$ of the Multiplier is taken.

For 6*s.* 8*d.* which is the $\frac{1}{3}$ of a Pound, the $\frac{1}{3}$ of the Multiplier is taken.

For 3*s.* 4*d.* which is the $\frac{1}{6}$ of a Pound, the $\frac{1}{6}$ of the Multiplier is taken.

For 1*s.* 8*d.* which is the $\frac{1}{12}$ of a Pound, the $\frac{1}{12}$ of the Multiplier is taken.

For 2*s.* 6*d.* which is the $\frac{1}{8}$ of a Pound, the $\frac{1}{8}$ of the Multiplier is taken.

For 6*d.* which is the $\frac{1}{40}$ of a Pound, the $\frac{1}{40}$ of the Multiplier is taken.

For 3*d.* which is the $\frac{1}{80}$ of a Pound, the $\frac{1}{80}$ of the Multiplier is taken.

For 8*d.* which is the $\frac{1}{30}$ of a Pound, the $\frac{1}{30}$ of the Multiplier is taken.

For 4*d.* which is the $\frac{1}{60}$ of a Pound, the $\frac{1}{60}$ of the Multiplier is taken.

For 2*d.* which is the $\frac{1}{120}$ of a Pound the $\frac{1}{120}$ of the Multiplier is taken.

For 1*d.* which is the $\frac{1}{240}$ of a Pound, the $\frac{1}{240}$ of the Multiplier is taken.

When a Number of Shillings is not exactly contained a certain Number of Times in a Pound, it is called an *aliquant Part* of a Pound. A Number of Pence which is not contained exactly a certain Number of Times in a Pound, is also called an *aliquant Part* of a Pound.

When the Number of Shillings or Pence in the Multiplicand is not an aliquot Part of a Pound, but only an aliquant Part, the Computists divide it into two or three Parts, each of which are aliquot Parts of a Pound.

LVI.

The Multiplication of Numbers consisting of Pounds, Shillings, Pence, and Farthings, according to the foregoing Method, being often Times tedious and perplexed, we shall proceed to explain the more compendious Rules, the Computists have found for remedying this Inconveniency.

The Pounds and Shillings of the Product, arising from the Multiplication of the Shillings of the Multiplicand by a whole Number, is obtained, at one Operation, by the following Rule. Multiply the Number of Shillings by the Figure in the Units Place of the Multiplier, and if this Product be less than 20, set it down in the Column of Shillings; but if this Product is greater than 20, retain 1*l*. for every twenty Shillings it contains, and set down the Remainder in the Column of Shillings. Then multiply one half of the Number of Shillings by all the other Figures of the Multiplier, removing each Figure of this Product one Place towards the right Hand, and adding to the first Part of the Product the Number of Pounds that were retained.

Method of finding at one Operation the Pounds and Shillings arising from the Multiplication of a Number of Shillings into a whole Number.

LVII.

*Let it be proposed to multiply ol. 1*s*. by 457.*

Having set down those Numbers as in the Margin.
1^o. To obtain the Shillings of the Product, I multiply the Figure 1*s*. by the Figure 7 of the Units of the Multiplier; and as the Product 7*s*. is less than 20*s*. I set it down in the Column of Shillings.

ol.	1 <i>s</i> .	First Example.
457		
22 <i>l</i> .	17 <i>s</i> .	

2^o. To obtain the Pounds of the Product, I multiply $\frac{1}{2}$, which is the one half of the Number of Shillings, by the remaining Figures 45 of the Multiplier, that is, I take the one half of 45, removing each Figure of the Product one Place towards the right Hand, which will be 22*l*. 10*s*. placing the first Part 22*l*. in the Rank of the Pounds, and for the 10*s*. placing 1 ten to the left Hand of 7*s*. already set down. By this Operation, I find 22*l*. 17*s*. for the Product of 1*s*. multiplied into 457.

LVIII.

*Let it be proposed to multiply ol. 18*s*. by 457.*

Having placed those Numbers as in the Margin, 1^o. I multiply 18*s*. by the Figure 7 of the Units of the Multiplier. But as it would be too difficult to perform this Multiplication at one Operation; I multiply first 8*s*. by 7, which gives 56*s*. I set down the Figure 6 in the Rank of the Shillings, and retain the 5 tens of Shillings, I then multiply the tens of Shillings by 7, and there results for Product 7 tens of Shillings, which added to the 5 tens retained, makes 12 tens of Shillings or 6*l*. there being therefore nothing to be set down in the Rank of the tens of Shillings, I retain 6*l*. to add them to the Pounds.

ol.	18 <i>s</i> .	Second Example.
457		
411 <i>l</i> .	6 <i>s</i> .	

ELEMENTS OF

2°. I multiply the one half of the Number of Shillings, that is, 9s. by the two remaining Figures 45 of the Multiplier, removing the Figures of the Product one Place towards the Right Hand, and adding to this Product the 6l. that was retained, there results 411l. So that 411l. 6s. will be the whole Product of 18s. multiplied into 457.

LIX.

Let it be proposed to multiply ol. 17s. by 457.

Third Example.

Having set down those Numbers as in the Margin, 1°. I multiply 17s. by the Figure 7 of the Units of the Multiplier, which gives 5l. 19s. I set down the Part 19s. in the Rank of Shillings, and retain the Part 5l.

ol. 17s.
457

2°. I multiply by the two remaining Figures 45 of the Multiplier, the one Half of the Number of Shillings, that is $8\frac{1}{2}$; but as this Operation cannot be performed but at twice, I first multiply 8 by 45, removing the Figures of the Product one Place towards the right Hand, which with the 5l. retained gives 365l. I afterwards multiply $\frac{1}{2}$ by 45, that is, I take the $\frac{1}{2}$ of 45 considered as a Number of Pounds which is 22l. 10s.

365 19
22 10
388l. 9s.

By this Operation I find the Product of 17s. multiplied by 457, in two Parts, the Sum of which is 388l. 9s.

LX.

Method of finding at one Operation the Pounds, Shillings, Pence, and Farthings arising from the Multiplication of a Number of Pence or Farthings into a whole Number.

The Pounds, Shillings, Pence, and Farthings of the Product arising from the Multiplication of the Number of Pence or Farthings of the Multiplicand, which is an aliquot Part of a Shilling, by a whole Number, is obtained at one Operation by the following Rule. Divide the Figure in the Units Place of the Multiplier, considered as a Number of Shillings, by the Number of Times that the Pence or Farthings are contained in a Shilling, and set down the Quotient in the Rank of the Shillings Pence and Farthings, if there be any; then divide the remaining Figures of the Multiplier by double the Number of Times that the Number of Pence or Farthings of the Multiplicand is contained in a Shilling; and the Quotient being removed one Rank towards the right Hand, will express the Pounds, and the Parts of the Pounds if there be any.

LXI.

Let it be proposed to multiply ol. os. 3d. by 457.

First Example.

Having disposed those Numbers as in the Margin, I observe that 3d. being the one Fourth of a Shilling, the one Fourth of the Figure in the Units Place of the Multiplier considered as a Number of Shillings, and the one half of the one Fourth, or the one Eighth of the remaining Figures of the Multiplier, must be taken, that is, the Figure 7 must be divided by 4, and the remaining Figures by 8. But as in Division, the Figures of the highest Degree

ol. os. 3d.
457
5l. 14s. 3d.

must be first divided, in order that the Remainders may be reduced and added to the following Figures; I begin by taking the one Eighth of the Figures of the Multiplier which precede that in the Units Place; removing the Figures of the Quotient one Rank towards the right Hand, to represent the Number of Pounds. I therefore take the one Eighth Part of 45 tens, which is 5 tens: and as this Quotient must be removed one Place towards the right Hand, I set it down in the Rank of the Pounds, the Remainder 5 tens being added to the Following Figure 7 makes 57, the one fourth Part of which is $14\frac{1}{4}$, which I set down in the Rank of the Shillings and Pence, under this Form 14s. 3d. whence there results 5l. 14s. 3d. for the Product required of 3d. multiplied into 457.

LXII.

Let it be proposed to multiply ol. os. 8d. by 457.

Having placed those Numbers as in the Margin, I observe that the 8d. in the Multiplier, being the two Thirds of a Shilling; the two Thirds of the Figure 7 Units of the Multiplier, or the one Third of 14 double of this Figure 7, and the one Third only of the remaining Figures of the Multiplier, must be taken. But as this Operation is a Division, we should begin by the Figures of the highest Degree, and take the one third of them, removing the Quotient one Place towards the Right Hand, I therefore take the one Third of 45 tens, which is 15 tens: and as those 15 tens which consist of 1 hundred and 5 Tens, should be removed one Place towards the right Hand; I set down 1 in the Rank of the tens, an 5 in the Rank of the Units of the Pound. I afterwards take the two thirds of 7 Units, or the one Third of 14 Units; which gives $4\frac{2}{3}$, which I set down in the Rank of the Shillings and Pence, under this Form 4s. 8d. whence 15l. 4s. 8d. will be the Product of 8d. multiplied into 457.

ol.	os.	8d.
457		
15l.	4s.	8d.

Second Example.

LXIII.

Let it be proposed to multiply ol. os. 11d. by 457.

Having set down those Number as in the Margin, and observing that the 11d. of the Multiplier is not an aliquot Part of a Shilling; I divide 11d. into two Parts which are aliquot Parts, or at least commodious Fractions of a Shilling. Those two Parts will be 8d. and 3d. which I multiply separately by 457.

	ol.	os.	11d.
	457		
8d.	15	4	8
3d.	5	14	3
	20l.	18s.	11d.

Third Example.

1^o. Multiplying 8d. by 457, I find as in the second Example 15l. 4s. 8d. for the Product.

2^o. Multiplying 3d. by 457, I find as in the first Example 5l. 14s. 3d.

3^o. Adding together those two Products, there results 20l. 18s. 11d. for the Product of 11 Pence multiplied into 457.

LXIV.

Let it be proposed to multiply ol. os. od. 3qrs. by 457.

Fourth Example.

Having placed those Numbers as in the Margin, and observing that 3qrs. is the $\frac{3}{4}$ of a Shilling, I divide 45 by 32, and place the Quotient 1 in the Rank of the Units of the Pounds, the Remainder 13, which with the Figure 7, makes 137s. I divide by 16, and set down the Quotient 8s. 6d. 3qrs. in the Rank of Shillings, Pence and Farthings.

ol.	os.	od.	3qrs.
457			
1l.	8s.	6d.	3qrs.

LXV.

The foregoing Method applied to an Example, containing Pounds, Shillings, Pence, and Farthings.

When a mixt Number, consisting of Pounds, Shillings, and Pence and Farthings, as 189l. 18s. 11d. 2qrs. is proposed to be multiplied by a whole Number, for Example by 457; first the whole Number 189 must be multiplied into 457, according to the Rules of the Multiplication of simple Numbers; then the 18s. must be multiplied into 457, according to the Rule for multiplying Shillings; and afterwards the 11d. 2qrs. by 457, according to the Rule for multiplying Pence and Farthings.

	189l.	18s.	11d.	2qrs.
	457 $\frac{3}{4}$			
4	1323			
50	945			
400	756			
18s.	411	6s.		
8d.	15	4	8d.	
3d.	5	14	3	
2qrs.		19	0	2qrs.
$\frac{3}{4}$	113	19	4	2
	86920l.	3s.	4d.	0qrs.

If the Multiplier contained also a Fraction, for Example if the Multiplier was 457 $\frac{3}{4}$; after the whole Multiplicand 189l. 18s. 11d. 2qrs. is multiplied by 457, it must be multiplied again by $\frac{3}{4}$, that is 3 Times the one Fifth of the Multiplicand must be taken; and when all the Parts of the Product are found, they must be added together, to obtain the whole Product of the Multiplication.

LXVI.

The Grounds of the foregoing Rule for multiplying mixt Numbers, consisting of Pounds, Shillings, Pence, and Farthings, will sufficiently appear from the following Considerations.

Grounds of the foregoing Method for multiplying Shillings by a whole Number.

1^o. by multiplying the Number of Shillings by the last Figure of the Multiplier, there will manifestly result a Number of Shillings; wherefore the Product arising from this Multiplication should be set down in the Rank of Shillings, when it does not exceed 20s. and when it exceeds 20s. one Pound should be retained for each 20s. and the Remainder set down in the Rank of Shillings.

The Rule next directs to multiply all the Figures of the Multiplicator, by one half of the Number of Shillings, and to remove each Figure of the Product one Place towards the right Hand. Now,

1°. The Product that results by multiplying by one half of the Number of Shillings will be only the one half of the Product arising from the Multiplication by all the Shillings. 2°. by removing each Figure of this Product one Place towards the right Hand, the tenth Part only of this Product is taken, and consequently only the one tenth Part of the one half of the Product resulting from the Multiplication of the Number of Shillings of the Multiplicand by the Multiplicator; but the tenth Part of the one half of this Product, that is, the twentieth Part of this Product is equal to the Number of Pounds it contains. Wherefore by multiplying all the Figures of the Multiplicator, except that in the Units Place, by the one half of the Number of Shillings, and removing each Figure of the Product one Place towards the right Hand; there will result a Quantity equal to the Number of Pounds contained in the Product arising from the Multiplication of the Multiplicator into the Number of Shillings of the Multiplicand.

2°. If the Multiplicand contains but 1s. according to this Rule, the Figure in the Units Place of the Multiplicator considered as representing Shillings, together with the one half of the remaining Figures of the Multiplicator considered as representing Pounds, will be the Product of 1s. multiplied into the Multiplicator; and as for the one half, the one fourth, the one third, the one sixth, the one twelfth, &c. of 1s. we should take but the one half, the one fourth, the one third, the one sixth, the one twelfth, &c. of what should be taken for 1s. it is manifest that for the one half, the one fourth, the one third, the one sixth, the one twelfth &c. of 1s. we should take but the one half, the one fourth, the one third, the one sixth, the one twelfth, &c. of the Figure in the Units Place of the Multiplicator, considered as representing Shillings, together with the one half of the one half, or of the one fourth, or of the one third, or of the one sixth, or of the one twelfth, &c. of the remaining Figures in the Multiplicator, considered as representing Pounds, removing each Figure of the Product one Place towards the right Hand; that is, we should divide the Figure in the Units Place of the Multiplicator by the Number of times that the Pence or Farthings of the Multiplicand are contained in a Shilling, and divide the remaining Figures of this Multiplicator considered as representing Pounds by double the Number of Times that the Pence of the Multiplicand is contained in a Shilling, removing each Figure of the Product one Place towards the right Hand; which is precisely what the Rule prescribes.

Grounds of
the foregoing
Method
for Multiplying
Pence
or Farthings
by a whole
Number.

LXVII.

Let it be proposed to multiply 24lb. 7oz. 6dwt. by 51

Multiplication of
Weights explained by
an Example.

Having disposed those Numbers as in the Margin. 1^o. I multiply 24lb. by 51, and set down the two particular Products, viz. 24 and 120, as in the Margin.

2^o. to multiply 7oz. by 51 I divide the 7oz. into the 3 aliquot Parts of a Pound, 4oz. 2oz. 1oz. and as the Product of 1lb. multiplied by 51 is a Number of Pounds equal to the Multiplier 51, the Product of 4oz. which is but the one

third of a Pound, will be only one third of the Multiplier 51. I therefore for 4oz. take the one third of 51lb. which is 17lb. For 2oz. which is the one sixth of a Pound, or the one half of 4oz. I take the one sixth of the Multiplier considered as 51 Pounds, or the one half of the Product 17lb. found for 4 Ounces which is 8lb. 6oz. for 1 Ounce which is the one twelfth of a Pound, or the one fourth of 4 Ounces, or the one half of 2 Ounces, I take the one twelfth of the Multiplier considered as 51 Pounds, or the one fourth of the Product 17lb. found for 4 Ounces, or the one half of the Product 8lb. 6oz. found for 2 Ounces, which is 4lb. 3oz.

3^o. having found the Product of 1 Ounce multiplied by 51, I divide the 6dwt. which remains to be multiplied, into the aliquot Parts of an Ounce, and those aliquot Parts will be 4dwt. and 2dwt.

For 4dwt. which is the one fifth of an Ounce, I take the one fifth of the Product 4lb. 3oz. found for 1 Ounce; which is 10oz. 4dwt.

For 2dwt. I take the one half of the foregoing Product 10oz. 4dwt. found for 4dwt. which is 5oz. 2 dwt.

Having thus found all the Parts of the Product, I add them together; and there results 1255lb. 0oz. 6dwt. for the Product required.

LXVIII.

Multiplication of Lines
and Surfaces.

Hitherto we have treated of Arithmetical Multiplication, we shall now proceed to treat of Geometrical Multiplication, so called because it relates to Extension, which is the Object of Geometry.

As every Quantity is measured by some other Quantity of the same Kind, the Computists are under the Necessity of employing three different Species of Measuring Units, for measuring the three different Species of Extension: Lineal measuring Units for measuring Distances and Lines; Superficial measuring Units for measuring Surfaces; and Solid measuring Units for measuring Solids.

	24lb.	7oz.	6dwt.
	51		
1	24lb.		
50	120		
4oz.	17		
2oz.	8	6oz.	
1oz.	4	3	
4dwt.		10	4dwt.
2dwt.		5	2
	1255lb.	0oz.	6dwt.

LXIX.

The Number which shews how often the measuring Unit is contained in the Extention measured, is called the Content of the Extention so measured. Let the Extention to be measured be a Rectangle $ABCD$, whose Base BC and Altitude AB have been measured, for Example, with a lineal Yard, simply called a *Yard*; let BG, GI, IL, LN, NP, PC , express the Yards contained in the Base BC , and let AQ, QR, RS, ST, TB , represent the Yards contained in the Breadth or Altitude AB . If through the Points G, I, L, N, P of the Base, be drawn GF, IH, LK, NM, PO , parallel to the Altitude AB ; the Rectangle $ABCD$ will be divided into as many Rectangles $ABGF, FGIH, HILK, KLN M, MNPO, OPCD$, as there are Yards in the Base BC .

Each of those Rectangles having one Yard in Breadth, and as many Yards in Length as there are Yards in the Altitude AB of the Rectangle, will manifestly contain as many square Yards as there are lineal Yards in the Altitude AB . Whence to obtain the Number of square Yards contained in the Rectangle $ABCD$, the Number of Rectangles $ABGF, FGIH, HILK$, &c. which stand on the Base BC , or the Number of Lineal Yards BG, GI , &c. contained in the Base BC , must be repeated as often as there are square Yards in each of those Rectangles, that is, as often as there are lineal Yards in the Altitude AB . But to repeat the Number of lineal Yards that are in the Base BC of the Rectangle, as often as there are lineal Yards in the Altitude AB of this Rectangle, is to multiply the Number of lineal Yards in the Base BC , by the Number of lineal Yards in the Altitude AB . Whence the Number of square Yards contained in a Rectangle is found, by multiplying the Number of lineal Yards in its Base by the Number of lineal Yards in its Altitude.

The Area of a Rectangle is found by multiplying the lineal Measures in the Base by those in the Altitude.

For Example if the Length of the Base BC of the Rectangle $ABCD$ be six Yards, and its Altitude AB 5 Yards, the Rectangle $ABCD$ will contain 6 Rectangles $ABGF, FGIH, HILK, KLN M, MNPO, OPCD$, each of which are 5 Yards long, and 1 Yard broad, and consequently contains 5 square Yards. Whence the Rectangle will contain 6 Times 5 or 30 square Yards.

Example.

LXX.

It is manifest that if the Base BC and Altitude AB of the Rectangle was measured in Feet, it would contain as many Rectangles 1 Foot broad as there are Feet in the Base AB , and that each Rectangle would contain as many square Feet as there are lineal Feet in the Altitude AB of the Rectangle $ABCD$. Whence the Number of square Feet contained in the Rectangle $ABCD$ will be found, by multiplying the Num-

Another Example.

ber of lineal Feet contained in the Base BC , by the Number of Feet contained in the Altitude AB .

The square Yard, for Example, having 3 Feet in Length and 3 Feet in Breadth, its Superficies will contain 3 Times 3 or 9 square Feet.

If the Square $ABCD$ represented a square Foot, its Base BC would contain 12 lineal Inches, and its Altitude AB 12 lineal Inches. Whence its Superficies would contain 12 Times 12 or 144 square Inches. In like Manner it will appear that a square Inch whose Base and Altitude are divided each into 12 Lines, contains 12 Times 12 or 144 square lines: and so of any other Measure that is employed.

LXXI.

It often happens that the same Measures are not employed for measuring the Base and the Altitude of the Rectangle. In this Case, the superficial Measures contained in the Superficies of the Rectangle, are not square Measures, but Measures whose Length is the Measure employed for measuring the Base, and whose Breadth is the Measure employed for measuring the Altitude.

The contiguous Sides of the superficial Measures contained in any Figure are the lineal Measures of its Dimensions.

Example.

For Example, if it was proposed to determine the Number of Bricks lying flat which are contained in a Rectangle: Since a Brick is 8 Inches long and 4 broad; the Length BC , should be measured with a Measure of 8 Inches, and the Altitude AB with a Measure of 4 Inches, and the Product arising from the Multiplication of the Number of Measures of 8 Inches contained in the Base BC , into the Number of Measures of 4 Inches contained in the Altitude AB , will be the Number of Bricks, or of the superficial Measures 8 Inches long and 4 broad, contained in Area of the Rectangle $ABCD$.

LXXII.

When the Length and Breadth of the Rectangle is measured by the Yard, and is not contained in them a certain Number of Times without a Remainder; the Computists measure this Remainder in Parts of a Yard, viz. in Feet, Inches, Lines, &c. and as the Products of those Measures do not always produce square Yards without a Remainder, they estimate this Remainder in Parts of a square Yard.

The superficial Measures are divided and subdivided into Parts analogous to those of the lineal Measures.

Example.

Though the most regular Parts of a square Yard, are square Feet, square Inches, and square Lines; those however are not the Parts that the Computists commonly employ, they find it more convenient to divide the square Yard into Parts analogous to those of the lineal Yard: and as the lineal Yard is divided into 3 lineal Feet, and the lineal Foot into 12 lineal Inches, and the lineal Inch into 12 lineal Lines, so they divide the square Yard into 3 Rectangles, each of which is 1 Foot broad and 1 Yard long, expressed thus, *Foot-Yard*, or *Yard-Foot*; on account of their two Dimensions; they divide the Rectangle *Yard-Foot* into twelve equal Parts each of which is 1 Yard long and 1 Inch broad,

expressed thus *Yard-Inch*; in fine they divide each of those new Rectangles into 12 equal Parts each 1 Yard long and 1 Line broad, expressed thus *Yard-Line* on Account of their two Dimensions. And so on of the other Measures employed.

LXXIII.

When the Sides of a Rectangle are measured by the Foot, and is not contained in them a certain Number of Times without a Remainder; Another Example. the Computists measure this Remainder in Inches and Lines; and as the Product of the two dimensions measured will not always produce an exact Number of square Feet without a Remainder, they have recourse to the Parts of a square Foot to measure this Remainder.

The most regular Parts of the square Foot with Respect to the Division of the Foot into Inches and Lines, are the square Inch, and square Line. But as the square Foot contains 144 square Inches; and the square Inch 144 square Lines; and that the lineal Foot is only divided into 12 Inches, and the lineal Inch into 12 Lines: The Computists chuse rather to divide the square Foot into 12 equal Parts, each of which is 1 Foot long and 1 Inch broad, expressed thus *Foot-Inch* on account of their two Dimensions. They divide also the *Foot-Inch* as the lineal Inch, into 12 equal Parts, each of which is 1 Foot long and 1 Line broad expressed thus *Foot-Line*.

LXXIV.

From what precedes it is easy to conclude, that two Numbers of equal lineal Measures, multiplied into one another, produce a Number of square superficial Measures, whose Sides are the lineal Measures contained in the Multiplicand and Multiplier. For Example, a Number of lineal Yards multiplied into a Number of lineal Yards produces a Number of square Yards; a Number of lineal Feet multiplied into a Number of lineal Feet produces a number of square Feet: And so on.

It is also manifest that a Number of equal lineal Measures, multiplied into a Number of other equal lineal Measures, different from the first, produces a Number of superficial Measures whose contiguous Sides are the two Sorts of Measures of the Multiplicand and Multiplier. For Example, a Number of lineal Yards multiplied into a Number of lineal Feet will produce a Number of superficial Measures each of which is 1 Yard long and 1 Foot broad; a Number of lineal Yards multiplied into a Number of lineal Inches or Lines, will produce a Number of superficial Measures each of which will be 1 Yard long and 1 Inch or 1 Line broad; And so on. The geometrical Multiplication therefore of a Line by a Line will be reduced to common Multiplication, by giving to the Units or Measures of the Multiplicand the two Dimensions which affect those of the Multiplicand and Multiplier, and considering the proposed Multiplier as an abstract Number.

The geometrical Multiplication of a Line by a Line is reduced to arithmetical Multiplication, by ascribing to the Multiplier and the two Dimensions affecting the Factors.

The Value of the different Units relative to the square Yard, with the Characters whereby they are distinguished from each other, are as follow.

<i>YY</i> Square Yard	<i>1YY</i> 9 <i>ff</i>	<i>Y''</i> Yard-Second	<i>1Y''</i> 12 <i>Y'''</i> or 3 <i>LL</i>
<i>ff</i> Square Foot	<i>1ff</i> 144 <i>II</i>	<i>Y'''</i> Yard-Tierce	<i>1Y'''</i> $\frac{1}{4}$ <i>LL</i>
<i>II</i> Square Inch	<i>1II</i> 144 <i>LL</i>		
<i>LL</i> Square Line		<i>fl</i> Foot-Inch	<i>1ff</i> 12 <i>fI</i>
	<i>1YY</i> 3 <i>Yf</i>	<i>fL</i> Foot-Line	<i>1fI</i> 12 <i>fL</i> or 12 <i>II</i>
<i>Yf</i> Yard-Foot	<i>1Yf</i> 12 <i>YI</i> or 3 <i>ff</i>	<i>f'</i> Foot-Prime	<i>1fL</i> 12 <i>f'</i> or 1 <i>II</i>
<i>YI</i> Yard-Inch	<i>1YI</i> 12 <i>YL</i> or $\frac{1}{4}$ <i>ff</i>	<i>f''</i> Foot-Second	<i>1f'</i> 12 <i>f''</i> or 12 <i>LL</i>
<i>YL</i> Yard-Line	<i>1YL</i> 12 <i>Y'</i> or 3 <i>II</i>		<i>1f''</i> 1 <i>LL</i>
<i>Y'</i> Yard-Prime	<i>1Y'</i> 12 <i>Y''</i> or $\frac{1}{4}$ <i>II</i>	<i>IL</i> Inch-Line	<i>1II</i> 12 <i>IL</i>
		<i>I'</i> Inch-Prime	<i>1IL</i> 12 <i>I'</i> or 12 <i>LL</i>
			<i>1I'</i> 1 <i>LL</i>

Method of finding the Area of a Rectangle whose Dimensions are expressed in Yards and Parts of a Yard explained by an Example.

Let it be proposed to find the Area of a Rectangle, whose Length is 58*Y.* 2*f* 8*I.* and Breadth 8*Y.* 1*f* 6*I.*

Though either of the Factors may indiscriminately be taken for Multiplier, yet that which contains the least Number of Yards is commonly assumed for Multiplier.

When the Number of Yards of the Multiplier is expressed by one Figure, as in this Example; the Computists first multiply each Part of the Multiplicand, beginning by the Parts of the lowest Denomination, by the Number of Yards of the Multiplier, afterwards taking for the Product arising from the Multiplication of the Multiplicand into the other Parts of the Multiplier, the same Parts of the Multiplicand that the Parts of the Multiplier are of a Yard. According to those Rules.

1°. I multiply 8 *Inches* by 8 *Yards*, which produces 64 *Yard-Inch*. As this Number of *Yard-Inch* contains 5 *Yard-Feet* and 4 *Yard-Inch* over, I therefore set down 4 *Yard-Inch* in the Column of Inches and retain 5 *Yf.* to add them to the *Yard-Foot* of the Product.

I next multiply 2*f.* by 8*Y.* which produces 16*Yf.* to which adding the 5*Yf.* I retained, there results 21*Yf.* as this Number of *Yard-Foot* contains 7*YY.* I set down 0 in the Column of Feet, and retain 7*YY.* to add them to the *Square Yards* of the Product.

Lastly multiplying 58*Y.* by 8*Y.* and adding to the Product the 7*YY.*

58 <i>Y.</i>	2 <i>f.</i>	8 <i>I.</i>	
8 <i>Y.</i>	1 <i>f.</i>	6 <i>I.</i>	
471 <i>YY.</i>	0 <i>Yf.</i>	4 <i>YI.</i>	
19	1	10	8 <i>YL.</i>
9	2	5	4
500 <i>YY.</i>	1 <i>Yf.</i>	8 <i>YI.</i>	0 <i>YL.</i>

I retained; there results $471\text{Yr. } 0\text{Yf. } 4\text{Yl.}$ for the Product of $58\text{Y. } 2\text{f. } 8\text{I.}$ multiplied by 8Y.

2°. To multiply the same Multiplicand $58\text{Y. } 2\text{f. } 8\text{I.}$ by the Part 1f. of the Multiplier; I observe that if this Multiplicand was multiplied by 1 Yard, it would produce $58\text{Yr. } 2\text{Yf. } 8\text{Yl.}$ that is, the Product would be equal to the Multiplicand, with this difference only that each Part would acquire a Dimension of 1 Yard: Whence multiplying by 1 Foot which is only the one third of a Yard, the Product will be the one third of the Multiplicand, each Part having a second Dimension of 1 Yard. Now this one third will be $19\text{Yr. } 1\text{Yf. } 10\text{Yl. } 8\text{Yl.}$ because the one third of 58Yr. is 19Yr. and 1Yr. over which is equal to 3Yf. which added to 2Yf. makes 5Yf. the one third of which is 1Yf. with 2Yf. or 24Yl. over, which added to 8Yl. makes 32Yl. the one third of which is 10Yl. and 2Yl. or 24Yl. over, the one third of which is 8Yl.

3°. To multiply the Multiplicand by the 6 Inches in the Multiplier, I observe that 6 Inches is the one Half of 1 Foot, and consequently should give the one Half of the Product $19\text{Yr. } 1\text{Yf. } 10\text{Yl. } 8\text{Yl.}$ which is $9\text{Yr. } 2\text{Yf. } 5\text{Yl. } 4\text{Yl.}$ because the one half of 19Yr. is 9Yr. with a Remainder of 1Yr. or 3Yf. which added to 1Yf. makes 4Yf. the one Half of which is 2Yf. and the one half of $10\text{Yl. } 8\text{Yl.}$ is $5\text{Yl. } 4\text{Yl.}$

Adding together those three particular Products there results $500\text{Yr. } 1\text{Yf. } 8\text{Yl.}$ for the Product required.

LXXVII.

Let it be proposed to find the Area of a Rectangle $68\text{Y. } 1\text{f.}$ long and $57\text{Y. } 2\text{f. } 8\text{I.}$ broad.

The Part 68 of the Multiplier by which the Multiplicand is first to be multiplied, consisting of several Figures; all the Parts of the Multiplicand cannot be multiplied after a direct Manner by this first Part of the Multiplier, as in the foregoing Example; but after the Part 57Y. of the Multiplicand is multiplied by 68Y. to obtain the Product of the remaining Parts of the Multiplicand multiplied into 68, the same Parts of 68 must be taken that the Parts 2 Feet 8 Inches of the Multiplicand are of a Yard; and the Sum of the several particular Products that result will be the Product of $57\text{Y. } 2\text{f. } 8\text{I.}$ into 68 Yards. Afterwards the same Multiplicand $57\text{Y. } 2\text{f. } 8\text{I.}$ must be multiplied by 2 Feet, after the Manner explained in the foregoing Example.

1°. Multiplying the Part 57Y. of the Multiplicand by 68Y. I find the two particular Products 456Yr. and 3420Yr.

57Y.	2f.	8I.
68Y.	1f.	
456Yr.		
342		
45	1Yf.	
15	0Yf.	4Yl.
19	0	$10\text{Yl. } 8\text{Yl.}$
3955	2Yf.	$2\text{Yl. } 8\text{Yl.}$

Application of the foregoing Method to another Example.

ELEMENTS OF

2°. to multiply 2 Feet by 68 Yards, I observe that 1 Yard multiplied into 68Y. will give 68 square Yards, that is, a Number of square Yards equal to the Multiplier. Whence 2 Feet which are only the two Thirds of a Yard, should only give the two Thirds of 68 square Yards: and each third Part being 22YY. 2Yf. I set down twice 22YY. 2Yf. or 45YY. 1Yf.

3°. to multiply 8 Inches by 68 Yards, I observe that 8 Inches are the one third of 2 Feet, which produced 45YY. 1Yf. Whence for 8 Inches I take the one third of 45YY. 1Yf. which is 15YY. 0Yf. 4Yl.

It remains to multiply the same Multiplier by 1 Foot, for which I take the 1 third of 57YY. 2Yf. 8Yl. which is 19YY. 0Yf. 10Yl. 8Yl.

Adding together all those particular Products, there results 3955YY. 2Yf. 2Yl. 8Yl. for the Product required.

LXXVIII.

When the Area of a Superficies is found in square Yards, and in Parts of the square Yard divided into 3 and subdivided continually into 12 equal Parts, as in the foregoing Examples. Those Parts may be reduced into square Measures, such as square Feet, square Inches, square Lines, after the following Manner.

Method of
reducing
the Parts of
superficial
Measures
analogous
to those of
the lineal
Measures;
into square
Measures.

1°. The Number of Yf. being multiplied by 3, will produce square Feet.

2°. Each Yard-Inch being 36 Inches long and 1 Inch broad, contains 36 square Inches, or the one fourth of a square Foot. Whence the one fourth of the Number of Yard-Inch, will be square Feet, and every 1Yf. that remains will be 36 square Inches.

3°. Each Yard-Line being the twelfth Part of a Yard-Inch, and the Yard-Inch being equal to 36 square Inches, consequently the Yard-Line is equal to 3 square Inches: Whence the Number of Yard-Line being multiplied by 3 will produce square Inches.

4°. each Unit of the Measures affected by the Mark Y' being equal to the twelfth Part of a Yard-Line, which is equal to 3 square Inches, it will be the one fourth of a square Inch, or 36 square Lines; whence the one fourth of the Number of Y' will be square Inches, and every 1Y' that remains will be equal to 36 square Lines.

5°. Each Y'' is equal to 3LL, because 1Y'' is the one twelfth of 1Y', which is equal to 36 square Lines. Whence the Number of Y'' multiplied by 3 will produce square Lines.

6°. It will appear in like Manner, that each Y''' is equal to $\frac{1}{4}$ LL: Wherefore the one fourth of the Number of Y''' will be square Lines: It is the same with respect of the other Parts continually 12 Times less which may be reduced into square Primes, which are the one hundred and forty fourth Parts of a square Line.

When the Parts of the lowest Denomination in the two Factors of the Multiplication are Lines, and the principal Parts are Yards, the

Parts of the lowest Denomination in the Product will be Y''' , 4 of which make a square Line: because when the two Factors of the Multiplication are reduced into Lines, the Product can only consist of square Lines.

LXXIX.

Let it be proposed to reduce into square Feet, square Inches, and square Lines, the Measures that are less than the square Yard in this Product.
120851YY 2Yf 5YI 9YL 0Y' 4Y'' 8Y'''

The first Part being composed of square Yards, which are the principal Measures, are not to be altered. To reduce the other

120851YY	2Yf	5YI	9YL	0Y'	4Y''	8Y'''
	3	$\frac{1}{4}$	3	$\frac{1}{4}$	3	$\frac{1}{4}$
	6ff	36II	12LL			
	1	27	2			
120851YY	7ff	63II	14LL			

The foregoing Method applied to an Example.

Parts, it suffices to set down 3 under the Yf, $\frac{1}{4}$ under the YI, 3 under the YL, $\frac{1}{4}$ under the Y', 3 under the Y'', and $\frac{1}{4}$ under the Y''', and multiply each Part of the Product by the Number wrote under it, viz. the first, third and fifth Parts after the square Yards, by 3; and the second, fourth, and sixth Parts after the square Yards, by $\frac{1}{4}$; observing that the first and second Parts after the Yards that are multiplied by 3 and $\frac{1}{4}$, produce square Feet; that the third and fourth, multiplied by 3 and by $\frac{1}{4}$, produce square Inches; and that the fifth and sixth Parts, multiplied by 3 and by $\frac{1}{4}$, produce square Lines.

Whence I say: 3 Times 2Yf is 6ff, which I set down under the Line. Then I say, the $\frac{1}{4}$ of 5YI is 1ff and 36II: I therefore set down 1ff under 6ff, and 36II in the following Column.

I then multiply 9YL by 3, which produces 27II which I set down in the Column of square Inches under the 36II; and multiplying 0Y' by $\frac{1}{4}$, which produces nothing more for the square Inches.

Lastly I multiply 4Y'' by 3, which produces 12LL; and 8Y''' by $\frac{1}{4}$ which produces 2LL.

Adding together the new square Measures of the same Species, I find that the proposed Product 120851YY 2Yf 5YI 9YL 0Y' 4Y'' 8Y''' is reduced to square Measures 120851YY 7ff 63II 14LL.

LXXX.

When a Parallelogram ADCB is a Rhomboid, if from the Extremities A and B of one its Sides, there be drawn two Perpendiculars AE, BF to the opposite Side the rectangular Parallelogram AEFB will be equal to the Rhomboid ADCB. (Euc. Prop. xxxv. B. 1) And as the Number of the square Measures contained in the Superfices of the rectan-

The Content of any Parallelo-gram is

found by multiplying the Base into Half the Altitude. gular Parallelogram AEFB, is found by multiplying the Number of Measures in the Base EF by the Number of Measures in the Side AE; also the Number of square Measures contained in the Rhomboid ADCB will be found, by multiplying the Number of Measures in EF or in AB or in DC, by the Number of Measures contained in the Perpendicular AE.

LXXXI.

Fig. 5.

The Content of any Triangle found by multiplying the Base into Half the Altitude.

Since a Triangle DAC is the one half of a Parallelogram ADCB of the same Base DC and Altitude AE (*Euc. Prop. XXI. B. 1*); and that the Number of square Measures contained in the Parallelogram ADCB is found by multiplying the Number of Measures in the Base DC by the Number of Measures in its Altitude AE; it is manifest that the Number of square Measures contained in the Triangle DAC will be found, by drawing a Perpendicular AE from its Vertex to its Base DC produced if necessary, and multiplying the Number of Measures in this Base DC by one half of the Number of Measures contained in the Altitude AE.

LXXXII.

Fig. 6.

Another Method of finding the Area of a Triangle.

The Area of any Triangle ABC may be also found, thus: add together the three Sides of the Triangle, take the one Half of the Sum; from this Sum subtract successively the three Sides of the Triangle; multiply the three Remainders into one another: multiply the Product again by the one Half of the Sum of the three Sides, and the Square Root of the Product will be the Content of the Triangle.

Let, for Example, the Length of the three Sides AB, BC, AC of the Triangle ABC, be 4 Yards, 15 Yards and 13 Yards. I take the one Half of the Sum of those three Sides which is 16 Yards, I then subtract successively from this Half Sum, each of the three Sides, that is, I subtract first 4 Yards from 16 Yards, and there remains 12 Yards, then 15 Yards from 16Y, and there remains 1 Yard, and lastly 13Y from 16Y and there remains 3 Yards. I afterwards multiply those four Numbers 16, 12, 1, 3 into one another, and extracting the Square Root of the Product 576, I find 24 for the Number of square Yards contained in the Triangle ABC.

LXXXIII.

To demonstrate this Rule, from the Angle A of the Triangle ABC let a Perpendicular AE be drawn to the opposite Side BC; then the foregoing Method, $AC^2 + 2BC \times BE = AB^2 + BC^2$ (*Euc. Prop. XII. B. II.*) Consequently

$$BE = \frac{AB^2 + BC^2 - AC^2}{2BC}. \text{ Squaring each Member of this Equation, there results } BE^2 = \frac{(AB^2 + BC^2 - AC^2)^2}{(2BC)^2}, \text{ consequently } AB^2 - BE^2 \text{ or}$$

$$AE^2 = AB^2 - \frac{(AB^2 + BC^2 - AC^2)^2}{(2BC)^2}; \text{ multiplying each Member by } (2BC)^2,$$

there results $AE^2 \times (2BC)^2 = AB^2 \times (2BC)^2 - (AB^2 + BC^2 - AC^2)^2$. Extracting the Square Root of each Member, and dividing them by 4, there results $\frac{AE \times BC}{2} = \frac{1}{4} \sqrt{[AB^2 \times (2BC)^2 - (AB^2 + BC^2 - AC^2)^2]}$ = Area

of Triangle ABC, but $AB^2 \times (2BC)^2 - (AB^2 + BC^2 - AC^2)^2$ is equal to $[AB \times 2BC + AB^2 + BC^2 - AC^2 \times AB \times 2BC - AB^2 - BC^2 + AC^2]$

Whence the Superficies or the Area of the Triangle ABC is equal to $\frac{1}{4} \sqrt{[(AB \times 2BC + AB^2 + BC^2 - AC^2) \times (AB \times 2BC - AB^2 - BC^2 + AC^2)]}$

but $AB \times 2BC + AB^2 + BC^2 - AC^2$ or $AB^2 + BC^2 + 2AB \times BC - AC^2$ is the Product of $AB + BC + AC$ multiplied into $AB + BC - AC$. And $AB \times 2BC - AB^2 - BC^2 + AC^2$, or $AC^2 - AB^2 - BC^2 + 2AB \times BC$ is the Product of $AC + AB - BC$ multiplied into $AC - AB + BC$, Whence the Superficies or the Area of the Triangle ABC is equal to $\frac{1}{4} \sqrt{[AB + BC + AC] \times [AB + BC - AC] \times [AC + AB - BC] \times [AC + BC - AB]}$

equal $\sqrt{[\frac{AB + BC + AC}{2} \times \frac{AB + BC - AC}{2} \times \frac{AC + AB - BC}{2} \times \frac{AC + BC - AB}{2}]}$

But $\frac{AB + BC - AC}{2} = \frac{AB + BC + AC}{2} - AC$, and $\frac{AC + AB - BC}{2} = \frac{AB + BC + AC}{2} - BC$, and $\frac{AC + BC - AB}{2} = \frac{AB + BC + AC}{2} - AB$.

Whence the Area of the Triangle ABC may be expressed thus $\sqrt{[\frac{AB + BC + AC}{2} \times (\frac{AB + BC + AC}{2} - AC) \times (\frac{AB + BC + AC}{2} - BC) \times (\frac{AB + BC + AC}{2} - AB)]}$

LXXXIV.

The Content of any quadrilateral Figure ABCD, having two parallel Sides AB, CD, is found by multiplying the Sum of those Sides by Half Fig. 7. their perpendicular Distance CE. For if the Diagonal AC be drawn, the Triangle ACB will be equal to $\frac{1}{2} CE \times AB$, and the Content of the Triangle DAC will be equal to $\frac{1}{2} CE \times CD$, and therefore the Content of the whole Quadrilateral ABCD will be equal to

$$\frac{1}{2} CE \times AB + \frac{1}{2} CE \times CD = \frac{1}{2} CE \times (AB + CD).$$

LXXXV.

From the Manner of finding the Area of a Triangle, the Area of any right-lined plane Figure, as ABCDE, may be determined, by dividing the whole into Triangles and finding the Content of each Triangle. Figure 8. Thus let the dividing Lines AC and AD, be 20 and 16 Inches, and the Perpendiculars BF, DG, EH falling thereon, 8, 12 and 16 respectively; then the Content of the Triangle ABC being 80, that of ACD 120,

Rules for finding the Content of a quadrilateral Figure whose opposite Sides are parallel,

Method of
finding the
Area of any
right lined
Figure by
dividing it
into
Triangles.

and that of ADE 80, it is evident that the Content of the whole Figure will be the Sum of all those, or 280 square Inches. But, when the given Lines are expressed by Fractions or very large Numbers, the Work will be some what shortened by finding the Content of every two Triangles, having the same Base, at one Operation; that is, by first adding the two Perpendiculars together, and then multiplying half their Sum by the common Base of the two Triangles. Thus, in the last Example, the half Sum of the two Perpendiculars BF and DG being 10, if this Number be therefore multiplied by 20, the Measure of the common Base AC, the Product which is 200 will be the Content of the Trapezium ABCDA, to which 80 the Content of the Triangle ADE being added, the Sum will be 280, the same as before.

LXXXVI.

Fig. 9.

Rule for
finding the
Area of any
regular Po-
lygon.

The Content of a regular Polygon ABCDEF, that is, one whose Sides and Angles are all equal, is found by multiplying half the Sum of its Sides by the Length of the Line KG drawn from the Middle of any Side AB to the Center of the Polygon.

Grounds of
the forego-
ing Rule for
finding the
Area of any
regular Poly-
gon.

For, from the Center K let there be drawn Rays to all the Angles of the Polygon, which will divide it into as many equal Triangles as it has Sides, and which will have all the same Altitude. (*Euc. Prop. xiv. B. III.*) Now if upon a same straight Line AH, the Bases AB, BC, CD, &c. of all those Triangles be extended, and upon AH be constructed the Triangle AKH having for Altitude KG, this Triangle AKH will be equal to the Sum of the Triangles which compose the regular Polygon ABCDEF, consequently the Superficies of any regular Polygon is equal to a Triangle AKH whose Base is equal to the Contour of the Polygon, and whose Altitude is the Line drawn from the Middle of any Side to the Center of the Polygon, but the Superficies of a Triangle AKH is equal to one Half of the Product of its Base AH into its Altitude KG, wherefore the Superficies of any regular Polygon ABCDEF is equal to the one Half of the Product arising from the Multiplication of its Contour into the Line drawn from the Middle of any Side to the Center of the Polygon.

LXXXVII.

The Area
of a Circle
found by
multiplying
its Periphery
by Half its
Radius.

By increasing the Number of Sides of a regular Polygon inscribed in a Circle, its Contour approaches continually nearer and nearer to the Periphery of the Circle, which is its Limit, and the Line drawn from the Middle of any Side to the Center of the Polygon, approaches nearer and nearer to the Ray, which is its Limit, wherefore since the Superficies of a regular Polygon, what ever the Number of Sides may be, is equal to a Triangle whose Base is the Contour of this Polygon, and Altitude the Line drawn from the Middle of any Side to the Center of the Polygon, it is evident that the Superficies of a Circle is equal to that of a Triangle

whose Base is the Circumference of the Circle, and Altitude the Ray of the same Circle.

If therefore a straight Line could be found exactly equal to the Circumference of a Circle whose Radius is given, a rectilineal Figure might be made equal to the Area of this Circle, and what is called the *Quadrature of the Circle* would be found. But though this Problem has not been solved, and is thought never will, yet several Ways have been invented by which the Value of the Circumference may be obtained to any assigned Degree of Exactness, whereof the following is one of the most simple.

LXXXVIII.

If from the Extremity A of any Arch AB, there be drawn a Ray CA, and an indefinit Tangent AD; and from the other Extremity B of the same Arch there be drawn a Line BE perpendicular to the Ray CA, and through the Points B and C, the Line BC which produced will meet the Tangent in D: BE is called the Sine of the Arch AB, AD its Tangent, CD its Secant, AE its Versè Sine.

Fig. 10

The Ray CF being supposed perpendicular to the Ray CA, that is, AF being a Quadrant of a Circle, and consequently the Arch BF being the Complement of the Arch BA, if from the Extremity B of the Arch BF the Line BG be drawn perpendicular to the Ray CF, and from the other Extremity F of the same Arch be drawn a Tangent FI meeting the Secant CI in I, BG or itsequal CE Sine of the Arch BF, is called the Co-sine of the Arch AB, FI its Co-tangent, CI its Co-secant, and GF its Versè co-sine.

Method of finding the Ratio of the Diameter to the Circumference.

These Notions being premised, let the Radius of the Circle be divided into ten Billions of equal Parts, or let $CA = 10000000000$, and let the Arch ALB be 30° , Consequently its Sine BE, which is equal to Half the Radius, will be 5000000000. Whence is deduced the Sine AK of

AL, half of the Arch ALB, that is, of 15° . for $EC = \sqrt{BC^2 - BE^2} = 8660254037$, 84438, and $CA - CE = AE = 1339745962$, 15562, and

Fig. 11.

$AB^2 = BE^2 + AE^2 = 26794919243112264579, 77747602$ consequently AB^2 or $AK^2 = 6698729810778066144$, 94436901, and of Course AK or $S. 15^\circ = 2588190451$, 0252.

In like Manner the Sine of $7^\circ 30'$ will be found to be equal to 1305261922, 2005, the Sine of $3^\circ 45'$ to be 654031292, 3014, the Sine of $1^\circ 52' 30''$ to be 327190828, 2177, the Sine of $56' 15''$ to be equal to 163617316, 2649, the Sine of $28' 7'' 30'''$ to be 81811396, 0394, the Sine of $14' 3'' 45'''$ to be 40906040, 2624, the Sine of $7' 1'' 52''' 30''''$

ELEMENTS OF

to be 10453062, 911641, the Sine of $3' 30'' 56''' 15''''$ to be equal to 10226536, 8034, the Sine of $1' 45'' 28''' 7''''$ to be 5113269, 0701, the Sine of $52'' 44''' 3'''' 45''''$ to be 2556634, 6186, the Sine of $26'' 22''' 1'''' 52'''' 30''''$ to be 1278317, 3198, the Sine of $13'' 11''' 0'''' 56'''' 15''''$ to be 639158, 66118383; the Sine of $6'' 35''' 30'''' 28'''' 7'''' 30''''$ to be 319579, 3307512: the Sine of $3'' 17''' 45'''' 14'' 3'' 45''''$ to be 159789, 665396; and consequently the Sine of $1'' 38''' 52'''' 37'' 1'' 52'''' 30''''$ will be found to be 79894, 8327005.

Now $1'' 38''' 52'''' 37'' 1'' 52'''' 30''''$ being represented by the Arch AL, half of the Arch AB, or of $3'' 17''' 45'''' 14'' 3'' 45''''$, if the Chord BF be drawn, which an account of the Angle FAB being a right Angle, will be perpendicular to AB, and consequently parallel to the Secant CH the two Triangles FEB, CAH will be therefore equiangular, and of Course $FE : BE = CA : AH$ (*Euc. Prop. IV. B. VI.*) but $FE = EC + CF = R + Cos. AB = 10000000000 + 9999999998, 723363 = 19999999998, 723363$, $BE = Sin. AB = 159789, 665396$ and $CB = 10000000000$.

Consequently AH or T. AL or T. $1'' 38''' 52'''' 37'' 1'' 52'''' 30'''' = \frac{159789.665396 \times 10000000000}{19999999998, 723363} = 79894, 832703$.

From whence it follows 1°. that since the Sin. $1'' 38''' 52'''' 37'' 1'' 52'''' 30''''$, viz 79894, 8327005 is less than its Arch, and this Arch being found by dividing the Arch 30° by 65536 or 16 Times successively by 2, and being consequently contained 65536 Times in 30° , or 6 Times 65536, that is, 393216 Times in the Semi-Circumference, it follows I say, that $79894, 8327005 \times 393216$, or 31415926535, 159808 will be less than the Semi-Circumference, whose Radius is expressed by 10000000000. 2°. Since the T. $1'' 38''' 52'''' 37'' 1'' 52'''' 30''''$, viz. 79894, 832703 is greater than its Arch, $79894, 832703 \times 393216$, or 31415926536, 339436 is greater than the Semi-circumference whose Radius is expressed by 10000000000, whence the Periphery of a Circle whose Radius is expressed by 1 is equal to 6, 283185308 very nearly.

LXXXIX.

From the Periphery thus found, the Area of the Circle will also be known, being equal to the Rectangle of the whole Periphery and half the Radius, that is, equal to 6, 283185308 $\times$ 0, 5 or 3, 141592654. Therefore since the Peripheries of Circles are as their Diameters (*Euc. Prop. III. Prop. XXXIII. B. VI.*) and the Circles themselves are as the Squares of those Diameters (*Euc. Prop. II. B. XII.*), it follows that as 2 to 6, 283185308, or 1 to 3, 141592654 :: the Diameter of any Circle to its Periphery, and as 4 to 3, 141592654 or as 1 to 0, 785398163 :: the Square of the Diameter to the Area very nearly.

Method of
finding the
Area of any
Circle by
Means of
the forego-
ing Propo-
sition of the
Diameter to
the Circum-
ference.

XC.

But when the Case proposed does not require any great Degree of Accuracy then those of *Archimedes* may be used, viz. 7 : 22 Diam : Circum. and 14 : 11 = square Diam : Area, which Proportions differ but little from those above as will appear from the following Example, wherein the Diameter of a Circle being 28, its Circumference and Area are required, here according to the first Proportions I multiply 28 by 3, 141592654 for the Circumference, and the Square of 28 or 784 by 0, 785398 163 for the Area, and there results 87, 964 &c. and 615, 75, &c. respectively. But according to the Proportions of *Archimedes* the Circumference will be found equal to 88 and the Area 616, which differ very little from the former. After the same Manner the Area of any other Circle will be found.

The Area of a Circle found more expeditiously but less accurately by the Proportions of *Archimedes*

XCI.

Since a Sector DKBCAD, or DKCAD, or DKA, contained by the three fourths, or the one half or the one fourth, &c. of the Circumference, is equal to the three fourths or to the one half or to the one fourth, &c. of the Circle, and consequently is equal to a Triangle whose Base is the three fourths or the one half or the one fourth, &c. of a Line equal to the Circumference of this Circle, and Altitude the Ray of the Circle; it is manifest that the Content of a Sector will be found, by multiplying its Arch by one half of its Ray.

Fig. 14.
Rule for finding the Area of a Segment of a Circle.

If, for Example, the Arch AD of the Sector be supposed to be 60 Degrees, and the Ray AK 14 Yards, the Diameter of the Circle of the Sector will be 28 Yards; Whence the Circumference of the Circle will be found to be 88 Yards; and 60 Degrees being the one sixth of 360 Degrees, or of the whole Circumference, the Number of Yards contained in the Arch AD will be 14 Y 2f, consequently the Content of the Sector AKD will be 14 Y 2f $\times$ 7 Y or 102 YY 2 Yf.

XCII.

Since the Superficies of any Sector AKDFL is determinable, as also the Area of the Triangle AKD contained by the two Rays AK, DK and the Chord AD, it is manifest that if the Area of the Sector AKDFL, and the Area of the Triangle AKD be separately found, and the Area of the Triangle be deducted from that of the Sector, the Remainder will be the Area of the Segment ADFL.

Method of finding the Area of a Sector of a Circle.

If, for Example, the Ray AK be 21 Yards, the Chord AD 30, and the Perpendicular KE 14 Y 2f 2l, and the Angle AKD 100 Degrees. I first determine the Superficies of the Sector AKDFL, which I find to be 385 square Yards. I afterwards determine the Superficies of the Triangle AKD, by multiplying AD by KE and taking the one half of the Product, which will be 220 YY 2 Yf 6 Yl, then subtracting from 385 YY the Area of the Sector AKDFL, 220 YY 2 Yf 6 Yl the Area of the Triangle AKD, the Remainder 164 YY 0 Yf 6 Yl will be the Area of the Segment ADFL.

XCIII.

From the Manner of computing the Areas of plane Figures, the Convex Superficies of solid Bodies may also be determined, thus.

Fig. 13 and
14.

Method of
finding the
convex Su-
perficies of
a Prism.

Grounds
of this
Method.

The Convex Superficies of a Prism AQPCB, is found by multiplying the Periphery of a Section RSTVXR perpendicular to one of the Sides AM, by the length of this Side.

That this may appear, let a Rectangle mr be constructed, whose Base ar is equal to the Periphery of the Section RSTVXR, and whose Altitude am is equal to the Side AM of the Prism; and let the Base ar of the Rectangle mr be divided into Parts af , st , tu , ux , xr equal to the Sides RS, ST, TV, VX, XR of the Section RSTVXR, and through the Points of Division let there be drawn Parallels to the Side am ; I say that the Parts F, G, H, I, K into which the Rectangle mr is divided, will be equal to the Parallelograms AN, BO, CP, DQ, EM which contain the Prism.

For since the Side AM is perpendicular to the Section RSTVXR, all its Parallels BN, CO, DP, EQ will be also perpendicular to the same Section, (*Euc. Prop. VIII. B. XI.*) from whence it follows 1^o. that the straight Line RS will be perpendicular to the two Parallels AM, BN, consequently the Parallelogram AN will be equal to the Product of AM into RS, or to the Rectangle F whose Altitude am and Base af are equal to AN, RS.

2^o. That the Straight Line ST will be perpendicular to the two Parallels BN, CO; whence the Parallelogram BO will be equal to the Product of ST multiplied by BN, or by its equal AM, and consequently equal to the Rectangle G, whose Base st and Altitude am are equal to ST, AM.

In like Manner it will appear, that the other Parallelograms CP, DQ, EM are equal to the Rectangles H, I, K; whence the Sum of the Parallelograms that contain the Prism, or the Superficies of the Prism, is equal to the Sum of the Rectangles F, G, H, I, K, that is, to the Rectangle whose Base is the Periphery RSTVXR of the Section, and whose Altitude is the Side AM of the Prism.

From whence it follows that the convex Superficies of an erect Prism, will be found, by multiplying the Periphery of the Base by the Altitude of the Solid, for in an erect Prism, the Section perpendicular to any of its Sides, is parallel to the Base, and consequently equal to it.

XCIV.

Fig. 15 and
16.

Rule for de-
termining
the Super-
ficies of a
Cylinder

Since the foregoing Conclusions hold universally whatever the Number of Faces of the Prism may be, and as the Prism by increasing the Number of its Faces approaches continually nearer and nearer to the inscribed Cylinder, which is its Limit, it is evident that the convex Superficies of a Cylinder is equal to a Rectangle mr , whose Base is a straight Line equal to the Periphery of the Section perpendicular to one of its

Sides AM, and whose Altitude is a Line equal to this Side : and if the Cylinder be erect, its convex Superficies is equal to a Rectangle whose contiguous Sides are equal to the Altitude and Periphery of the Base of this Cylinder.

xcv.

The upper Superficies of a regular Pyramid SABCDE is found by multiplying the Periphery of its Base by half the Length of the Line SG drawn from the Vertex to the Middle of any Side AB of this Base.

Fig. 17 and 18. Method of finding the Superficies of a regular Pyramid.

For the Pyramid SABCDE being a regular one, its Base ABCDE will be a regular Polygon, and consequently all its Sides AB, BC, CD, &c. will be equal, and the straight Line SF drawn from the Vertex to the Center F of the Circle described about this Polygon will be perpendicular to the Plane of this Polygon, consequently all the straight Lines SA, SB, SC, &c. drawn from the Vertex of the Pyramid to the Angles of its Base, will be equal. (*Euc. Prop. iv. B 1.*) Wherefore all the Triangles ASB, BSC, CSD, &c. whose Vertices coincide at the Vertex of the Pyramid will be equal, and isosceles Triangles, and their Altitudes will be equal to the Line SG drawn from the Vertex of the Pyramid to the Middle of any Side of the Base, as AB, and consequently perpendicular to this Side.

Grounds of this Method.

Wherefore the Sum of all the Triangles ASB, BSC, CSD, &c. which compose the upper Superficies of the regular Pyramid SABCDE, is equal to the Triangle ZHI, having for Base the sum of all their Bases, and whose Altitude ZH is equal to the straight Line SG drawn from the Vertex of the Pyramid to the Middle of any Side AB of the Base.

xcvi.

The Superficies of any Frustum ABCDEMNO PQ of a Pyramid, is found by multiplying the Sum of the Peripheries of the two Ends ABCDE, MNOPQ, by Half the Length of the Line drawn through the Middle of any two corresponding Sides AB, MN.

Rule for finding the Superficies of a Frustum of a Pyramid.

For the two Ends of the Frustum MNOPQ and ABCDE being parallel, the Sides MN, NO, OP, &c. of the one will be parallel to the Sides AB, BC, CD, &c. of the other, wherefore $AB : MN = SB : SN$ and $SB : SN = BC : NO$ (*Euc. Prop. II. B VI.*) Consequently $AB : MN = BC : NO$, and alternately $AB : BC = MN : NO$; but the End ABCDE being a regular Polygon, $AB = BC$, consequently $MN = NO$. By a similar reasoning it will appear that $NO = OP$, &c. and the Lines SA, SB, SC, SD, &c. which are all equal, will be cut proportionally, consequently SM, SN, SO, SP, &c. will be also equal, wherefore the Triangles MSN, NSO, OSP which compose the Superficies of the Pyramid will have their Sides equal each to each, and consequently all their Altitudes will be equal to that ST of the Triangle MSN; whence the Sum of all the Triangles MSN, NSO, OSP, &c. or the upper

Grounds of this Rule.

Superficies of the Pyramid, is equal to the Triangle ZKL, whose Base KL is equal to the sum of the Bases of all those Triangles, and whose Altitude ZK is equal to the Line ST; but the Superficies of the Pyramid SABCDE is equal to the Triangle ZHI; wherefore the Superficies of the Frustum contained between the parallel Bases ABCDE MNO PQ, is equal to the Trapezium HILK, whose two parallel Sides are equal to the Peripheries of the two Ends of the Frustum, wherefore &c.

How the Superficies of an irregular Pyramid is determined.

As to the Superficies of an irregular Pyramid, there is no other Rule for finding its Content, but to compute separately the Area of each of the Triangles which meet at the Vertex of this Pyramid.

XCVII.

Fig. 19.

Rule for determining the Superficies of a Cone.

Since the foregoing Conclusions hold universally whatever the Number of Sides of the Pyramid may be, and as the Pyramid by increasing the Number of its Sides approaches continually to the inscribed Cone which is its Limit, it is evident that the convex Superficies of a Cone is equal to the Product of the Circumference of the Base of the Cone into half the slant Side thereof, and that the convex Superficies of any Frustum of this Solid is equal to the Product of the Peripheries into half the Slant-side of the Frustum.

XCVIII.

Method of finding the Superficies of a Sphere.

Fig. 20.

The Superficies of a Sphere, is found by multiplying the Periphery of the greatest or generating Circle, by its Diameter; or by multiplying the Square of the Diameter by 3, 1416.

Let a Circle RQsq be inscribed in a regular Polygon ABCDE &c. of an even Number of Sides, from the Center O to the Point of contact Q of any Side BC, let the Radius OQ be drawn; also draw BbM, QPq, CcL &c. perpendicular to AF, and BN perpendicular to CL, and let the Polygon and its inscribed Circle be supposed to revolve about the produced Diameter AF.

Grounds of this Method.

Because the Solid generated by the Plane bc CB is the Frustum of a Cone, the convex Superficies thereof generated by BC, is equal to the Rectangle under $\frac{1}{2}$ BC and the Sum of the two Peripheries described by Bb and Cc: but the Sum of these two, as Qq is an arithmetical mean between them, is equal to twice the Periphery Qq, and therefore the convex Superficies of the said Frustum is equal to $\frac{1}{2}$ BC $\times$ 2 Periph. Qq = BC $\times$ Periph. Qq, but because of the similar Triangles OPQ, BNC, we have BC : BN(bc) = OQ : PQ = Periph. RQsq : Periph. Qq and consequently BC $\times$ Periph. Qq = bc $\times$ Periph. RQsq = Superficies generated by BC. By the very same Argument it will appear that the Superficies generated by any other Side CD is cO $\times$ Periph. RQsq: Whence it is manifest that the Superficies of the whole Solid is = Ab + bc + cO +, &c. $\times$ Periph. RQsq, and as the whole Superficies is

equal to $AF \times \text{Periph. } RQSq$, let the Number of Sides of the generating Polygon be what it will, and as the said Superficies, by increasing the Number of its Sides, approaches nearer and nearer continually to the Superficies of the inscribed Sphere, which is its Limit; it is evident that the Superficies of the Sphere itself is likewise equal to a Rectangle under its Axis RS and Periphery $RQSq$; wherefore, &c.

X C I X.

The convex Superficies of any Segment of a Sphere BALDB, is found by multiplying the Square of twice the Altitude LB of the Segment, together with the Square of the Diameter of the Base of the Segment, by 0,7854; as being equal to the Superficies of the Circle, having for Ray the Altitude LB of the Segment, together with the Superficies of the Base of the Segment.

Fig. 21.
Rule for determining the Superficies of a Segment of a Sphere

For it is evident from the foregoing Article, that the convex Superficies of any Segment BALDB of a Sphere, is equal to the Product of the Altitude multiplied into the Periphery of the greatest Circle ABDGA of the Sphere, but if the Chords AB, AG, be drawn, the Triangles ABL, BAG being Equiangular, $LB : AB = AB : BG$, and dividing the Consequents of this Proportion by 2, then $LB : \frac{AB}{2} = AB : BC$:

Grounds of this Rule

But $AB : BC = \text{Circum. } AB : \text{Circum. } BC$: Wherefore $LB : \frac{AB}{2} =$

$\text{Circum. } AB : \text{Circum. } BC$, whence $\text{Circum. } AB \times \frac{AB}{2} = LB \times \text{Circum. } BC$.

But the Product $LB \times \text{Circum. } BC$, representing the convex Superficies of the Segment BALDB, & $\text{Circum. } AB \times \frac{AB}{2}$ representing the Superficies of a Circle whose Ray is AB, which on Account of the Triangle ALB, being right angled in L, is equal to the Sum of the Circles, whose Rays are the Sides AL, LB of this Triangle, it is manifest that the convex Superficies of the Segment BALDB is equal to the Superficies of the Circle, whose Ray is the Altitude LB of the Segment, together with the Superficies of the Base of the Segment.

C.

Having explained how the Superficies of solid Bodies are determined, it remains to shew how the Contents of the Solids themselves may be found. Let the Solid to be measured be a Parallelepiped A O, whose Length AB, Breadth BC, and Height AM, have been measured, for Example, with the lineal Yard, and let this Solid be cut by Planes DP, EQ, FR, GS, HT, IV, &c. parallel to its Base, into as many equal Solids one Yard high, as there are Yards in its Height AM.

Fig. 22.
The Content of a Parallelepiped

Each Solid as M-V contained between two parallel Planes MO, IV,

its found by being one Yard high, will contain as many cubical Yards as there are multiplying the Area of square Yards in the Superficies of the Base M O; because a cubical the Area of Yard may be placed upon each of the square Yards in the Base M O, its Base by and that the cubical Yards that occupy all the square Measures of the its Altitude. Base, will be exactly contained between the two parallel Planes M O, I V, distant one Yard from each other.

But the Number of square Yards contained in the Rectangle M O, is equal to the Product of its Length M N multiplied by its Breadth N O. Wherefore the Number of cubical Yards contained in each of the Solids one Yard high, into which the Parallelepiped A O is divided, is equal to the Product of its Length M N multiplied by its Breadth N O; and as the Parallelepiped A O contains as many Solids one Yard high as there are Yards in its Altitude A M; it follows that the Number of cubical Yards contained in the Parallelepiped A O will be found by multiplying the Product of its Length M N and Breadth N O, measured in lineal Yards by the Number of Yards contained in its Altitude A M.

CI.

Example

Let, for Example, the Length of the Base A B be 6 Yards; the Breadth B C 5 Yards; and the Height A M of the Solid 3 Yards; the Number of cubical Yards contained in the Parallelepiped A O, will be equal to the Product of the three Numbers 6, 5, 3; and consequently will be 90 cubical Yards.

CII.

Another Example.

If the Length A B of the Base, the Breadth B C, and the Height A M of the Parallelepiped A O were measured in lineal Feet, Inches, or Lines, it is manifest that the Content of the Parallelepiped would be expressed in cubical Feet, cubical Inches, or cubical Lines.

A cubical Yard, being a Parallelepiped 6 Feet long, 6 Feet broad, and 6 Feet high, the cubical Yard will contain $6 \times 6 \times 6$ or 216 cubical Feet.

A cubical Foot being a Parallelepiped 12 Inches long, 12 Inches broad, and 12 Inches high, the cubical Foot will contain $12 \times 12 \times 12$, or 1728 cubick Inches:

In like Manner it will appear that a cubical Inch which is 12 Lines long, 12 Lines broad, and 12 Lines high, contains 1728 cubical Lines, and so of the other Measures that are employed.

CIII.

When the Dimensions of a Parallelepiped are measured by the Yard, and is not contained in them a certain Number of Times without a Remainder, the Computists measure the Remainder in Parts of a Yard, viz. in Feet, Inches, Lines, and as the Product of those three Dimen-

sions is not always an exact Number of cubical Yards without a Remainder, they estimate this Remainder in Parts of the cubical Yard.

The solid Measures are divided and subdivided into Parts analogous to those of the lineal Measures.

Though the cubical Foot, the cubical Inch, and cubical Line, are the most regular Parts of the cubical Yard, those however are not the Parts that the Computists commonly employ in estimating the Parts of Solids that are less than the cubical Yard, after the greatest Part of the Solid has been computed in cubical Yards, to render the Computation more easy, they divide the cubical Yard into Parts analogous to those of the lineal Yard.

They divide therefore the cubical Yard into 3 equal Parallelepipeds having each a square Yard for Base, that is 1 Yard long, 1 Yard broad, and 1 Foot high, which on account of their three Dimensions is expressed thus *Yard-Yard-Foot*.

They divide the *Yard-Yard-Foot*, or the third Part of the cubical Yard, into 12 equal Parts, having each for Base a square Yard, that is, 1 Yard long, 1 Yard broad, and 1 Inch high, expressed thus *Yard-Yard-Inch*.

Example

They divide also the *Yard-Yard-Inch*, or the twelfth Part of the third Part of the cubical Yard into 12 equal Parts, having each a square Yard for Base and 1 Line for Height, which on account of their three Dimensions is expressed thus, *Yard-Yard-Line*; and so of the other Measures that are employed.

CIV.

When the Dimensions of a Parallelepiped are measured by the Foot, and is not contained in them a certain Number of Times without a Remainder, the Computists measure this Remainder in Inches and Lines. In this Case the Parallelepiped contains a certain Number of cubical Feet with a Remainder, which the Computists estimate in Parts of a cubical Foot. To this End they divide the cubical Foot into 12 equal Parts, and those Parts which are 1 Foot long, 1 Foot broad, and 1 Foot high, are expressed thus, *Foot-Foot-Inch*, on account of their three Dimensions; they subdivide afterwards the *Foot-Foot-Inch* into 12 equal Parts, each 1 Foot long, 1 Foot broad, and 1 Line high, expressed thus, *Foot-Foot-Line*, to distinguish them by their three Dimensions.

Another Example.

In fine, whatever Measures are employed to measure a Solid, it is usual to assume cubical Measures for the principal Measures of the Solid, and to measure the Part which is less than the principal solid Measure with lesser Measures, formed by subdividing the principal cubical Measure into as many equal Parts as are contained in the lineal Measure; so that all the solid Measures resulting from those Divisions, have two Dimensions equal to those of the principal Measure.

CV.

The Length,
Breadth, and
Altitude of
the solid
Measures
contained in
any Solid,
are the line-
al Measures
of its Di-
mensions.

From what precedes it is easy to conclude that a Number of equal superficial Measures of any Length and Breadth, multiplied by a Number of lineal Measures of any Length, produces a Number of solid Measures, having for Bases the Measures of the Multiplicand, and for Height the Measures of the Multiplier.

For Example, if a Number of square Yards 1 Yard long, and 1 Yard broad, be multiplied by a Number of lineal Yards, the Product will be composed of a Number of solid Measures, 1 Yard long, 1 Yard broad, and 1 Yard high, that is, of cubical Yards.

If a Number of square Yards be multiplied by a Number of lineal Feet, or of Inches, or of Lines, the Product will contain a Number of solid Measures, 1 Yard long, 1 Yard broad, and 1 Foot or 1 Inch or Line high.

If a Number of superficial Measures that have two different Dimensions, be multiplied by a Number of lineal Measures that differ also from those Dimensions, for Example, if a Number of superficial Measures 8 Inches long, and 4 Inches broad, be multiplied by a Number of lineal Measures of two Inches, there will result for Product, a Number of solid Measures, each 8 Inches long, 4 Inches broad, and 2 Inches high, and so on.

CVI.

The Value of the different Units relative to the cubical Yard, and the Characters whereby they are distinguished from each other, are as follow.

YYY	Cubical Yard	1YYY	27fff	ffL	Foot-Foot-Line	1ffL	12ff	12III
fff	Cubical Foot	1fff	1728III	ff	Foot-Foot-Prime	1ff	12ff	1III
III	Cubical Inch	1III	1728LLL	ff'	Foot-Foot-Second	1ff'	12ff'	144LLL
LLL	Cubical Line	1LLL	1728LLL	ff''	Foot-Foot-Tierce	1ff''	12ff''	12LLL
		1YYY	3YYf	ff'''	Foot-Foot-Quart	1ff'''	12ff'''	1LLL
YYf	Yard-Yard-Foot	1YYf	12YYI or 9fff					
YYI	Yard-Yard-Inch	1YYI	12YYL 3fff			1III	12IIL	
YYL	Yard-Yard-Line	1YYL	12YY' 18fff	IIL	Inch-Inch-Line	1IIL	12II'	144LLL
YY'	Yard-Yard-Prime	1YY'	12YY'' 9III	II'	Inch-Inch-Prime	1II'	12II''	12LLL
YY''	Yard-Yard-Second	1YY''	12YY''' 3III	II''	Inch-Inch-Second	1II''		1LLL
YY'''	Yard-Yard-Tierce	1YY'''	12YY ^v 18III					
YY ^v	Yard-Yard-Quart	1YY ^v	12YY ^v 9LLL	Yff	Yard-Foot-Foot	1Yff	4YYI	3fff
YY ^v	Yard-Yard-Quint	1YY ^v	12YY ^v 12LLL	YfI	Yard-Foot-Inch	1YfI	4YYL	3fff
YY ^v	Yard-Yard-Sext	1YY ^v	12YY ^v 18LLL	YfL	Yard-Foot-Line	1YfL	4YY'	36III
				YfI	Yard-Inch-Inch	1YfI	4YY''	36III
				YfL	Yard-Inch-Line	1YfL	4YY'''	3III
ffI	Foot-Foot-Inch	1ffI	12ffI	YLL	Yard-Line-Line	1YLL	4YY ^v	3III
		1ffI	12ffL 144III					

CVII.

Let it be proposed to find the Content of a Parallelepiped the Area of whose Base is 3957YY 2Yf 8YI, and its Altitude 22Y if 6I.

Having disposed those Numbers as in the Margin, I first multiply the Number of YY of the Multiplicand, by 22Y, and afterwards take for the Products of the other Parts of the Multiplicand, multiplied by 22Y, the same Parts of 22YYY, that the Parts 2Yf 8YI of the Multiplicand are of a square Yard.

1°. Multiplying the Part 3957YY of the Multiplicand by 22Y, I find those two particular Products to be 7914YYY and 79140YYY.

2°. To multiply 2Yf by 22Y, I observe that 1YY multiplied by 22Y, will produce 22YYY, whence 2Yf which are only the two thirds of 1YY should give for Product the two thirds of 22YYY, and each third Part being 7YYY 1YYf, I set down twice 7YYY 1YYf or 14YYY 2YYf.

3°. To multiply the 8YI of the Multiplicand by 22Y, I observe that 8YI are the one third of 2Yf, which produced 14YYY 2YYf, whence for 8YI I take the $\frac{2}{3}$ of 14YYY 2YYf, which is 4YYY 2YYf 8YYI.

4°. If the Multiplicand was to be multiplied by 1Y, the Product would be 3957YYY 2YYf 8YYI, but as it is only to be multiplied by 1 Foot, or the $\frac{1}{3}$ of a Yard, the Product will be the $\frac{1}{3}$ of 3957YYY 2YYf 8YYI, viz. 1319YYY 0YYf 10YYI 8YYL.

5°. As 6I is only the one half of 1f, it should give for Product the one half of the Product 1319YYY 0YYf 10YYI 8YYL, which is 659YYY 1YYf 10YYI 4YYL.

Adding together all those particular Products, there results 89052YYY 1YYf 6YYL for the Content required of the Parallelepiped.

CVIII.

Let it be proposed to find the solid Content of a Parallelepiped 87Y 2f 8I long, 68Y 2f broad, and 32Y if 6I high.

As the Length is to be multiplied by the Breadth, and that Product by the Height of the Parallelepiped, I multiply 87Y 2f 8I, by 68Y 2f, and the Product 6035YY 0Yf 1YI 4YL by 32Y if 6I, from whence there results 197138YYY 2YYf 1YYI 4YYL for the solid Content required of the Parallelepiped.

3957YY	2Yf	8YI	
22Y	1f	6I	
7914YYY			
7914			
14	2YYf		
4	2	8YYI	
1319	0	10	8YYL
659	1	11	4
89052YYY	1YYf	6YYI	0YYL

Method of finding the Content of a Parallelepiped, whose Base is expressed in square Yards and Parts of a square Yard, and Altitude in Yards and Parts of a Yard.

87Y	2f	8I	
68Y	2f		
6035YY	0Yf	1YI	4YL
6035YY	0Yf	1YI	4YL
32Y	1f	6I	0YYL
197138YYY	2YYf	1YYI	4YYL
			0YYL

Method of finding the Content of a Parallelepiped, whose Dimensions are expressed in Yards and Parts of a Yard.

12III
1III
144LLL
12LLL
1LLL

144LLL
12LLL
1LLL

3fff
1fff
36III
36III
3III
2III

CIX.

Method of
reducing the
Parts of so-
lid Measures
analogous to
those of the
lineal Mea-
sures into
cubical
Measures.

When the solid Content of a Body is found in cubical Yards, and in other solid Measures that have all a square Yard for Base, and for Height the Parts into which the lineal Yard is divided, those Measures may be reduced into cubical Measures, such as cubical Feet, cubical Inches, and cubical Lines, after the following Manner.

1°. Each square Yard containing nine square Feet, and 1 YYf being the Product of 1 square Yard or of 9 square Feet multiplied by 1 Foot, which produces 9 cubical Feet, it follows that the Number of YYf being multiplied by 9, will produce cubical Feet.

2°. Each YYI being the one-twelfth Part of 1 YYf, or of 9 cubical Feet, it follows that 1 YYI is the $\frac{1}{12}$ of a cubical Foot; whence the Number of YYI being multiplied by $\frac{1}{12}$, will produce cubical Feet, and if there remains 1 or 2 or 3 Units, they will be equal to 432 or 864 or 1296 cubical Inches.

3°. Each YYL being the one-twelfth Part of 1 YYI or of the $\frac{1}{12}$ of a cubical Foot, it follows that 1 YYL is the $\frac{1}{144}$ of a cubical Foot, or 108 cubical Inches, whence the one-sixteenth Part of the Number of 1 YYL will be cubical Feet, and every 1 YYL that remains will be equal to 108 cubical Inches.

4°. Each YY' being the one twelfth Part of 1 YYL, or of 108 cubical Inches, it follows that 1 YY' is equal to 9 cubical Inches, consequently the Number of YY' being multiplied by 9, will produce cubical Inches.

5°. Each YY'' being the one twelfth Part of 1 YYY', or of 9 cubical Inches, it follows that 1 YY'' is the $\frac{1}{12}$ of a cubical Inch, whence the Number of YY'' being multiplied by $\frac{1}{12}$ will produce cubical Inches, and if there remains 1 or 2 or 3 Units they will be equal to 432, 864, 1296 cubical Lines.

6°. Each YY''' being the one twelfth Part of 1 YY'', or of $\frac{1}{12}$ of a cubical Inch, it follows that 1 YY''' is the $\frac{1}{144}$ Part of a cubical Inch, or 108 cubical Lines, whence the one sixteenth Part of the Number of 1 YY''' will be cubical Inches, and every 1 YY''' that remains will be equal to 108 cubical Lines.

7°. Each YY'''' being equal to the twelfth Part of 1 YY''', or of 108 cubical Lines, it follows that 1 YY'''' is equal to 9 cubical Lines, whence the Number of 1 YY'''' being multiplied by 9, will produce cubical Lines.

8°. Each YY^v being the twelfth Part of 1 YY'''' or of 9 cubical Lines it follows that 1 YY^v is equal to the $\frac{1}{12}$ of a cubical Line, wherefore the number of YY^v being multiplied by $\frac{1}{12}$ will produce cubical Lines.

9°. Lastly, 1 YY^v being the twelfth Part of 1 YY^v, or of $\frac{1}{12}$ of a cubical Line, it follows that 1 YY^v will be the $\frac{1}{144}$ of a cubical Line, and

NUMERAL ARITHMETICK.

III

consequently the number of YY^v being divided by 16, will produce cubical Lines.

When the Parts of the lowest Denomination, in the three mixt Numbers that are proposed to be multiplied into one another, are Lines, the $\frac{3}{4}$ of the Number of $1YY^v$ will always be an exact Number of cubical Lines, and the Number of $1YY^v$ will be always divisible by 16; because the three Numbers when reduced into Lines, and multiplied into one another, can only produce cubical Lines.

CX.

Let it be proposed to reduce into cubical Feet, cubical Inches, cubical Lines, the Numbers of solid Measures in the following Product, off off $2YYL$ $3YY$ $5YY''$ $5YY'''$ $2Y'''$ $4Y^v$ $48YY^v$ that have all for Base a square Yard, and for Height the Parts into which the lineal Yard is divided.

To reduce those Measures it suffices, to set down under the three first, the Numbers 9, $\frac{3}{4}$, $\frac{1}{8}$, which will produce cubical Feet, likewise under the three following the same Numbers 9, $\frac{3}{4}$, $\frac{1}{8}$, which will produce cubical Inches, and under the three last the same Numbers 9, $\frac{3}{4}$, $\frac{1}{8}$, likewise, which will produce cubical Lines, and multiply each Number of Measures by the Number wrote under it.

off	off	$2YYL$	$3YY$	$5YY''$	$5YY'''$	$2Y'''$	$4Y^v$	$48Y^v$
9	$\frac{3}{4}$	$\frac{1}{8}$	9	$\frac{3}{4}$	$\frac{1}{8}$	9	$\frac{3}{4}$	$\frac{1}{8}$
			216III			1296LLL		
			27			540		
			3			18		
						3		
						3		
			247III			132LLL		

The foregoing Method applied to an Example.

As the three first cannot produce an Unit, the Reduction will not produce cubical Feet, but because there remains 2 Yard-Yard-Lines, which multiplied by $\frac{1}{8}$ or divided by 16, will produce the $\frac{1}{4}$ of a Foot, or the $\frac{1}{8}$ of 1728 cubical Inches, I set down 216 in the Rank of cubical Inches.

Multiplying the three following Numbers $3YY$, $5YY''$, $5YY'''$, which should produce cubical Inches by the Numbers 9, $\frac{3}{4}$, $\frac{1}{8}$, wrote under them, I find for the first Product 27 II, for the second 3 III, 1296 LLL and for the third 540 LLL.

Multiplying the three last Numbers of the Measures that should produce cubical Lines, by 9 $\frac{3}{4}$ $\frac{1}{8}$, the first Product will be 18 LLL, the second 3 LLL, and the third will be 3 LLL.

Having reduced all the Parts of the proposed Product, into cubical Measures, I add them together, and there results off 247 III 132 LLL.

CXI.

From the Content of a Parallelepiped thus known that of a Prism is also known. Let it be proposed for Example to find the Content of a triangular Prism ABCDEF, because this Prism is the Half of its Parallelepiped (*Euc. Prop. xxviii B. xi*) it will be equal to a Parallelepiped of equal Base and Altitude. Let therefore in its Base ABC, and from one of its Angles B, a Perpendicular BG be let fall upon the Side AC opposite

to this Angle; then $AC \times \frac{BG}{2}$ will represent a rectangular Parallelogram MNO whose contiguous Sides MN, NO are equal to the Side AC and to one half of the Altitude BG of the Base ABC of the Prism ABCDEF. Let a Perpendicular DI be drawn from any Point D of the upper Base of the Prism to the Base ABC: then $AC \times \frac{BG}{2} \times DI$ will represent the solid Parallelepiped MR of equal Base and Altitude with the triangular Prism ABCDEF.

The Content of any Prism is found by multiplying the Area of its Base by its Height.

As all Polygons may be divided into Triangles, and that each Triangle may be transformed into a rectangular Parallelogram, it is manifest that every Prism is equal to a Parallelepiped of equal Base and Altitude; therefore the Content of any Prism will be found by multiplying the Area of the Base, (found by the Rules for Surperficies) by the Height of the Prism, and the Product will be the required Content.

CXII.

Fig. 24.

The Content of a Cylinder found by multiplying the Area of its Base by its Altitude.

After the very same Manner the Content of a Cylinder is found (it being also equal to a Parallelepiped of equal Base and Altitude): Therefore, if for Example, the Diameter of the Base be supposed to be 20 Inches, and the Height of the Cylinder 15 Inches, then the Content of the Solid will be 4712, 4 cubical Inches, very nearly. For the Area of the circular Base being 314,16, this multiplied by 15, gives 4712,4, as before

CXIII.

Fig. 25.

The Content of a Pyramid or Cone, found by multiplying the Measure of its Base into 1 third of its Altitude.

Hence also the Solidity of a Pyramid or Cone is likewise known; every such Solid being $\frac{1}{3}$ of a Prism or Cylinder of equal Base and Altitude (*Euc. Prop. vii. Cor. ii. Prop. x. B. xii*). Therefore the Content of a Pyramid or Cone is found by multiplying the Area of the Base by $\frac{1}{3}$ of the Altitude.

CXIV.

The Content of a Frustum of a Pyramid is found by multiplying the Area of the two Ends, and a geometrical Mean between them by $\frac{1}{3}$ of the Height of the Frustum.

The Area of the two Ends of the Frustum is found by the Rules for Surperficies, but as those two Ends are similar Figures, and consequently

proportional to the Squares of their homologous Lines, (*Euc. Prop. xx. Fig. 26. Cor. 11. B. vi.*) when one of the two Ends of this Frustrum is found, the other may be determined by the following Proportion.

As the Square of any Line in one Base, whose Length is known, is to the Square of the homologous Line in the other Base; so is the Area of the first Base to the Area of the second Base.

Rule for finding the solid Content of a Frustrum of a Pyramid or Cone

When the Area of the two Bases of the Frustrum is found, the Area of the Base, which is a geometrical Mean between them, is determined by the following Proportion.

As a Line in one of the Bases, is to the homologous Line in the other Base; so is the Area of one of the Bases to the Area of the geometrical Mean between them.

CXV.

That this may appear let ABCDEF be the Frustrum of a triangular Pyramid, from an Angle A of the upper Base of the Frustrum, let there be drawn in the lateral Faces, BD, CD of this Frustrum, two Diagonals AE, AF, and let the Frustrum be cut by a Plane EAF, passing through those Diagonals. It is manifest that this Frustrum will be divided into two Pyramids AEDF, ABCFE, the first AEDF having for Base, the Base EDF of the Frustrum, as also the same Altitude.

Grounds of the foregoing Rule for determining the Solidity of a Frustrum of a Pyramid or Cone.

Let the second quadrangular Pyramid ABCFE, be cut by a Plane CAE passing thro' AC, AE, which will divide it into two Pyramids ABCE, ACFE; the first ABCE having for Base the Triangle BAC, and its Vertex in E, consequently the same Altitude as the Frustrum.

To find the Expression of the third Pyramid ACFE, I compare it with the Pyramid ABCE, and observing that the Vertices of those two Pyramids coincide in A, and that their Bases BCE, CFE, are in the same Plane BCFE, I conclude that they have the same Altitude, and consequently that they are proportional to their triangular Bases BCE, CFE.

Now the two opposite Bases BAC, EDF, of the Frustrum of the Pyramid, being parallel; BC and EF, will be parallel; whence the Triangles BCE, CFE, contained between those two Parallels BC, EF, will have the same Altitude, and will be proportional to their Bases BC, EF; and consequently the Pyramids ABCE, ACFE, will be proportional to the same Lines BC, EF.

But instead of taking the Point A for the common Vertex, and the two Triangles BCE, CFE for the Bases of the two Pyramids, ABCE, ACFE; let the Point E and the Triangle ABC, be considered as the Vertex and Base of the Pyramid ABCE, and let the Pyramid ACFE, be supposed to be reduced to another Pyramid of the same Height as the Frustrum

tum; the two Pyramids ABCE, ACFE having the same Altitude, will be proportional to their new Bases, BAC & X: and as those Pyramids have been demonstrated to be proportional to the two straight Lines BC, EF, their new Bases will be also proportional to the Lines BC, EF, or $BAC : X = BC : EF$, but the two opposite Bases EDF, BAC, of the Frustrum being similar Figures $EDF : BAC = EF \times EF : BC \times BC$; wherefore by the Composition of Ratios $EDF : X = EF \times EF \times BC : BC \times BC \times EF = EF : BC$, and invertendo, $X : EDF = BC : EF$; wherefore $BAC : X = X : EDF$.

From whence we may conclude that the Frustrum of a Pyramid or Cone is equal to the Sum of three Pyramids, or of three Cones, of the same Altitude as the Frustrum, two of which have for Bases the two Ends of the Frustrum, and the third having for Base a geometrical Mean between the two Ends of the Frustrum. Consequently its solid Content is equal to one third of the Product, arising from the Multiplication of the Altitude of the Frustrum into a Base composed of the two opposite Bases of this Frustrum, and a geometrical Mean between them.

CXVI.

Rule for
finding the
solid Con-
tent of a
Sphere.

The solid Content of a Sphere is found by multiplying the Area of its greatest or generating Circle by $\frac{2}{3}$ of its Diameter; or because the Area of such a Circle is to the Square of the Diameter, as 0,7854 to Unity; multiply the Cube of the Diameter by ,5236 which is $\frac{2}{3}$ of 0,7854 and the Product will be the Content of the Sphere.

Thus if the Diameter of a Sphere be 20, the Cube thereof will be 8000, which multiplied by the Fraction ,5236, gives 4188,8 for the Solidity of the Sphere very nearly.

CXVII.

Grounds of
the forego-
ing Rule.

Fig. 27.

To demonstrate this Rule, let AB be the Axis, about which a Sphere and Cylinder are generated by the Rotation of a Semi-circle AGB, and a Rectangle ADCB; let HL be any right Line perpendicular to AB, meeting the Periphery of the Semi-circle in K, and from the Center O let OK and OD (intersecting HL in I) be drawn because $AD = OA$, therefore is $HI = OH$, and consequently $HI^2 (OH^2 = OK^2 - HK^2) = HL^2 - HK^2$; whence because all Circles are as the Squares of their Radii (*Euc. Prop. 2. B. XII.*) it is evident that the Circle described by HI, or the Section of the Cone generated by the Triangle AOD is equal to the Difference of the Circles described by HL & HK, that is equal to the Annulus described by KL, or the Section of the Solid, which remains when the Sphere is taken out of the Cylinder; wherefore seeing the Sections are every where equal, the Solids themselves must be also equal or the Cone EOD generated by the Triangle AOD, equal to the Ex-

cess of the Cylinder, GDEg, above the Hemisphere GAg, whence, as the Cone or Excess is $\frac{1}{3}$ of the Cylinder, the Hemisphere must consequently be the other two thirds, and so the whole Sphere equal to $\frac{2}{3}$ of its circumscribing Cylinder CDEF.

CXVIII.

The Content of a Segment of a Sphere is found by multiplying the Square of twice the Height or Thickness of the Segment by the Ray of the Sphere less by $\frac{1}{3}$ of the said Height, and that Product again by 0,7854. Rule for determining the solid Content of a Segment of a Sphere.

For to obtain the Content of a Segment BAD of a Sphere, we must subtract the Cone CAD, from the spherical Sector CABD.

But 1^o. The spherical Sector CABD is equal to a Cone, having the Radius BC of the Sphere for Altitude, and the spherical Superficies of the Segment BAD for Base, consequently is equal to the Sum of two Cones, Fig. 28. both having the Ray BC of the Sphere for Altitude, and whose Bases have for Rays the two straight Lines AL, BL, that is, Sector

$$CABD = \frac{\text{Cir. AL} \times \text{BC}}{3} + \frac{\text{Cir. BL} \times \text{BC}}{3}$$

2^o. The Cone CAD to be subtracted from the Sector, having AL for Ray and LC for Altitude, this Cone CAD = $\frac{\text{Cir. AL} \times \text{LC}}{3}$, consequent- Grounds of the foregoing Rule.

$$\text{ly Sector CABD} - \text{Cone CAD or Segment BAD} = \frac{\text{Cir. AL} \times (\text{BC} - \text{LC})}{3} + \frac{\text{Cir. BL} \times \text{BC}}{3} = \frac{\text{Cir. AL} \times \text{BL}}{3} + \frac{\text{Cir. BL} \times \text{BC}}{3}$$

But $\text{Cir. BL} : \text{Cir. AL} = \text{BL}^2 : \text{AL}^2$ and $\text{BL} : \text{AL} = \text{AL} : \text{LG}$ and $\text{BL}^2 : \text{AL}^2 = \text{BL} : \text{LG}$. Wherefore $\text{Cir. BL} : \text{Cir. AL} = \text{BL} : \text{LG}$; consequently $\text{Cir. AL} \times \text{BL} = \text{Cir. BL} \times \text{LG}$ and $\frac{\text{Cir. AL} \times \text{BL}}{3} = \frac{\text{Cir. BL} \times \text{LG}}{3}$.

$$\text{Hence the spherical Segment BAD} = \frac{\text{Cir. BL} \times \text{LG}}{3} + \frac{\text{Cir. BL} \times \text{BC}}{3}$$

$$\text{But } \frac{\text{Cir. BL} \times \text{LG}}{3} + \frac{\text{Cir. BL} \times \text{BC}}{3} = \frac{\text{Cir. BL} \times (\text{LG} + \text{BC})}{3} = \frac{\text{Cir. BL} \times (3\text{BC} - \text{BL})}{3} = \text{Cir. BL} \times (\text{BC} - \frac{\text{BL}}{3})$$

But Cir. BL being the Area of a Circle whose Ray is the Thickness of the Frustrum BAD, it is manifest that $\text{Cir. BL} \times (\text{BC} - \frac{\text{BL}}{3})$ is the solid Content of a Cylinder, whose Ray is BL, and Altitude is Equal to the Ray BC of the Sphere, less the one third of the Height BL of the Segment. Wherefore, &c.

CXIX.

Having fully explained the Method of multiplying mixed Numbers, and shewn how Superficies and Solids are produced by the Multiplication of

Numbers, substituted for Lines, we now proceed to explain the Method of dividing mixed Numbers, and to shew how Superfices and Solids are resolved by Division.

Method of dividing a mixed Number by a simple one.

The Divisor given to divide a mixed Number by, may be either a simple or compound Number. When the Divisor given is a simple Number, the Division of mixed Numbers does not differ from that of simple Numbers, and is performed by dividing each Part of the mixed Number by the given simple Divisor, beginning the Division by the Parts of the highest Denomination. For Example, if the mixed Number that is proposed to be divided, consists of Pounds, Shillings, Pence, and Farthings, the Pounds are first to be divided, then the Shillings, after adding to them the Value of the Pounds that could not be divided; afterwards the Pence, after adding to them the Value of the Shillings that could not be divided; and lastly the Farthings are to be divided, after adding to them the Pence that could not be divided.

Method of dividing a mixed Number by another mixed Number.

When the given Divisor is a compound Number, the Computists reduce it to a simple Number, by multiplying it by such Numbers as will make all the Parts, whose Units are less than the principal Unit, disappear; and, that the Quotient may be the same as would result if the Dividend was divided by the compound Divisor, they multiply the Dividend by the same Numbers that the Divisor was multiplied by, to reduce it to a simple Number; the Reason of this Operation is, that a Dividend and Divisor being multiplied by the same Number, gives the same Quotient as they would give if they had not been multiplied.

CXX.

How to distinguish when the Quotient should be a concrete or abstract Number.

When the Divisor is an abstract Number, the Units of the Quotient are of the same Species as those of the Dividend, because the abstract Divisor denotes, by the Number of its Units, that the Dividend should be divided into a certain Number of equal Parts, and it is manifest that the Parts of the Dividend are of the same Species as this Dividend.

When the Divisor is a concrete Number, its Units should be always of the same Species as those of the Dividend, except the Dividend be a Number of superficial or solid Measures, for in this Case the Divisor may be a concrete Number of Measures that have one or two Dimensions less than those of the Dividend.

If the Dividend and Divisor consist of the same Species of Units, the Quotient is always an abstract Number, since it should express how often the Divisor is contained in the Dividend.

Dimensions of the Units of the Quotient, when

If the Dividend contains a Number of square Measures, and the Divisor contains a Number of Measures that are the Sides of those square Measures, the Quotient will be a Number of Measures that will be the

Sides of the same square Measures; and in general, when the Dividend consists of a Number of Measures of a certain Number of Dimensions, and that the Units of the Divisor have some of the Dimensions of the Units of the Dividend, the Units of the Quotient always will have the Dimensions of the Units of the Dividend, that the Units of the Divisor have not: All which will be explained in the following Examples.

XCXI.

Let it be proposed to divide the mixed Number 38386 £. 6 s. 10 d. $\frac{1}{2}$ by the abstract Number 74.

Having set down the Divisor to the Right Hand of the Dividend, and drawn a Line, under which the Figures of the Quotient are to be set down, according as they are found.

1°. I divide the Part 38386 £. of the Dividend by the given Divisor 74, and I find 518 £. for the Quotient, with a Remainder 54 £. which cannot be divided by 74 under the Form of Pounds.

2°. This first Division being performed, I reduce the 54 £. remaining into Shillings, by multiplying it by 20, which will produce 1080 s. to which adding the 6 s. in the Dividend, there results 1086 s. to be divided by the given Divisor 74, the Quotient will be found to be 14 s. with a Remainder of 50 s. that cannot be divided by 74.

3°. Having set down this second Quotient 14 s. to the Right Hand of 518 £. already found, I reduce the 50 s. remaining into Pence, by multiplying it by 12, which with the 10 d. in the Dividend, will produce 610 Pence, to be divided by 74, the Quotient will be found to be 8 d. with a Remainder 18 d. which reduced into Farthings, and adding to the Product, the $\frac{1}{2}$ in the Dividend, there results 74 Farthings, which divided by 74 gives $\frac{1}{4}$ for the Quotient without a Remainder; whence 38386 £. 6 s. 10 d. $\frac{1}{2}$ divided by the abstract Number 74, will give 518 £. 14 s. 8 d. $\frac{1}{4}$ for the Quotient required.

38386 £. 6 s. 10 d. $\frac{1}{2}$	}	74	The Divi on of Pounds, Shillings, and Pence, by an ab stract Num ber, explain ed by an Ex ample.
370		518 £. 14 s. 8 d. $\frac{1}{4}$	
138			
74			
646			
592			
54 £. 6 s. or 1086 s.			
20		74	
		346	
		296	
		50 s. 10 d. or 610 d.	
		12	
		592	
		18 d. $\frac{1}{2}$ or 74 qrs.	
		74	
		00	

Let it be proposed to divide the mixed Number 1267 lb. 8 oz. 5 dwt. 15 gr. by $51 \frac{1}{4}$.

The Division of Weights by an abstract Number explained by an Example.

$$1267 \text{ lb. } 8 \text{ oz. } 5 \text{ dwt. } 15 \text{ gr. } \left\{ 51 \frac{1}{4} \right.$$

$$5070 \text{ lb. } 9 \text{ oz. } 2 \text{ dwt. } 12 \text{ gr. } \left\{ \begin{array}{l} 205 \\ 24 \text{ lb. } 8 \text{ oz. } 16 \text{ dwt. } 12 \text{ gr.} \end{array} \right.$$

410

970

820

150 lb. 9 oz. or 1809 oz.

12

1640

169 oz. 2 dwt. or 3382 dwt.

20

205

1332

1230

102 dwt. 12 gr. or 2460 gr.

24

205

410

410

000

The Divisor $51 \frac{1}{4}$ being a mixed Number, I multiply it by 4, to make the Fraction $\frac{1}{4}$ disappear, and there results 205 for a new Divisor quadruple of the one proposed; and that the Quotient may be the same as if the Dividend was divided by the proposed Divisor $51 \frac{1}{4}$, I multiply the Dividend also by 4, which will produce 5070 lb. 9 oz. 2 dwt. 12 gr. for a new Dividend, and having placed the new Divisor to the right Hand of the new Dividend.

1°. I divide the first Part 5070 lb. by 205 and I find 24 lb. for the Quotient, with a Remainder 150 lb. which cannot be divided by 205.

2°. I reduce the Remainder 150 lb. into Ounces, by multiplying it by 12, and adding to the Product the 9 Ounces in the Dividend, there results 1809 Ounces for a new Dividend, which I divide by 205, and I find 8 oz. for the Quotient, with a Remainder 169, that cannot be divided by 205.

3°. I reduce the Remainder 169 oz. into Pennyweights, by multiplying it by 20; and adding to the Product the 2 dwt. in the Dividend, there results 3382 dwt. for a new Dividend, which I divide by 205, and I find 16 dwt. for the Quotient, with a Remainder 102 dwt. which cannot be divided by 205.

4°. I reduce the Remainder 102 dwt. into Grains, by multiplying it by 24, and adding to the Product the 12 gr. in the Dividend, there results 2460 gr. for a last Dividend, to be divided as the rest by 205, and I find 12 for the Quotient, without a Remainder, whence the Quotient required will be 24 lb. 8 oz. 16 dwt. 12 gr.

CXXIII.

Let it be proposed to divide 38515 £. 19 s. by 518 £. 14 s. 8 d.

The Divisor being a mixed Number, I multiply it by such Numbers as will make the Shillings and Pence disappear; to find those Numbers I observe that 8 d. is the one third of 2 s. consequently by multiplying the Dividend and the Divisor by 3, there will result a new Dividend 115547 £. 17 s. and a new Divisor 1556 £. 4 s. in which there are no Pence. I observe further that the 4 s. in the new Divisor being the one fifth of 1 £. by multiply-

$$\begin{array}{r}
 38515 \text{ £. } 19 \text{ s. } \left\{ \begin{array}{l} 518 \text{ £. } 14 \text{ s. } 8 \text{ d.} \\ 115547 \text{ £. } 17 \text{ s. } \left\{ \begin{array}{l} 1556 \text{ £. } 4 \text{ s.} \\ 577739 \text{ £. } 5 \text{ s. } \left\{ \begin{array}{l} 7781 \text{ £.} \\ 74 \frac{1}{4} \end{array} \right. \\ 54467 \\ 33069 \\ 31124 \\ 1945 \text{ £. } 5 \text{ s.} \end{array} \right. \end{array} \right.
 \end{array}$$

The Division of Pounds, Shillings, and Pence, by Pounds, Shillings, and Pence, explained by an Example.

ing the Dividend and Divisor by 5, there will result a new Dividend 577739 £. 5 s. and a new Divisor 7781 £. in which there are neither Shillings or Pence.

The Dividend and Divisor being thus prepared, I divide one by the other. 1°. Dividing 577739 £. the first Part of the prepared Dividend, by the prepared Divisor 7781 £. I find the abstract Number 74 for the Quotient, which denotes that the Divisor 7781 £. is contained 74 Times in the Dividend 577739 £. and there remains 1945 £. 5 s. that cannot be divided by 7781 £. but as this Remainder is precisely the one fourth of the Divisor 7781 £. I set down $\frac{1}{4}$ in the Quotient; whence $74\frac{1}{4}$ is the exact Quotient required.

CXXIV.

When the Dividend and Divisor consist of Units of the same Species, as in the foregoing Example, the Division may be performed after the

**Another
Method of
dividing
mixed Num
bers of the
same Spe
cies.**

following Manner: Reduce the Dividend and Divisor to the lowest Denomination they contain: Thus the lowest Denomination in the foregoing Example being Pence, reduce the Dividend and Divisor into Pence, by multiplying the Pounds by 240, and the Shillings by 12, and then divide the Dividend reduced to one Term, by the Divisor reduced also to one Term, and the Quotient will be an abstract Number, expressing how often the Divisor is contained in the Dividend.

CXXV.

Let it be proposed to divide $512\overline{rr}$ $1\overline{r}$ f $8\overline{r}$ l $1\overline{r}$ l by $57\overline{r}$ $2\overline{r}$ f $8\overline{r}$ l.

**Division of
Square
Yards and
Parts of a
Square Yard
by lineal
Yards, and
Parts of a
lineal Yard,
explained
by an Ex
ample.**

$$\begin{array}{r}
 512rr \quad 1rf \quad 8rI \quad 1rL \left\{ \begin{array}{l} 57r \quad 2f \quad 8I \end{array} \right. \\
 \hline
 1537rr \quad 2rf \quad 0rI \quad 3rL \left\{ \begin{array}{l} 173r \quad 2f \end{array} \right. \\
 \hline
 4613rr \quad 0rf \quad 0rI \quad 9rL \left\{ \begin{array}{l} 521r \\ \hline 8r \quad 2f \quad 6I \quad 9L \end{array} \right. \\
 4168 \\
 \hline
 445rr \quad 0rf \quad or \quad 1335rf \\
 3 \qquad \qquad \qquad 1042 \\
 \hline
 293f \quad 0rI \quad or \quad 3516rI \\
 12 \qquad \qquad \qquad 3126 \\
 \hline
 390rI \quad 9rI \quad or \\
 12
 \end{array}$$

To make the 8 Inches in the Divisor disappear, I multiply it by 3; I likewise multiply the Dividend by 3, and there results 1537YY 2Yf 0YI 3YL for a new Dividend, and 173Y 2f for a new Divisor.

To make the 2 Feet in the new Divisor disappear, I multiply likewise the new Dividend and Divisor by 3, and there results 4613YY oYf, oYI 9YL for the prepared Dividend, and 521Y for the prepared Divisor; the Quotient of which will be the same as that of the proposed Dividend, divided by the proposed Divisor.

As a Number of Yards, Feet, and Inches, multiplied by a Number of Yards, produce a Number of square Yards and Parts of a square Yard,

divided into 3, and subdivided continually into 12 equal Parts, and that the Extention produced by Multiplication is resolved by Division; it is manifest that 4613 YY 0 Yf 0 YI 9 YL, divided by 521 Y, will give for Quotient a mixed Number, composed of Yards, and Parts of a Yard.

Dividing therefore the Part 4613 YY, of the Dividend by the prepared Divisor 521 Y, the Quotient will be 8 Y, with a Remainder 445 YY, which cannot be divided by the Divisor 521 Y.

To continue on the Division, I reduce into Yard-Foot the 445 square Yards remaining, by multiplying it by 3, which will produce 1335 Yf, to be divided by 521 Y, and the Quotient will be 2 Feet with a Remainder 293 Yf; because a Number of Yard-Foot, arising from the Multiplication of a Number of Feet into a Number of Yards, when divided by a Number of Yards, will give a Number of Feet for Quotient.

To continue on the Division, I reduce into Yard-Inches the Remainder 293 Yf, of the foregoing Division, by multiplying it by 12, which will produce 3516 YI, to be divided by 521 Y, and the Quotient will be 6 I, with a Remainder 390 YI; because a Number of Yard-Inch divided by a Number of Yards, should give a Number of Inches for Quotient.

To continue on the Division, I reduce into Yard-Line the Remainder 390 YI, by multiplying it by 12, and adding to the Product the 9 YL in the Dividend, there results 4689 YL, to be divided by 521 Y, and the Quotient will be 9 L, and there being no Remainder, the Quotient required will be 8 Y 2f 6 I 9 L.

CXXVI.

In the foregoing Example, the Yard being the principal Dimension of the Measures of the Dividend, we were under the Necessity of reducing the mixt Divisor into Yards; and we found for the Quotient a Number composed of Yards, Feet, Inches, &c.

If the Foot was the principal Dimension of the Parts of the Dividend, the Divisor must be reduced into Feet, and the Quotient would be a Number composed of Feet, Inches, Lines, &c.

For Example, if it was proposed to divide 60ff 1fI 10fL, by 2f 6I 6L. To reduce the Divisor to a Number of Feet, I multiply the Dividend and Divisor by 24 and there results 1443ff 8fI to be divided by 61f, and performing the Division, as in the Margin, I find the Quotient to be 23f 8I.

$$\begin{array}{r}
 60ff \ 1fI \ 10fL \quad \left\{ \begin{array}{l} 2f \ 6I \ 6L \end{array} \right. \\
 \hline
 1443ff \ 8fI \quad \left\{ \begin{array}{l} 61f \\ 23f \ 8I \end{array} \right. \\
 \hline
 122 \\
 223 \\
 183 \\
 \hline
 40ff \text{ or } 488fI \\
 12 \quad \underline{488} \\
 000
 \end{array}$$

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CXXVII.

Let it be proposed to divide 90372YYY oYYf 4YYI 1YYL by 22Y 2f 6l.

$$90372YYY \text{ oYYf } 4YYI \text{ 1YYL} \left\{ \begin{array}{l} 22Y \text{ 2f } 6l. \end{array} \right.$$

$$180744YYY \text{ oYYf } 8YYI \text{ 2YYL} \left\{ \begin{array}{l} 45Y \text{ 2f } \end{array} \right.$$

Division of
cubical
Yards and
Parts of a
cubical
Yard by li
neal Yards
and Parts
of a lineal
Yard, ex
plained by
an Example

$$542232YYY \text{ 2YYf } 0YYI \text{ 6YYL} \left\{ \begin{array}{l} 137Y \\ 3957YY \text{ 2Yf } 8YI \text{ 6YL} \end{array} \right.$$

411

1312

1233

793

685

1082

959

$$123 \text{ 1YY } 2Yf \text{ or } 371YYf$$

3

274

$$97 \text{ YYf } 0YYI \text{ or } 1164YYI$$

12

1096

$$68YYI \text{ 6YYL } \text{ or } 822YYL$$

12

822

000

To make the 2f 6l in the Divisor disappear, I multiply the Dividend and Divisor successively by 2 and by 6, and there results 542232YYY 2YYf 0YYI 6YYL for a prepared Dividend, and 137Y for a prepared Divisor.

As a Number of square Yards and Parts of a square Yard divided into 3, and subdivided continually into 12 equal Parts, multiplied by a Number of lineal Yards, produce a Number of cubical Yards, cubical Feet, cubical Inches, &c. and that the Extension produced by Multiplication, is resolved by Division, it is evident that 542232YYY 2YYf 0YYI 6YYL; divided by 137Y, will give for Quotient a Number of square Yards, square Feet, square Inches, &c.

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Dividing therefore the Part 542232YYY, by 137Y, the Quotient will be 3957YY, with a Remainder 123YYY, which I reduce into Yard-Yard-Feet, and adding to the Product the 2YYf in the Dividend, there results 371YYf, which I divide by 137Y, and the Quotient will be 2Yf, with a Remainder 97Y.Yf. Reducing this Remainder into Yard-Yard-Inch, and continuing on the Division, I find the Quotient required to be 3957YY 2Yf 8YI, 6YL.

CXXVIII.

Let it be proposed to divide 2747YII 1fII 4III by 72II.

The Dividend and Divisor of a Division, being divided by the same Quantity, the Quotient arising from the Division of the new Dividend, by the new Divisor, will be the same as that of the first Dividend, divided by the first Divisor. Now the proposed Dividend 2747YII 1fII 4III, being the Product of 2747Y if 4I into 1H, it may be divided by 1II, and consequently reduced to 2747Y if 4I, and the Divisor 72II, being the Product of 1II, multiplied by the abstract Number 72, may be also divided by 1II, and consequently reduced to the absolute Number 72; wherefore the Quotient of the Division will be the same, whether the proposed Dividend be divided by the proposed Divisor, or the abridged Dividend 2747Y if 4I, be divided by the abridged Divisor 72; and as this new Dividend and Divisor are more simple than the former, they should be employed in the Division.

Division of a Number of Yard-Inch-Inch, Feet-Inch-Inch, Inch-Inch-Inch by a Number of Inch-Inch, explained by an Example.

1^o. Dividing 2747Y by 72, there results 38Y for Quotient, with a Remainder 11Y.

$$\begin{array}{r} 2747Y \text{ if } 4f \\ 216 \\ \hline 587 \\ 576 \\ \hline 11Y \text{ if or } 34f \text{ or } 412I \\ 3 \quad 12 \quad 360 \\ \hline 52I \text{ or } 624L \\ 12 \quad 576 \\ \hline 48L \text{ or } 576' \\ 12 \quad 576 \\ \hline 000 \end{array}$$

2^o. I reduce the Remainder 11Y into Feet, by multiplying it by 3, and adding to it the 1f in the Dividend, there results 34f, which cannot be divided by 72; I, therefore, set down of in the Quotient, and reduce the 34 Feet into Inches, by multiplying it by 12, and adding to it the 4 Inches in the Dividend, there results 412 Inches; which divided by 72, gives 5I for Quotient, with a Remainder 52I, which being reduced into Lines, and

ELEMENTS OF

divided by 72, gives 8L 8' for the Quotient, and there being no Remainder left, the Quotient required will be 38Y of 51 8L 8'.

CXXIX.

Let it be proposed to find the Side of a Square, whose superficial Content is 24YY of 2YI 3YL.

Extraction
of the Square
Root of
compound
Numbers,
explained by
an Example

1°. I extract the Square Root of 24YY, and I find 4Y for the first Part of the Root, with a Remainder 8YY.

2°. To find the Number of Feet of the Root, I reduce this 8YY into Yard-Feet, and adding to them

24YY of 6YI 3YL
16

8 24
5 2 4

2YY 1Yf 2YI
86

2YY 1Yf 2YI 3YL
0 0 0 0

4Y	2f	9I
8Y	2f	
2YY	2Yf	8YI
2	2	8
5YY	2Yf	4YI
9Y	1f	9I
2YY	1Yf	2YI 3YL

the Yf in the proposed Number, there results 24Yf, which I divide by Double of 4Y, or by 8Y, and set down the Quotient in the Root, and to the right Hand of 8Y; cancelling the 24Yf that I have no further Occasion for, then multiplying 8Y 2f by 2f, and the Product 5YY 2Yf 4YI subtracted from 8YY of 6YI, leaves 2YY 1Yf 2YI for a Remainder. Continuing on the Operation, as in the Margin, I find 4Y 2f 9I for the Side required of the Square.

CXXX.

Let it be proposed to find the Side of a Cube, whose solid Content is 118YYY 2YYf 6YI 8YI 9YI.

Extraction
of the Cube
Root of
compound
Numbers,
explained by
an Example

1°. I extract the Cube Root of 118YYY, and I find 4Y for the first Part of the Root, with a Remainder 54YYY.

2°. To find the Number of Feet of the Root, I reduce this Remainder into Yard-

118YYY 2YYf 6YI 8YI 9YI
64

54 164

101YYY 1YYf 10YI 8YI

17YYY 0YYf 8YI 0YI 9YI
620

118YYY 2YYf 6YI 8YI 9YI
0 0 0 0 0

Yard-Feet, and continuing on the Operation, as in the Margin, I find 4Y 2f 9I for the Side required of the Cube.

4Y	2f	9I
48YY		
65YY	1Yf	

C H A P. III.

Of Proportion, and the principal Rules of Arithmetick which depend thereon.

THE Comparifon of one Quantity with another, is called *Ratio*. I. What is meant by a Ratio.

When two Quantities are compared, if it be confidered how much the one is greater than the other, and what is their Difference; this Difference is called their *Arithmetical Ratio*. For Example, comparing 12 with 3, if it be confidered how much 12 is greater than 3, or that 3 is exceeded by 12 by 9 Units; thofe 9 Units which is the Difference between 12 and 3, or 3 and 12, is the Arithmetical Ratio of 12 to 3. But when two Quantities are compared, it be confidered how many Times the one is contained in the other, this *Number of Times* is called their *geometrical Ratio*. For Example, comparing 12 with 3, if it be confidered that 12 contains 4 Times 3, or that 3 is contained 4 Times in 12, this Number 4 is called the geometrical Ratio of 12 to 3. Difference between an arithmetical Ratio and a geometrical Ratio.

From thefe two Definitions of Ratios, it follows that only Quantities of the fame Kind can have a Ratio to one another.

1°. The *Arithmetical Ratio* being the Difference of two Quantities, this Ratio or Difference cannot be obtained, but by fubtracting the leffer from the greater; whence the leffer Quantity fhould be a Part of the greater, and confequently of the fame Kind with it. Quantities of the fame Kind can only have a Ratio to one another

2°. The *geometrical Ratio* of two Quantities, being the Number of Times that one contains the other, fupposes manifeftly that the leffer is a Part of the greater, and confequently of the fame Species with it.

II.

From what precedes it is alfo eafy to conclude, that the Arithmetical Ratio of two Quantities is a Quantity of the fame Species with thofe which are compared; becaufe the arithmetical Ratio being the Difference of the two Quantities compared, or the Excefs of the greateft above the leaft, is neceffarily a Part of the greateft, and is confequently of the fame Kind with it. The arithmetical Ratio is of the fame Species with the Quantities compared.

It is not fo with refpect to the geometrical Ratio. This Ratio is always an abftract Number, becaufe it represents a Number of Times, that is, the Number of Times, that one of the two Magnitudes compared, contains the other. The geometrical Ratio is always an abftract Number.

III.

Since the geometrical Ratio of two Quantities is the Number of Times that one contains the other; and that this Number of Times is found

The geometrical Ratio is expressed by the Quotient of the Division of the Terms of the Ratio one by the other.

by dividing one by the other ; it is manifest that the geometrical Ratio of two Quantities, is the Quotient arising from the Division of one of the Quantities by the other. For Example, the Ratio subsisting between 12 and 3, is the Quotient of the Division of 12 by 3.

When two Quantities are compared, for Example, 12 and 3, the Quantity 12 expressed or set down the first, is called the *Antecedent*, and the other 3 is called the *Consequent* ; if 3 was compared with 12, the Number 3 expressed the first, is the *Antecedent* of the Ratio, and the other Number 12 is the *Consequent*.

Because the geometrical Ratio of two Quantities is the Quotient of the Division of one by the other, the two Terms of a Ratio may be set down after the Manner of a Fraction, that is, the *Antecedent* may be set down above a small Line, and the *Consequent* under it, for Example, the Ratio of 12 to 3, may be set down thus $\frac{12}{3}$, which denotes 12 divided by 3, or rather the Quotient of 12 divided by 3, and the Ratio of 3 to 12 is set down thus $\frac{3}{12}$ which denotes the Quotient of 3 divided by 12.

IV.

Two equal Ratios for Example. The Ratio of 2 to 3, and that of 4 to 6, form a *geometrical Proportion*. Whence a geometrical Proportion consists of 4 Terms ; the first of which contains the Second as many Times as the Third contains the Fourth ; or of four Terms, of which the First is contained in the Second, as often as the Third is contained in the Fourth.

Two equal Ratios form a Proportion.

To represent a geometrical Proportion, for Example, that, composed of the two equal Ratios $\frac{2}{3}$ and $\frac{4}{6}$, the Computists set it down thus, $2 : 3 = 4 : 6$; which denotes that 2 is to 3 as 4 is to 6, or that 2 is contained in 3 as 4 is contained in 6.

The first and fourth Terms of a geometrical Proportion, are called the *Extremes*, and the second and third are called the *Mean Terms*.

V.

The fourth Term of a geometrical Proportion is found by multiplying the third Term by the Quotient of the second divided by the first.

To make this appear, we shall distinguish two Cases ; either the first Term is contained in the second, or the second is contained in the first. In the first Case, it is manifest that the fourth Term of a geometrical Proportion will be found by multiplying the third Term by the Number of Times it is contained in the fourth ; but from the Nature of Proportion, the third Term is contained in the fourth as many Times as the

Method of finding the fourth Term of a Proportion.

first is contained in the second, and this Number of Times that the first Term is contained in the second, is equal to the Quotient of the second Term divided by the first; wherefore the fourth Term of a geometrical Proportion is found, by multiplying its third Term by the Quotient of the second Term divided by the first.

For Example, if a geometrical Proportion begins by those three Terms $2 : 3 = 7 :$, and that the fourth Term is required. I divide the second Term 3 by the first Term 2; and there results for Quotient the Fraction $\frac{3}{2}$, which expresses the Number of Times that the first Term 2 is contained in the second 3, or that the third Term 7 is contained in the fourth sought. Whence, by multiplying 7 by $\frac{3}{2}$, the Product $\frac{21}{2}$ or $10 \frac{1}{2}$ will be the fourth Term required, and the whole Proportion will be $2 : 3 = 7 : 10 \frac{1}{2}$.

In the second Case it is manifest that the fourth Term of a Proportion will be found, by dividing the third Term by the Number of Times that the fourth is contained in it. But from the Nature of Proportion, the fourth Term is contained in the third as many Times as the second is contained in the first; and this Number of Times is equal to the Quotient of the Division of the first Term by the second. Wherefore the fourth Term of a geometrical Proportion will be obtained, by dividing the third Term by the Quotient arising from the Division of the first Term by the second, that is by a Fraction, having the first Term for Numerator, and the second for Denominator. But to divide by a Fraction, having for Numerator the first Term, and for Denominator the second Term, is to multiply it by the converse Fraction, having for Numerator the second Term, and for Denominator the first, and which is consequently the Quotient arising from the Division of the second Term by the first. Wherefore the fourth Term of a geometrical Proportion will be obtained by multiplying its third Term by the Quotient of the second Term divided by the first, as in the first Case.

Grounds of
this Method

For Example, if a geometrical Proportion begins by those three Terms $12 : 8 = 20 :$, and the fourth Term be required, I divide the first Term 12 by the second 8; and there results for Quotient the Fraction $\frac{3}{2}$, which will express the Number of Times that the second Term 8 is contained in the first 12, or that the fourth Term sought is contained in the third 20. Whence, by dividing 20 by $\frac{3}{2}$, the fourth Term required will be obtained. But to divide 20 by the Fraction $\frac{3}{2}$ is to multiply 20 by the converse Fraction $\frac{2}{3}$. Wherefore, the fourth Term of a Proportion, whose three first Terms are $12 : 8 = 20 :$, will be found, by multiplying the third Term 20 by the Fraction $\frac{2}{3}$, which is the Quotient of the second Term divided by the first; and this fourth Term being $13 \frac{1}{3}$, the whole Proportion will be $12 : 8 = 20 : 13 \frac{1}{3}$.

VI.

The fourth Term of a Proportion found by multiplying the third and second together, and dividing the Product by the first Term.

To multiply a Number by a Fraction, is to multiply by the Numerator of the Fraction, and divide the Product by the Denominator of the same Fraction; since therefore the fourth Term of a Proportion is found, by multiplying its third Term by a Fraction, having the second Term for Numerator, and the first for Denominator; this fourth Term will be obtained, by multiplying the third by the second, and by dividing the Product by the first Term; that is, the fourth Term will be equal to the Product of the mean Terms of the Proportion divided by the first Term.

For Example, if a geometrical Proportion begins by those three Terms $2 : 3 = 7 :$ and the fourth Term be required; the third Term 7 must be multiplied by the Fraction $\frac{2}{3}$; that is 7 must be multiplied by 2, and the Product divided by 3, which will give $\frac{14}{3}$ or $10\frac{2}{3}$ for the fourth Term required, whence the Proportion will be $2 : 3 = 7 : 10\frac{2}{3}$.

VII.

The Transposition of the mean Terms of a Proportion do not alter the fourth Term.

If there be three Terms of a geometrical Proportion given, for Example, those three Numbers $2 : 3 = 7 :$, the third 7 may be put in the Place of the second 3, and the second 3 in the Place of the third 7, thus $2 : 7 = 3 :$, without altering the fourth Term required.

For the fourth Term is equal to the Product of the Means, divided by the first, and in those two Arrangements $2 : 3 = 7 :$ and $2 : 7 = 3 :$, the Means being the same, as also the first Term, the Quotient of the Division of the Product of the Means by the first Term will be the same.

VIII.

In every geometrical Proportion the Product of the Means is equal to the Product of the Extremes.

Since the fourth Term of a geometrical Proportion, for Example, of the following $2 : 3 = 4 : 6$, is found by multiplying the third Term by the second, and dividing the Product by the first; it follows that the Product of the Extremes of a geometrical Proportion is equal to the Product of the Means of the same Proportion, that is 6×2 and 4×3 are equal Products, as will appear when it is observed, that the fourth Term considered as the Product of the Means divided by the first Term, being multiplied by the first Term, will produce the Product of the Means; because the Division of the Product of the Means by the first Term is destroyed by the Multiplication by the first Term. Wherefore the Product of the Extremes of a geometrical Proportion is equal to the Product of the Means of the same Proportion.

IX.

The Product of the Extremes of a geometrical Proportion being equal to the Product of the Means, if those two Products be divided by an extreme or by a mean Term, it will appear that each extrem Term of

a Proportion is equal to the Product of the Means divided by the other Extream, and that each mean Term is equal to the Product of the Extreams divided by the other Mean. Wherefore when three Terms of a geometrical Proportion are given, and the Order in which they are disposed in the Proportion is known, the Term which is wanting in this Proportion may be found.

Any three Terms of a Proportion being given, how to find the Term that is wanting.

X.

When three Terms of a geometrical Proportion are given; the Operation to be performed to find the Term that is wanting in this Proportion, is called the *Rule of Three*; it is also called the *Rule of Proportion*, and by some Computists *The Golden Rule*, on account of its great Utility in Trade.

What is meant by the Rule of Three, or Golden Rule.

Having shewn how the Term which is wanting in a Proportion of which three Terms are given may be discovered; we shall now proceed to give several Examples of this Operation, and to explain how the given Terms are to be considered.

XI.

The Computists distinguish two Species of Rules of Three, the *Rule of Three direct*, and the *Rule of Three inverse*, which are both either *Simple* or *Compound*. Whence there are four Species of Rules of Three; the *Simple Rule of Three direct*, and the *Simple Rule of Three inverse*, the *Compound Rule of Three direct*, and the *Compound Rule of Three inverse*.

The different Species of Rules of Three.

The simple Rule of Three direct, is that whose three given Terms are the three first Terms of a geometrical Proportion. The Object therefore of this Rule is to find the fourth Term of a geometrical Proportion, of which the three first Terms are given.

The Simple Rule of Three Direct.

The simple Rule of Three inverse, is that in which three Terms are given, of which two are the Extreams of a geometrical Proportion, and the other a mean Term of the same Proportion; so that the Object of this Rule is to find a mean Term of a Proportion of which three Terms are given. But because the Computists do not put the unknown Term sought in its Place, but set down the three given Numbers one after the other, as if those three Numbers were the three first Terms of a Proportion; the last Ratio of the Proportion is inverted, when the unknown Term required is really a mean Term of the Proportion; and it is for this Reason that the Operation to be performed to find the mean Term sought, is called *Rule of Three inverse*.

The Simple Rule of Three Inverse.

The Compound Rule of Three is that in which more than three Terms are given. But all those given Terms may be always reduced to three, and the Term sought is always the fourth Term, or a Factor of

The Compound Rule of Three Direct and Inverse.

the fourth Term of a Proportion, when the Rule is direct, and is always a mean Term, or a Factor of a mean Term, when the Rule is inverse.

XII.

The Simple Rule of Three Direct explained by an Example.

It has been said that the Object of a *Simple Rule of Three direct*, is to find the fourth Term of a geometrical Proportion of which the three first Terms are given. For Example, in this Question,

If 37 Pieces of Cloth cost 148 £. what will 30 Pieces of the same Cloth be worth?

It is manifest that the Price of 30 Pieces of Cloth, which is the Answer to this Question, is the fourth Term of a geometrical Proportion, of which 37 Pieces, 30 Pieces and 148 £. are the three first Terms; because it is manifest that 37 Pieces of Cloth should contain 30 Pieces of the same Cloth, as the Price 148 £. of 37 Pieces, contains the Price required of 30 Pieces.

The like may be said of this other Question, which is the Converse of the foregoing one.

If 148 £. will buy 37 Pieces of Cloth, how many Pieces of Cloth may I buy for 120 £?

It is manifest that the Number of Pieces of Cloth, which is the Answer to this Question, is the fourth Term of a geometrical Proportion, of which the three first Terms are 148 £. 120 £. and 37 Pieces of Cloth; because the two Numbers of Pieces of the same Cloth should be proportional to the two Sums 148 £. and 120 £. which is to be paid for them, that is,

148 £. to be paid for 37 Pieces of Cloth, is to 120 £. to be paid for the Number sought of Pieces of the same Cloth; as 37 Pieces of Cloth, to the Number of Pieces required of the same Cloth.

XIII.

To perform those Rules of Three, that is, to discover the fourth Terms of those two Proportions, whose three first Terms are given, we have seen, (Art VI.) that the second Term must be multiplied by the Third, or the Third by the Second, and that Product divided by the first Term; but here there occurs a Difficulty which however is easily solved.

Of the three Terms given in a Rule of Three the two which are of the same Species should

In the Operation to be performed to find the fourth Terms of the Proportions proposed for Examples, we find concrete Numbers to be multiplied one by the other, which is contrary to the Rules of Multiplication, it having been proved that the Multiplier is always an abstract Number.

But if it be considered that the two first Terms whose Units are of the same Species, affect the fourth Term, but by the Number of Times

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that one contains the other, since to obtain the fourth Term, it suffices to multiply the third by the Quotient of the second, divided by the first, and that this Quotient is an abstract Number similar to that resulting from the Division of an abstract Number by another abstract Number, it will appear, that those two concrete Terms may be considered as abstract Numbers.

be considered as abstract Numbers.

In Effect, since 37 Pieces contains 30 Pieces after the same Manner that 37 abstract Units contains 30 abstract Units, and that 148 £. contains 120 £. after the same Manner that 148 abstract Units contains 120 abstract Units; it is manifest that the two Proportions proposed, as Examples may be reduced to the two following, whose two first Terms are abstract Numbers.

37 is to 30 as 148 £. to a fourth Term required,

And 148 is to 120 as 37 Pieces of Cloth is to a fourth Term required.

whence of the three Terms given in a Rule of Three, the two which are of the same Species should be considered as abstract Numbers.

XIV.

With Respect to the Units of the fourth Term which is the one required; it is manifest that those Units are of the same Species with those of the mean Term which is not considered as an abstract Number. For in order to find the fourth Term, the concrete mean Term must be multiplied by the other abstract mean Term, and that Product divided by the first, which is an abstract Number, and it has been proved that the Nature of the Units of a concrete Number is not altered whether it be multiplied or divided by an abstract Number.

The fourth Term required of a Proportion is of the same Species with that which is not considered as an abstract Number.

From what has been said it appears, that if the three first Terms of a Rule of Three be abstract Numbers, the last Term required will be also an abstract Number.

XV.

In order that the two first Terms of each Proportion might be of the same Species, and form a geometrical Ratio, we were under the Necessity of inverting the Order of the second and third Terms of the Rules of Three proposed as Examples. But it is manifest that this Inversion does not alter the Value of the fourth Term required, (Art. vii.) as it is therefore unnecessary for discovering the fourth Term, the Computists set down the Terms of the Rules of Three, in the Order they are expressed, even when the two first Terms are not of the same Species, and in this Case the first and third Terms, which are of the same Species, are considered as abstract Numbers.

for the first Term, and I had 814 for the Quotient, that is, for the Answer to the proposed Question.

XVI.

First Exam-
ple of the
Simple Rule
of Three
Direct.

If 37 Pieces of Cloth cost 148 l. what will 30 Pieces of Cloth be worth?

To perform this Rule of Three, I consider the first and third Terms as abstract Numbers, that is as if the Question had been proposed thus:

If 37 cost 148 £. what will 30 be worth?

I multiply therefore 148 £. by 30, which produces 4440 £. which I divide by the first Term 37, and there results 120 £. for the Quotient, or for the Value required of the 30 Pieces of Cloth.

$$\begin{array}{r} 148 \text{ £.} \\ \times 30 \\ \hline 4440 \text{ £.} \\ \div 37 \\ \hline 120 \text{ £.} \end{array}$$

XVII.

If 148 l. will buy 37 Pieces of Cloth, how many Pieces of the same Cloth will 120 l. buy?

I consider the first and third Terms, which are of the same Species, as abstract Numbers, and as if the Question was proposed thus:

If 148 Units will buy 37 Pieces of Cloth, how many Pieces of the same Cloth will 120 Units buy?

I multiply the second Term 37 Pieces of Cloth, by the third 120, which will produce 4440 Pieces of Cloth; and I divide this Product by the first Term 148, which gives 30 Pieces of Cloth, for the Answer to the proposed Question.

$$\begin{array}{r} 37 \text{ Pieces} \\ \times 120 \\ \hline 4440 \text{ Pieces} \\ \div 148 \\ \hline 30 \text{ Pieces} \end{array}$$

Second Ex-
ample of
the Simple
Rule of
Three Di-
rect.

XVIII.

If 7l. 13s. 4d. gained 20l. 6s. 9d. what will 30l. 13s. 4d. gain?

I consider the Pounds of the first and third Terms as abstract Numbers, and their Shillings and Pence as Fractions of abstract Units; so that 13s. 4d. being the two-thirds of a Pound, I operate as if the Question had been proposed thus.

If $7\frac{2}{3}$ gain 20l. 6s. 9d. what will $30\frac{2}{3}$ gain?

The Question being thus reduced to more simple Terms, I multiply the second Term 20l. 6s. 9d. by the third $30\frac{2}{3}$, and there results 623l. 13s. 8d. for the Product, which I divide by the first Term $7\frac{2}{3}$, and I find 81l. 7s. for the Quotient, that is, for the Answer to the proposed Question.

$$\begin{array}{r} 20 \text{ l. } 6 \text{ s. } 9 \text{ d.} \\ \times 30 \frac{2}{3} \\ \hline 623 \text{ l. } 13 \text{ s. } 8 \text{ d.} \\ \div 7 \frac{2}{3} \\ \hline 81 \text{ l. } 7 \text{ s.} \end{array}$$

Third Ex-
ample of the
Simple Rule
of Three
Direct.

XIX.

It often happens that the Shillings and Pence of the first and third Terms are not easily reduced into Fractions; in this Case, it will be more ready to reduce those Terms into Pence; for Example, in the proposed Question I might have reduced the first Term 7*£*. 13*s*. 4*d*. and the third 30*£*. 13*s*. 4*d*. into Pence, whereby the two Terms will be transformed into 1840*d*. and 7360*d*. and the proposed Question into the following:

*If 1840 Pence gained 20*l*. 6*s*. 9*d*. what will 7360 Pence gain?*

Considering the first and third Terms in this Question as abstract Numbers, I multiply the second 20*l*. 6*s*. 9*d*. by the third 7360, which produces 149684*l*. and divide this Product by the first Term 1840, and I find, as before, 81*l*. 7*s*. for the Quotient, and for the Answer to the proposed Question.

The three first Terms of the Proportion might have been reduced into Pence, and after the fourth Term was found in Pence, it might have been reduced into Pounds and Shillings; but this Method of operating occasions two useless Reductions.

XX.

Since to find the fourth Term of a Rule of Three, the second Term must be multiplied by the third, and their Product divided by the first, and that a Number divided by Unity gives a Quotient equal to this Number, it is manifest, that if Unity be the first Term of a Rule of Three, the fourth Term required will be found by multiplying the second Term by the third, or the third by the second: However, it is to be observed, that when the first Term is a concrete Unit, it is not to be suppressed as useless; because the concrete Unit of the first Term serves to determine the Nature of the Units of the Product of the two other Terms, by indicating which of the mean Terms of the Proportion should be considered as an abstract Number, viz. that which is of the same Species with it. For Example, let the two following Rules of Three be proposed.

*If 1 Yard of Silk cost 4*l*. 10*s*. 6*d*. what will 27 Yards cost at that Rate?*

*If 1*l*. will buy 27 Yards of Galloon, how many Yards will 4*l*. 10*s*. 6*d*. buy?*

Unity being the first Term, and the two other given Terms being the same in those two Rules of Three, viz. 27 Yards, and 4*l*. 10*s*. 6*d*. the fourth Terms of each of those Proportions will be found by multiplying 27 Yards by 4*l*. 10*s*. 6*d*. but the fourth Terms of those Rules of Three, tho' produced by the Multiplication of the same Terms, do not consist of the same Units; because the first and third concrete Terms

In what Cases the first and third Terms of a Rule of Three should be reduced to the lowest Denomination they contain.

If the first Term of a Rule of Three be a concrete Unit it is not to be suppressed as useless.

of the first Rule, consisting of Yards, and the same Terms of the second Rule consisting of Pounds, they are rendered abstract Numbers by suppressing the Denomination of *Yards* in the first Rule, and the Denomination of *Pounds* in the second. In the first Rule therefore the Product of 27 Yards into 4*l.* 10*s.* 6*d.* is reduced to 4*l.* 10*s.* 6*d.* repeated 27 Times, which gives 122*l.* 3*s.* 6*d.* for the fourth Term of this Rule; but in the second, the Product of 27 Yards into 4*l.* 10*s.* 6*d.* is reduced to 27 Yards repeated a Number of Times, expressed by $4\frac{1}{2}$ and $\frac{1}{20}$, or by $4\frac{21}{20}$, which gives 122 Yards $\frac{7}{20}$ for the fourth Term of this second Rule. Wherefore, when Unity is the first Term of a Rule of Three, and that consequently the fourth Term is found by multiplying the second by the third, the Unity that constitutes the first Term is not to be suppressed as useless, since it is the Species of this Unit which determines that of the fourth Term.

Whence if it be proposed to multiply a concrete Number by another concrete Number; for Example, 27 Y by 4*l.* 10*s.* 6*d.* Since from the Multiplication of those two Numbers there results Products consisting of Units of different Species, as appears from the foregoing Rules of Three; it is manifest that the Units of the Product of a similar Multiplication cannot be determined, and consequently the Multiplicand and Multiplier of a Multiplication cannot be both concrete Numbers.

XXI.

How a Rule
of Three In-
verse is per-
formed.

It has been said that a Rule of Three is inverse, when of the three given Terms there are two that are the Extremes of a Proportion, so that the Term required is a mean Term of the same Proportion. And as the Terms of the *Rule of Three* are expressed one after the other; in order to obtain this mean Term, the first Term must be multiplied by the second which in Reality is the fourth Term of the Proportion, and that Product divided by the third Term, as will appear by the following Example.

XXII.

If 30 Men can do a Piece of Work in 40 Days, how many Days must 10 Men require to do the same Work?

Example.

As 30 Men and 10 Men are to do the same Work, so much more Time is required as the Number of Men is less; that is, a Number of Men twice less, would require twice as much Time; a Number of Men three times less, would require three times more Time, &c. wherefore 30, which is the first Number of Men, will be to the second 10, as the unknown Time, required by the second Number of Men, will be to the Time, 40 Days, employed by the first Number of Men.

The Proportion being thus expressed, the Time required will be the third Term; the Value therefore of this Term will be obtained by mul-

tipling into one another the Extremes 30 and 40 Days, and dividing the Product by the mean Term 10 which is given, that is, the Time required will be $\frac{30 \times 40 \text{ Days}}{10}$, or 120 Days.

But the two Terms 30 and 40 Days that have been multiplied, are the first and second Terms of the proposed Rule of Three Inverse, whose three Terms, 30 Men, 40 Days, 10 Men, are set down one after the other; wherefore the unknown Term of a Rule of Three Inverse is found by multiplying the first and second Terms together, and dividing the Product by the third Term.

The foregoing Question, and all others of the same Kind, may be so stated, that its Terms will be the three first Terms of a geometrical Proportion, whence we are dispensed from considering two Species of Rules of Three. The Rule for placing the Terms is as follows; first set down the Quantity that is of the same Kind with the Quantity sought, then consider, from the Nature of the Question, whether that which is given is greater or less than that which is sought; if it is greater, then place the greatest of the other two Quantities on the left Hand; but if it is less, place the least of the other two Quantities on the left Hand, and the other on the right: Then shall the Terms be in due Order, as will appear by the following Example.

General Rule for performing a single Rule of Three, either direct or inverse.

XXIII.

If a Penny white Loaf ought to weigh 8 Ounces, Troy Weight, when Wheat is sold for 6s. 6d. the Bushel, what must it weigh when Wheat is sold for 4s. the Bushel? Example.

Because it is a Number of Ounces that is sought, I first set down 8, the Number of Ounces that are given; I easily see that the Number that is given is less than the Number that is sought, therefore I place 4 on the left Hand, and $6\frac{1}{2}$ on the right, and say $4 : 8 \text{ oz.} = 6\frac{1}{2} : \frac{6\frac{1}{2} \times 8}{4}$, or 13 oz. consequently $\frac{6\frac{1}{2} \times 8}{4}$, or 13 oz. is the Answer required to the proposed Question.

$$\begin{array}{r} 4 : 8 \text{ oz.} = 6\frac{1}{2} \\ 6\frac{1}{2} \\ \hline 48 \\ 4 \\ \hline 52 \left\{ \begin{array}{l} 4 \\ 13 \text{ oz.} \end{array} \right. \\ 4 \\ \hline 12 \\ 12 \end{array}$$

XXIV.

A compound Rule of Three is when the Question proposed contains more than three known Quantities; and though it consists of more than three Terms, it is still called Rule of Three, because it may be reduced to a Rule of three Terms resulting from the Multiplication of those expressed in the Question.

How a compound Rule of Three is reduced to a simple Rule of Three.

To reduce a compound Rule of Three to three Terms, the Computists consider it as consisting of two Causes, and of two Effects; set down under one another all the Terms which relate to the first Cause; they also set down under one another all those that compose the first Effect; they dispose in like Manner the Terms which belong to the second Cause, as also those which compose the second Effect: Observing to dispose alternatively the Causes and their Effects, and to begin by the first Cause, or by the first Effect, according as the Term sought belongs to the second Effect, or to the second Cause. They afterwards multiply together all the Quantities that compose each Term, whence results a simple Rule of Three, whose three first Terms are known Quantities, so that the fourth Term may be found (Art. XIII.)

But after the Terms of the compound Rule of Three are thus reduced, there may happen two Cases; either the Quantity sought will be the fourth Term of the Proportion, or only a Factor of this fourth Term. In the first Case, the Resolution of the Rule of Three (Art. XIII.) will manifestly give the Quantity required. In the second Case, when the fourth Term is found, it must be divided by the given Factors that enter into its Composition, to obtain the Quantity required. All which will be explained in the following Examples.

XXV.

If 20 Persons spend 99 l. in 15 Days, how much will 60 Persons spend in 25 Days?

The foregoing Method explained by an Example.

In this Question there are five Terms given to find a sixth Term, which is the unknown Expence of 60 Persons in 25 Days. Of those six Terms, there are two (20 Persons and 15 Days) which are the Cause of a first Effect, or of the first Expence 99 l. and two others (60 Persons and 25 Days) which are the Cause of a second Effect, or of a second unknown Expence. I therefore set down under one another, for a first Term, 20 Persons and 15 Days which compose the first Cause; I then set down for a second Term the first Effect 99 l. I afterwards set down under one another, for a third Term, the two Quantities 60 Persons and 25 Days which compose the second Cause, and the fourth Term will be the unknown Expence of 60 Persons in 25 Days.

1st Cause	1st Effect.	2d Cause.	2d Effect.
20 Persons 15 Days } 300 :	99 l.	60 Persons 25 Days } 1500 :	Unknown. £ 495 l.

As the Causes are proportional to their Effects, those four Terms, of which the last is unknown, compose a geometrical Proportion; whence

the fourth Term unknown, viz. the Expence of 60 Persons in 25 Days, will be found by multiplying the second Term by the third, and dividing the Product by the first.

But before this Rule of Three can be performed, we must find what the first and third Terms of the Proportion are reduced to. To this End I observe that 20 Persons will spend in 15 Days as much as 15 times 20 Persons in a Day; and that 60 Persons will spend in 25 Days as much as 25 times 60 Persons in a Day: So that the first and third Terms are reduced to those two Products, 15 times 20 Persons, and 15 times 60 Persons; wherefore the proposed Question will be solved by this Rule of Three.

If 15 times 20 Persons spend 99l. how much will 25 times 60 Persons spend?

The Rule being thus reduced, the fourth Term required will be found to be 495l. that is, the Expence of 60 Persons in 25 Days.

XXVI.

If 20 Persons spend 99l. in 15 Days, in how many Days will 60 Persons spend 495l.

In this Example, the Cause of the first Expence 99l. is composed of 20 Persons and of 15 Days, and the Cause of the second Expence 495l. is composed of 60 Persons and of a certain unknown Number of Days.

The unknown Number of Days sought being in the Cause of the second Expence, this Number of Days cannot be found in the fourth Term of the Proportion, except the Cause of the Expence be set down after the Effect; whence I set down the given Terms of the Question, as follows:

1st Effect.	1st Cause.	2d Effect.	2 Cause.
99l. }	20 Persons.	495l. }	60 Persons.
	15 Days.		Unknown Number
			of Days.
99	: 35 Days	= 495	: 4th Term.

As the two Quantities which produce each compound Term should be multiplied one by the other, and that the fourth Term is compounded of 60 Persons, and of the Number of Days required; this fourth Term must be investigated, and, when found, divided by 60, to obtain the unknown Number of Days required.

To discover this fourth Term, I consider all the other Terms, except 15 Days, as abstract Numbers; and after having multiplied 15 Days by 20, which produces 300 Days for the second Term, I multiply this

second Term 300 Days by the third 495, and there results 148500 Days for the Product. In fine, this Product being divided by the first Term 99, I find 1500 Days for the Quotient, and for the fourth Term of the Proportion; and as this Number 1500 Days is made up of the required Number of Days multiplied by 60, I divide 1500 Days by 60, and there results 25 Days for the Number of Days required.

XXVII.

Third Example of the compound Rule of Three direct.

If 60 Men, working 8 Hours a Day, can dig in 12 Days a Ditch 60 Yards long, 5 Feet broad, and 7 Feet deep; what will be the Length of a Ditch 4 Feet broad and 6 Feet deep, that 50 Men will dig in 15 Days in the same Soil, working 6 Hours a Day?

1°. It is manifest that 60 Men, 12 Days during which they work, and 8 Hours they are employed each Day, compose by their Multiplication the Cause of the first Ditch; because 60 Men will do 12 times more Work in 12 Days than they would do in one Day, 60 Men should be multiplied by 12; and as they would perform 8 times more Work by working 8 Hours a Day, than they would perform by working 1 Hour a Day; the first Product of 60 Men multiplied by 12 should be again multiplied by 8.

2°. The Ditch 60 Yards long, 5 Feet broad, and 7 Feet deep, being considered as a Parallelepiped, is an Effect compounded of those three Dimensions, 60 Yards, 5 Feet, and 7 Feet.

3°. Fifty Men, 15 Days during which they work, and 6 Hours they are employed each Day, compose by their Multiplication the Cause of the second Ditch.

4°. In fine, the second Ditch, composed of the Multiplication of its unknown Length by its Breadth and by its Depth, is the Effect of the second Cause.

I therefore dispose under one another the Factors of each Cause; I range likewise under one another the Factors of each Effect, as follows.

1st. Cause.	1st. Effect.	2d. Cause.	2d. Effect.
60 Men	60 Y	50 Men	Length required.
12 Days	5 F	15 Days	4 F
8 Hours	7 F	6 Hours.	6 F
5760	2100 Y	4500	4th Term.

I afterwards multiply together the Factors of the three first Terms, considering all those Factors as abstract Numbers, except that, 60 Yards of the first Effect, in order to obtain the Length of the Ditch in Yards; lastly, I multiply the second Term reduced 2100 Y by the third Term

reduced 4500, which produces 9450000, and dividing this Product by the first Term reduced 5760, there results 1640Y if 10L 6L for the fourth Term. But this fourth Term is the Length of the Ditch multiplied into the Product 4×6 , or 24 of its Breadth and Depth; wherefore dividing 1640Y if 10L 6L by 24, there results 68Y if 0L 11L 3Y for the Length required of the Ditch.

XXVIII.

Three Men are employed to make a Ditch; the first could make the Ditch in 11 Days, the second in 22 Days, and the third in 33 Days; it is required in what Time those three Men together would make the Ditch?

Here it is plain that the first Workman will dig the $\frac{1}{11}$ of the Ditch in 1 Day, the second Workman will dig $\frac{1}{22}$ of the same Ditch in 1 Day, and the third Workman will dig $\frac{1}{33}$ of the Ditch also in 1 Day, wherefore the three Workmen will dig $\frac{1}{11}$, $\frac{1}{22}$ and $\frac{1}{33}$ of the Ditch in a Day; adding together those three Fractions, or Parts of the Ditch, after having reduced them to the same Denominator, I find their Sum to be $\frac{3}{22}$ or $\frac{1}{6}$, that is, the three Workmen together will dig $\frac{1}{6}$ of the Ditch in a Day.

Fourth Example of the compound Rule of Three direct.

As those Workmen the longer they are employed the more Work they will perform, the Quantities of Work will be directly proportional to the Times in which they are performed; wherefore assuming the Ditch for Unity, I say,

As $\frac{1}{6}$ of the Ditch is to 1 which represents the whole Ditch; so is 1 Day, Time employed to dig $\frac{1}{6}$ of the Ditch, to the Number of Days that the three Workmen will employ to dig the whole Ditch.

Wherefore to find the Number of Days that the three Workmen will require to dig the Ditch, I multiply the third Term 1 Day by the second 1 representing the Ditch, which will produce 1 Day, and divide this Product 1 Day by the first Term, that is, by the Fraction $\frac{1}{6}$, or multiply it by 6, which will produce 6 Days for the Time required.

XXIX.

The Compound Rule of Three Inverse does not differ from the simple Rule of Three Inverse, except that the given Terms of the latter are simple, and those of the former arise from the Multiplication of several others; as will appear by the following Example.

The Compound Rule of Three Inverse explained by an Example

If 30 Men can finish a Piece of Work in 40 Days, when they work 8 Hours a Day, in how many Days will 10 Men finish the same Piece of Work, when they work 6 Hours each Day?

As the 30 Men and the 10 Men are to do the same Piece of Work, the 10 Men will require more Time in Proportion as their Number is

less, and the Number of Hours they work each Day is less; whence there results the following Proportion.

As 30 Men working 8 Hours, are to 10 Men working 6 Hours; so is the unknown Number of Days employed by the 10 Men, to the 40 Days employed by the 30 Men.

The first Term of this Proportion is reduced to 8 times 30 Men, or to 240 Men, and the second is reduced to 6 times 10 Men, or to 60 Men; because 30 Men working 8 Hours, perform the same Work that 8 times 30 Men working 1 Hour; and that 10 Men working 6 Hours, perform the same Work that 6 times 10 Men working 1 Hour; whence the Proportion is reduced to the following one.

240 Men, are to 60 Men; as the unknown Number of Days, is to 40 Days.

Whence the Number of Days required will be obtained by multiplying 40 Days by 240, and dividing the Product 9600 by 60; which gives 160 Days for the Answer to the proposed Question.

xxx.

Direct Method of solving Questions in the compound Rule of Three Inverse.

Any Question that falls under the compound Rule of Three inverse may be solved, by considering in it two Causes and two Effects, and comparing directly the Causes with their Effects.

In the proposed Question: — *If 30 Men, working eight Hours each Day, can do a Piece of Work in 40 Days; in how many Days will 10 Men do the same Piece of Work, when they work 6 Hours each Day?*

It is easy to observe that 30 Men, 8 Hours, and 40 Days, compose by their Multiplication the first Cause of the Piece of Work; and that 10 Men, 6 Hours, and the Number of Days sought, compose also by their Multiplication the Cause of the same Work which those 10 Men should perform. Now the two Effects, or the two Pieces of Work being equal, their Causes are equal; wherefore all the Numbers given being considered as abstract Numbers, except that of the Days, $30 \times 8 \times 40$ Days, or 9600 Days, are equal to 10 times 6, or 60 times, the Number of Days sought; whence the sixtieth Part of 9600 Days, that is, 160 Days, will be the Number of Days required. Wherefore when all the Terms of a compound Rule of Three Inverse can be reduced to two Causes which produce a same Effect, you may observe the following Rule: Multiply together all the Numbers that compose the first Cause, and divide the Product by the Product of the Terms that are given in the second Cause, and the Quotient will be the Term required.

xxxi.

Rule of Conjunction.

When it is proposed to join several Statings in the Rule of Proportion into one, and by the Relation that the several Antecedents have to their Consequents, to discover the Proportion between the first Antecedent

and the last Consequent, this Operation is called the *Rule of Conjunction*.

To perform a Rule of Conjunction, the Computists range the Antecedents in the left Hand Column, and the Consequents in the right Hand one; so that the first Antecedent and last Consequent, whose Antecedent is sought, is of the same Species, as also the second Consequent and third Antecedent: This Order being continued throughout the whole. They then divide the Product arising from the Multiplication of all the Consequents into one another, by the Product of all the Antecedents multiplied into one another, and the Quotient resulting from this Division gives the Antecedent required, as will appear by the following Examples.

XXXII.

Suppose 100lb. of Venice weighs 70lb. of Lyons, and 120lb. of Lyons 100lb. of Rouen, and 80lb. of Rouen 100lb. of Toulouse, and 100lb. of Toulouse 74lb. of Geneva, how many Pounds of Geneva will equiponderate 100lb. of Venice?

Let the Pound	lb.		lb.	
Weight of Venice	100 of Venice	=	70 of Lyons.	
be expressed by 1V,	120 of Lyons	=	100 of Rouen.	
that of Lyons by 1L,	80 of Rouen	=	100 of Toulouse.	
that of Rouen by 1R,	100 of Toulouse	=	74 of Geneva.	
that of Toulouse by 1T,	How many of Geneva	=	100 of Venice.	
that of Geneva by 1G, because	$\frac{70 \times 100 \times 100 \times 74 \times 100}{100 \times 120 \times 80 \times 100}$	=	$53 \frac{23}{24}$ lb.	
$100 \times 1V = 70 \times 1L,$				

Application
of the Rule
of Conjunction
to an
Example.

$120 \times 1L = 100 \times 1R, 80 \times 1R = 100 \times 1T, 100 \times 1T = 74 \times 1G$, it follows, that
 $1V : 1L = 70 : 100, 1L : 1R = 100 : 120, 1R : 1T = 100 : 80, 1T : 1G = 74 : 100$.

Wherefore multiplying in order all the Antecedents by the Antecedents, and the Consequents by the Consequents,

$1V \times 1L \times 1R \times 1T : 1L \times 1R \times 1T \times 1G = 70 \times 100 \times 100 \times 74 : 100 \times 120 \times 80 \times 100$,

consequently $\frac{1V \times 1L \times 1R \times 1T}{1L \times 1R \times 1T \times 1G} = \frac{70 \times 100 \times 100 \times 74}{100 \times 120 \times 80 \times 100}$, which is reduced

to $\frac{1V}{1G} = \frac{70 \times 74}{120 \times 80} = \frac{7 \times 74}{12 \times 80} = \frac{7 \times 37}{6 \times 80} = \frac{259}{480}$. Whence

$1V : 1G = 259 : 480$, which denotes that the Pound Weight of Venice is to the Pound Weight of Geneva as 259 is to 480, or that the Pound Weight of Venice is only the $\frac{259}{480}$ Parts of the Pound Weight of Geneva. I now say, if a Pound Weight of Venice is reduced to $\frac{259}{480}$ of a Pound Weight of Geneva, what will 100 lb. of Venice be reduced to?

or $1 : \frac{259}{480} = 100 : \frac{259 \times 100}{480} = 53 \frac{23}{24}$ lb. that is $53 \frac{23}{24}$ lb. will equiponderate 100 lb. of Venice.

XXXIII.

If 1 French Crown be equivalent to 80 Pence of Holland, 415 Pence of Holland to 240 Pence English, 240 Pence English to 420 Pence of Hambourg, 64 Pence of Hambourg to 1 Florin of Frankfort, how many Florins of Frankfort are equal to 166 French Crowns?

Another
Example.

Let the French Crown be expressed by 1F, the Penny of Holland by 1H, that of England by 1A, that of Hambourg by 1h, the Florin of Frankfort by 1f, then,

1 French Crown	=	80 Pence of Hol.
415 Pence of Hol.	=	240 Pence Engl.
240 Pence English	=	420 Pence of Ham.
64 Pence of Hamb.	=	1 Florin of Frank.
How many F. of Frank.	=	166 French Crowns.

$$\text{or } \frac{80 \times 240 \times 420 \times 166}{415 \times 240 \times 64} = 210 \text{ F. of Frank.}$$

1F : 1H = 80 : 1, 1H : 1A = 240 : 415, 1A : 1h = 420 : 240, 1h : 1f = 1 : 64, hence $\frac{1F \times 1H \times 1A \times 1h}{1H \times 1A \times 1h \times 1f} = \frac{80 \times 240 \times 420}{415 \times 240 \times 64}$ or $\frac{1F}{1f} = \frac{80 \times 420}{415 \times 64} = \frac{8 \times 10 \times 105 \times 4}{415 \times 8 \times 4 \times 2}$
 $= \frac{1050}{830} = \frac{105}{83}$. Whence 1F : 1f = 105 : 83, and consequently

1F × 83 = 1f × 105, that is, 83 French Crowns are equivalent to 105 Florins of Frankfort; I now say, if 83 French Crowns are equivalent to 105 Florins of Frankfort, what will 166 French Crowns be equivalent to, or $83 : 105 = 166 : \frac{105 \times 166}{83} = 210$, that is, 210 Florins of Frankfort are equivalent to 166 French Crowns.

XXXIV.

Varieties in
Proportion.

Having fully explained the different Rules of Proportion, it remains to shew their Use in the common Affairs of Life and Commerce, which, for greater Perspicuity, the Computists range under distinct Heads.

XXXIV.

Things to
which the
Questions
in Interest
relate.

Interest is an Allowance made by the Borrower to the Lender for the Loan of Money, or any other Goods. In Questions relating to Interest there are four Things considered. 1°. The Sum lent, which is called the Principal. 2°. The common Standard at which the Interest is fixed, which by common Consent is at so much for the Forbearance of 100/. for 1 Year, and is called the Rate. 3°. The Time during which the Principal is lent out. 4°. The Amount of the Principal and Interest at the Expiration of the stipulated Time, which is simply called the Amount. Any three of these Things being given, we shall shew how the other may be found by the Rules of Proportion.

XXXVI.

CASE I. The Principal, Rate, and Time being given, to find the Interest. Multiply the Principal by the Rate, and that Product by the Time, the last Product divided by 100 is the Interest required, as will appear by the following Example.

Let the Interest, for Example, of 426*l.* 5*s.* 9*d.* lent out at 4 $\frac{1}{2}$ per Cent. for 6 $\frac{3}{4}$ Years be required.

To solve this Question I say, if 100*l.* in 1 Year gives 4 $\frac{1}{2}$ *l.* what will 426*l.* 5*s.* 9*d.* give in 6 $\frac{3}{4}$ Years? wherefore by the compound Rule of Three, $100 \times 1 : 4 \frac{1}{2} = 426,2875 \times 6 \frac{3}{4} : \pounds 129*l.* 9*s.* 8*d.* \frac{1}{4}$.

From the Principal, Rate, and Time given, to find the Interest.

Example.

XXXVII.

CASE II. The Amount, Rate, and Time being given, to find the Principal. Say, as the Amount of 100*l.* at the Rate and Time given, is to 100*l.* so is the given Amount to the Principal required.

Let it be required to find what Sum in ready Money is equivalent to 3998*l.* 12*s.* 10 $\frac{1}{4}$ *d.* due 3 Years and 145 Days hence.

To solve this Question, I first find the Amount of 100*l.* in 3 Years and 145 Days, at 3 per Cent. per Annum, saying $1 : 3,4 = 3*l.* : \pounds 10,2*l.*$ and adding 10,2*l.* to 100*l.* I say $110,2 : 100 = 3998,6427*l.* : \pounds 3628,53239*l.*$

From the Amount, Rate, and Time given to find an Principal.

Example.

XXXVIII.

CASE III. The Amount, Principal, and Time being given, to find the Rate of Interest. Say, as the Principal multiplied by the Time, is to the whole Interest, so is 100*l.* to the Rate per Cent. required.

Let it be required to find at what Rate of Interest will 500*l.* become 537*l.* 10*s.* in 1 Year and 6 Months?

To solve this Question, I say if 500*l.* in 1 Year and 6 Months gives 37*l.* 10*s.* what will 100*l.* in 1 Year give, whence

$$500 \times 1,5 : 100 = 37*l.* 10*s.* : \pounds 5*l.*$$

From the Amount, Principal, and Time given, to find the Rate of Interest.

Example.

XXXIX.

CASE IV. The Principal, Amount, and Rate of Interest being given, to find the Time. Say, as the Interest of the Principal for 1 Year at the given Rate is to 1 Year, so is the whole Interest to the Time required.

Let it be required to find in what Time 500*l.* will become 537*l.* 10*s.* at 5 per Cent. per Annum?

To solve this Question, I say if 100*l.* in 1 Year gives 5*l.* in what Time will 500*l.* give 37*l.* 10*s.* whence $5 : 100 \times 1 Y = 37,5 : x \times 500$; wherefore the Time required x equal $\frac{37,5 \times 100}{5 \times 500}$ equal 1 Year and 6 Months.

From the Principal, Amount, and Rate given, to find the Time.

Example.

XL.

Things to which the Questions in Discount relate.

Discount is an Allowance made for the Paying of Money before it falls due.

In Questions relative to Discount there are four Things to be considered. 1°. The Sum due. 2°. The Rate *per Cent. per Annum* of the Discount. 3°. The Time that the Payment is anticipated. 4°. The present Worth. Any three of these Things being given, the fourth will be found by the Rules of Proportion.

XLI.

From the Sum due, Rate and Time given to find the present Worth.

CASE I. The Sum due, Rate and Time being given, to find the present Worth. Say, as the Amount of 100*l.* at the Rate and Time given, is to 100*l.* so is the Sum due to its present Worth; because the present Worth is the Principal which laid out at Interest at the given Rate of the Discount, as long as the Sum is paid before it is due, will amount to the Debt.

Example.

*A Bill, for Example, of 2000*l.* which has 17 Months to run, is presented to the Bank to be discounted, Interest at 5 per Cent. how much Money is the Holder to receive?*

To solve this Question, I find the Amount of 100*l.* for 17 Months, which is $\frac{85}{12}$ *l.* and adding it to 100*l.* I say $\frac{1285}{12} : 100 = 2000 \text{ l.} : x$ 1867*l.* $\frac{181}{257}$.

XLII.

*A young Gentleman going to travel, deposits with a Banker 1500*l.* who furnishes him with Bills of Exchange to the Amount of that Sum, on Condition of his paying 3 per Cent. of the Money he shall receive. The Value of the Bills is required.*

To solve this Question, I say, if 103*l.* are reduced to 100*l.* what will 1500*l.* be reduced to? Whence $103 : 100 = 1500 \text{ l.} : 1456 \text{ l. } 6 \text{ s. } 2 \text{ d. } \frac{1}{2}$, which is the Value required of the Bills.

XLIII.

From the Sum due, present Worth and Rate given, to find the Time.

Example.

CASE II. The Sum due, present Worth and Rate of Interest being given, to find the Time. Say, as the Interest of the present Worth for 1 Year at the given Rate is to 1 Year, so is the whole Discount to the Time required.

*A Gentleman who contracted a Debt of 1344*l.* that becomes due a certain Time hence, discharges it by paying 1200*l.* in ready Money; how many Years did he anticipate the Payment, supposing the Discount to be 3 per Cent. per Annum?*

To solve this Question I say, if 100*l.* in 1 Year produces 3*l.* in what Time would 1200*l.* produce 144*l.* whence $3 : 100 \times 1 \text{ Y} = 144 : x \times 1200$; wherefore the Time required x equal $100 \times \frac{144}{3 \times 1200}$ equal $\frac{144}{36}$, or 4 Years.

XLIV.

CASE III. The Sum due, present Worth and Time being given, to find the Rate of Discount. Say, as the present Worth multiplied by the Time is to the whole Discount, so is 100*l.* to the Rate per Cent.

*A Gentleman who contracted a Debt of 2000*l.* payable in two Years, discharges it at the End of 7 Months by paying 1867 $\frac{13}{17}$ *l.* ready Money, what was the Rate per Cent. per Annum of the Discount?* *Ans.* 5*l.* per Cent.

From the Sum due, present Worth and Time given, to find the Rate.

XLV.

In weighing several Commodities, the Weight of the Package is included in the Weight of the Goods, and the whole upon that Account is called gross Weight; the Allowance for which is regulated either by Custom, or by some express Stipulation between the Buyer and Seller, and goes under the following Denominations.

What is meant by Tare and Tret.

1^o. *Tare*, which is an Allowance for the Weight of the Cask, Chest, Box, &c. in which the Goods are packed, allowed to be either so much per Bag, Barrel, Chest, &c. or so much per Cwt.

2^o. *Tret*, which is an Allowance for Dust contracted by keeping, Waste by Freight, Carriage, &c. When a Deduction is made for the Allowance of Tare from the gross Weight, the Remainder is called Nett, unless Tret is likewise allowed, and then it is called Subtle, out of which the Tret is deducted, and the last Remainder is called nett Weight.

XLVI.

When the Allowance is made at so much per Cwt. the Nett is found by the following Proportion: As 112*lb.* is to the Difference betwixt the given Tare or Tret, and 112*lb.* so is the given Gross to the Nett required.

Rule for deducting the Tare.

What is the Net of 410 Cwt. 2 qrs. 12 lb. at 20 per Cwt.?

To solve this Question, I subtract 20 from 112, the Remainder is 92, and then say, if 112*lb.* is reduced to 92*lb.* what will 410 Cwt. 2 qrs. 12*lb.* be reduced to? *Ans.* 337 Cwt. 1 qr. 4*lb.*

When Tare and Tret are both allowed, to find the Nett: Say, as 112 $\times$ 112 is to the Product of the Difference betwixt the given Tare, and 112 multiplied into the Difference of the given Tret and 112, so is the given Gross to the Nett required.

Rule for deducting Tare and Tret at one Operation, explained by an Example.

*Tare being allowed at 4*lb.* to 112*lb.* and Tret at 5*lb.* to 112*lb.* what is the nett Weight in 87*lb.* gross.?*

To solve this Question, I say as 112*lb.* is to 108*lb.* (viz. 112 less 4) so is 87*lb.* to the Subtle. I then say, as 112*lb.* to 107*lb.* (viz. 112 less 5) so is the Subtle to the Nett. Multiplying therefore the Antecedents of the Ratios of those two Proportions together, as also their Consequents together, 112 $\times$ 112 : 108 $\times$ 107 = 87 $\times$ Subtle : Nett $\times$ Subtle, that is, 112 $\times$ 112 : 108 $\times$ 107 = 87 : Nett.

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XL.

Things to
which the
Questions
in Discount
relate.

Discount is an Allowance made for the Paying of Money before it falls due.

In Questions relative to Discount there are four Things to be considered. 1°. The Sum due. 2°. The Rate *per Cent. per Annum* of the Discount. 3°. The Time that the Payment is anticipated. 4°. The present Worth. Any three of these Things being given, the fourth will be found by the Rules of Proportion.

XLI.

From the
Sum due,
Rate and
Time given
to find the
present
Worth.

CASE I. The Sum due, Rate and Time being given, to find the present Worth. Say, as the Amount of 100*l.* at the Rate and Time given, is to 100*l.* so is the Sum due to its present Worth; because the present Worth is the Principal which laid out at Interest at the given Rate of the Discount, as long as the Sum is paid before it is due, will amount to the Debt.

Example.

*A Bill, for Example, of 2000*l.* which has 17 Months to run, is presented to the Bank to be discounted, Interest at 5 per Cent. how much Money is the Holder to receive?*

To solve this Question, I find the Amount of 100*l.* for 17 Months, which is $\frac{85}{12}$ *l.* and adding it to 100*l.* I say $\frac{1285}{12} : 100 = 2000 \text{ l.} : x$ 1867*l.* $\frac{181}{257}$.

XLII.

*A young Gentleman going to travel, deposits with a Banker 1500*l.* who furnishes him with Bills of Exchange to the Amount of that Sum, on Condition of his paying 3 per Cent. of the Money he shall receive. The Value of the Bills is required.*

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XLIII.

From the
Sum due,
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Rate given,
to find the
Time.

CASE II. The Sum due, present Worth and Rate of Interest being given, to find the Time. Say, as the Interest of the present Worth for 1 Year at the given Rate is to 1 Year, so is the whole Discount to the Time required.

Example.

*A Gentleman who contracted a Debt of 1344*l.* that becomes due a certain Time hence, discharges it by paying 1200*l.* in ready Money; how many Years did he anticipate the Payment, supposing the Discount to be 3 per Cent. per Annum?*

To solve this Question I say, if 100*l.* in 1 Year produces 3*l.* in what Time would 1200*l.* produce 144*l.* whence $3 : 100 \times 1 \text{ Y} = 144 : x \times 1200$; wherefore the Time required x equal $100 \times \frac{144}{3 \times 1200}$ equal $\frac{144}{36}$, or 4 Years.

XLIV.

CASE III. The Sum due, present Worth and Time being given, to find the Rate of Discount. Say, as the present Worth multiplied by the Time is to the whole Discount, so is 100l. to the Rate per Cent.

A Gentleman who contracted a Debt of 2000l. payable in two Years, discharges it at the End of 7 Months by paying 1867 $\frac{1}{11}$ l. ready Money, what was the Rate per Cent. per Annum of the Discount? R 5l. per Cent.

From the Sum due, present Worth and Time given, to find the Rate.

XLV.

In weighing several Commodities, the Weight of the Package is included in the Weight of the Goods, and the whole upon that Account is called gross Weight; the Allowance for which is regulated either by Custom, or by some express Stipulation between the Buyer and Seller, and goes under the following Denominations.

What is meant by Tare and Tret.

1^o. *Tare*, which is an Allowance for the Weight of the Cask, Chest, Box, &c. in which the Goods are packed, allowed to be either so much per Bag, Barrel, Chest, &c. or so much *per Cwt.*

2^o. *Tret*, which is an Allowance for Dust contracted by keeping, Waste by Freight, Carriage, &c. When a Deduction is made for the Allowance of Tare from the gross Weight, the Remainder is called Nett, unless Tret is likewise allowed, and then it is called Subtle, out of which the Tret is deducted, and the last Remainder is called nett Weight.

XLVI.

When the Allowance is made at so much per Cwt. the Nett is found by the following Proportion: As 112lb. is to the Difference betwixt the given Tare or Tret, and 112lb. so is the given Gross to the Nett required.

Rule for deducting the Tare.

What is the Net of 410Cwt. 2qrs. 12lb. at 20 per Cwt.?

To solve this Question, I subtract 20 from 112, the Remainder is 92, and then say, if 112lb. is reduced to 92lb. what will 410Cwt. 2qrs. 12lb. be reduced to? R 337Cwt. 1qr. 4lb.

When Tare and Tret are both allowed, to find the Nett: Say, as 112X112 is to the Product of the Difference betwixt the given Tare, and 112 multiplied into the Difference of the given Tret and 112, so is the given Gross to the Nett required.

Rule for deducting Tare and Tret at one Operation, explained by an Example.

Tare being allowed at 4lb. to 112lb. and Tret at 5lb. to 112lb. what is the nett Weight in 87lb. gross.?

To solve this Question, I say as 112lb. is to 108lb. (viz. 112 less 4) so is 87lb. to the Subtle. I then say, as 112lb. to 107lb. (viz. 112 less 5) so is the Subtle to the Nett. Multiplying therefore the Antecedents of the Ratios of those two Proportions together, as also their Consequents together, 112 X 112 : 108 X 107 = 87 X Subtle : Nett X Subtle, that is, 112 X 112 : 108 X 107 = 87 : Nett.

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XLVII.

Rule of Mixtures.

It is usual in Trade to mix several Sorts of Wares together for the Convenience of Sale. The Operation to be performed, to proportion the Price of the Mixture to the several Prices of the Simples, or to find the Quantity of each Ingredient that will proportion the Mixture to a certain Price, is called the *Rule of Mixtures*.

XLVIII.

First Case.

The Quantities and Prices of the Ingredients given, to find the Rate.

When the Quantities as well as the Prices of the Ingredients are given, to find the Rate of the Mixture. Multiply the Value of an Unit of each Ingredient by the Number of Units of that Ingredient, whence there will result as many particular Products as there are Sorts of Wares to be mixed; divide the Sum of those Products by the Sum of the Ingredients, the Quotient will give the Rate of the Mixture, as will appear by the following Example.

XLIX.

First Example.

A Farmer mixes three Sorts of Grain of different Prices, viz. 10 Sacks of Wheat at 12s. per Sack, 8 Sacks of Wheat at 14s. per Sack, and 6 Sacks of Oats at 8s. per Sack; what is a Sack of the Mixture worth?

To solve this Question, I say, 1°. If 1 Sack cost 12s. what will 10 Sacks cost? $\text{R } 6\text{l.}$ 2°. If 1 Sack cost 14s. what will 8 Sacks cost? $\text{R } 5\text{l. } 12\text{s.}$ 3°. If 1 Sack cost 8s. what will 6 Sacks cost? $\text{R } 2\text{l. } 8\text{s.}$

Whence the 24 Sacks of Corn mixed together are worth 14l. I now say, if 24 Sacks of Corn mixed together are worth 14l. what will 1 Sack of the Mixture be worth? $\text{R } 11\text{s. } 8\text{d.}$

L.

Second Example.

If 278 Gallons of Rum, at 11s. 6d. per Gallon, were mixed with 174 Gallons, at 9s. 3d. per Gallon, what would a Gallon of the Mixture be worth?

Multiplying 11s. 6d. by 278, I find the Price of 278 Gallons, at 11s. 6d. per Gallon, to be 159l. 17s. Multiplying 9s. 3d. by 174, I find the Price of 174 Gallons, at 9s. 3d. per Gallon, to be 82l. 13s.

Consequently the 278 Gallons and 174 Gallons mixed together, or the 452 Gallons of Rum, are worth 242l. 10s.

Wherefore dividing 242l. 10s. the whole Price of the 452 Gallons of Rum, by 452, the Quotient 10s. 8 $\frac{1}{2}$ d. will be the Price of a Gallon of the Mixture.

LI.

Third Example.

A Goldsmith melted down together 60lb. of Silver of different Standards, viz. 32lb. of 110z. fine, 20lb. of 110z. 12dwt. fine, 8lb. of 100z. 10dwt. fine. The Standard of the Mixture is required.

If Silver is pure and free from Mixture, that is, if the 12 Parts into which the Pound of Silver is divided are fine, the Silver is said to be

12oz. fine. If the lb. is composed of 11 Parts of pure Silver, and of 1 of another Metal, the Silver is said to be 11oz. fine. If the lb. of Silver is composed of 10oz. and 10dwt. of pure Silver, and of 1oz. 10dwt. of another Metal, the Silver is said to be 10oz. 10dwt. fine, and so on. These Notions being premised, the Question is solved thus :

As there are 32lb. of 11oz. fine, 20lb. of 11oz. 12dwt. fine, 8lb. of 10oz. 10dwt. fine. Multiplying 11oz. by 32, there results 352oz. Multiplying 11oz. 12dwt. by 20, there results 232 oz. Lastly, multiplying 10oz. 10dwt. by 8, there results 84oz.

The 60lb. therefore will contain in all 668oz. fine ; whence dividing those 668oz. fine by 60, the Quotient 11oz. 2dwt. 16gr. will be the Quantity of fine Silver contained in a lb. of the Mixture, and consequently will be the Standard of this Mixture.

LII.

The particular Rates of two Simples to be mixed, and the Rate of the Mixture being given, to find how much of each Ingredient must be taken to compose the Mixture.

Form two Fractions that shall have for common Denominator the Difference of the Rates of the two Ingredients; the one having for Numerator the Difference between the mean Rate, and the lowest Rate will express the Portion to be taken of an Unit of the Ingredient of the highest Rate; and the other having for Numerator the Difference between the mean Rate, and the highest Rate will express the Portion to be taken of an Unit of the Ingredient of the lowest Rate, as will appear by the following Example.

LIII.

How much Wheat, at 13s. the Sack, and Rye at 10s. the Sack, will compose a Mixture that may be sold for 12s. the Sack ?

To solve this Question, let the three Sacks, that of 10s. the lowest Rate, that of 13s. the highest Rate, and that of the Mixture to be sold at the mean Rate 12s. be conceived to be divided into a Number of equal Parts.

The Difference 2s. between the mean Rate 12s. and the lowest Rate 10s. being double of the Difference 1s. between the highest Rate and the mean Rate; it is manifest that every Part that is taken of the Corn of the lowest Rate to compose the Mixture, will diminish the mean Rate twice more than each Part of the Corn of the highest Rate will increase it, we should therefore take two Parts of the Corn of the highest Rate for every one that we take of the lowest Rate, in order that the mean Price may not be altered, that is, of the three Parts that are taken in all to compose a Sack of Corn of the mean Rate, two should be of the Corn

ELEMENTS OF

of the highest Price, and one of the Corn of the lowest Price; whence the Sack of the Mixture to be sold for 12s. will consist of $\frac{2}{3}$ of the Sack at 13s. and $\frac{1}{3}$ of the Sack at 10s.

Now it is easy to perceive that the Denominator 3, common to the two Fractions $\frac{2}{3}$ and $\frac{1}{3}$, which express the two Portions of the Sack of the Mixture, arose from the Difference 3 in the Prices of the two Sorts of Corn, and that the Numerators 2 and 1 of those Fractions are the Differences between the mean Rate and the particular Rates.

LIV.

Let it be proposed to mix two different Substances, a cubical Foot of one of which weighs 650lb. and of the other 480lb. so as a cubical Foot of the Mixture shall weigh 500lb.

Second Example.

The heaviest cubical Foot weighing 650lb. the lightest 480lb. and the cubical Foot of the mean Weight weighing 500lb. the Difference between the lightest and the heaviest will be 170lb. the Difference between the mean Weight and the heaviest 150lb. the Difference between the mean Weight and the lightest 20lb. whence two Fractions having 170lb. or simply 170 for common Denominator, and 20lb. and 150lb. or simply 20 and 150 for their Numerators, will express the Portions to be taken of the two given cubical Feet to compose the cubical Foot required, weighing 500lb. that is,

1°. $\frac{20}{170}$ or $\frac{2}{17}$ will be the Part to be taken of the cubical Foot weighing 650lb.

2°. $\frac{150}{170}$ or $\frac{15}{17}$ will be the Part to be taken of the cubical Foot weighing 480lb.

LV.

Use of the foregoing Example in Gunnery.

The foregoing Question is of Use in Gunnery. The Metal employed in the Construction of Cannon is a Mixture of Copper and Block-Tin. Some Founders, to every 100lb. of Copper, add 11lb. of Block-Tin; others, to every 100lb. of Copper, add 12lb. of Block-Tin. This last Proportion being supposed to be the best, when old Pieces of Cannon, composed of those two Metals, are to be melted down, it will be necessary to determine the Quantity of each Metal they contain, in order to discover whether those Metals have been mixed in the Proportion, which Experience has proved to be the best.

It is easy to perceive, that if the Weight of a Quantity of Copper, as also of a Quantity of Block-Tin of the same Bulk with the Mixture, could be determined, the Quantities of each Metal in the Mixture would be easily found.

A heavy Body immersed in Water loses a Part of its Weight equal to the Weight of the Quantity of Water that it puts out of its Place; whence Bodies that lose, when weighed in Water, equal Parts of their

Weight, put equal Quantities of Water out of their Place, and consequently are of the same Bulk, agreeable to this Principle.

Let a Portion of the Mixture, weighing for Example 80*lb.* be taken; likewise a Piece of Copper and one of Block-Tin, each of the same Weight, and let the Portion of the Mixture, when weighed in Water, lose $9\frac{1}{3}$ *lb.* Copper losing in Water the 9th Part of its Weight, and Block-Tin the 7th Part, the Piece of Copper will lose $8\frac{2}{3}$ *lb.* and the Piece of Block-Tin $11\frac{1}{3}$ *lb.*

Now to find the Quantity of Copper and Block Tin separately that compose the Mixture, it suffices to find how much Water the Copper and Block-Tin should put out of its Place separately, so that the Mixture shall put $9\frac{1}{3}$ *lb.* out of its Place.

To find which, I take the Difference between $11\frac{1}{3}$ *lb.* and $8\frac{2}{3}$ *lb.* which is $2\frac{2}{3}$ or $\frac{8}{3}$, the Difference between $\frac{28}{3}$ and $8\frac{2}{3}$, which is $\frac{8}{3}$ or $\frac{28}{33}$, and the Difference between $\frac{28}{3}$ and $11\frac{1}{3}$, which is $2\frac{8}{33}$; wherefore the Fractions having for common Denominator 160, and for Numerators 28 and 132, will express the Parts of the $9\frac{1}{3}$ *lb.* of Water, which the Copper and Block-Tin in the proposed Mixture put out of its Place; wherefore the Quantity of Copper in the proposed Mixture is 66*lb.* and of Block Tin 14*lb.*

Having discovered after this Manner the Quantity of each Metal in the Portion of the Mixture, it will be easy to find how much of each the whole Mixture contains. Let the Piece, for Example, be a 24 Pounder, weighing 5100*lb.* to find the Quantity of Copper in this Piece. I say, if 80*lb.* of the Mixture contain 66*lb.* of Copper, how much will 5100*lb.* contain? $\approx 4207\frac{1}{2}$ *lb.* In like Manner the Quantity of Block Tin will be found to be $892\frac{1}{2}$, that is, $21\frac{1}{3}$ *lb.* of Block-Tin for every 100*lb.* of Copper; but as the Proportion of the Quantity of Copper and Tin should be as 100 : 12, I say, if 12*lb.* requires 100*lb.* what will $892\frac{1}{2}$ *lb.* require? $\approx 7437\frac{1}{2}$ *lb.* Consequently there should be added 3230*lb.* of Copper to the proposed Mixture, that it may be of the Quality required.

LVI.

The Total of the Compound of two Simples, with the total Value of that Composition, and the Value of an Unit of each Simple being given, to find the Quantity of each simple Ingredient in the Composition.

To find the Quantity of the higher priced Simple, 1^o. multiply the lesser Price of the Unit by the total Quantity of the Composition, deduct the Product from the total Value of the Composition, and divide the Remainder by the Difference in Value of an Unit of the two Simples given, and the Quotient is the Quantity of the higher priced Simple.

Third Case.
A Compound
of two Sim-
ples, its to-
tal Value,
and that of

an Unit of
each Ingre-
dient given,
to find the
Quantity of
each.

2^o. To find the Quantity of the lowest priced Simple, multiply the greatest Price of the Unit by the total Quantity of the Composition, deduct the Product from the total Value of the Composition, and divide the Remainder by the Difference in Value of an Unit of the two Simples given, and the Quotient is the Quantity of the lowest priced Simple; as will appear by the following Example.

LVII.

*Suppose there are 20 Ounces of Gold melted into one Mass, consisting of Gold at 4*l*. per oz. and Gold at 4*l*. 5*s*. per oz. the Value of the whole being 82*l*. it is required to find how much of each was taken to make the Composition?*

Example.

If the 20oz. consisted of Gold at 4*l*. per oz. the total Value of it would be 80*l*. and consequently 2*l*. less than the proposed Value 82*l*. This Product 80*l*. must therefore be increased by 2*l*. without increasing the Number of Ounces. Now it is manifest that it is this Increase of 2*l*. that is found by the Rule, when it directs to multiply the lesser Price of the Unit by the total Quantity of the Composition, and deduct the Product from the total Value.

As 4*l*. 5*s*. exceeds 4*l*. by $\frac{1}{4}$ of a Pound, every Ounce at 4*l*. 5*s*. that is taken in the Room of an Ounce at 4*l*. will increase the Product 80*l*. by $\frac{1}{4}$ *l*. without increasing the Number of Ounces; wherefore, in order to increase the Product 80*l*. of 20oz. at 4*l*. per oz. by 2*l*. we must take in the Room of the Gold at 4*l*. per oz. as many Ounces at 4*l*. 5*s*. as $\frac{1}{4}$ *l*. is contained in 2*l*. that is, we must take 8oz. of the Gold at 4*l*. 5*s*. per oz. Now it is this Number 8oz. that is found by the Rule, when it directs to divide the Remainder by the Difference in Value of an Unit of the two Simples given; wherefore there were 12oz. of Gold at 4*l*. per oz. and 8oz. at 4*l*. 5*s*. in the proposed Mixture.

2^o. If the 20 Ounces consisted of Gold at 4*l*. 5*s*. per oz. the total Value of it would be 85*l*. and consequently 3*l*. more than the given Value 82*l*. This Product 85*l*. must therefore be diminished by 3*l*. without increasing the Number of Ounces; to effect which, we must take as many Ounces of Gold at 4*l*. per oz. as $\frac{1}{4}$ *l*. is contained in 3*l*. the Difference in Value of an Unit of the Ingredients, that is, we must take 12oz. of the Gold at 4*l*. per oz. and consequently 8oz. of the Gold at 4*l*. 5*s*. per oz.

LVIII.

A Com-
pound of
any Num-
ber of Sim-
ples, its to-
tal Value,
and that of
an Unit of
each Ingre-

The Total of the Compound of any Number of Simples, with the total Value of that Composition, and the Value of an Unit of each Simple being given, to find the Quantity of each Ingredient in the Composition. Multiply the lowest Price of the Unit by the total Quantity of the Composition, and deduct this Product from the total Value of the Composition; divide the Remainder into as many Parts less one as there are Ingredients, divisible by the Differences between the lowest Price of the Unit, and all

the other Prices Those Parts being divided by those Differences, the Quotient will express the Quantities of each Ingredient, except the lowest priced one. dient given,
to find the
Quantity of
each.

LIX.

How many Pounds of Gun-Powder at 2s. 1s. and at 6d. per lb. must be taken to compound a Mixture of 22lb. worth 30s.?

If the 22lb. consisted of Powder at 6d. per lb. the total Value of it would be 132d. and consequently 228d. less than the proposed Value 360d. whence this Product 132d. must be increased by 228d. without increasing the Number of Pounds of the Mixture. Now this Augmentation of 228d. found by deducting the Product 132d. from the proposed Value 360d. cannot be made but by taking a Number of lb. of Powder at 2s. and at 1s. per lb. in the Room of a similar Number of lb. of Powder at 6d. per lb. Example, |

1°. Each lb. of Powder at 2s. taken in the Room of a lb. at 6d. will give an Augmentation of 18d. equal to the Difference in Value of an Unit of the highest rated and lowest rated Powder; whence the Part of the Augmentation produced by taking a Number of Pounds of Powder at 2s. per lb. in the Room of a like Number of Pounds at 6d. per lb. will be a Number of Pence multiple of 18d. and consequently divisible by 18d. Difference in Value of an Unit of the highest priced and lowest priced Powder.

2°. Every lb. of Powder at 1s. taken in the Room of a lb. at 6d. will produce an Augmentation of 6d. which is the Difference in Value of an Unit of the mean priced and lowest priced Powder; whence the Part of the Augmentation, produced by substituting a Number of lb. of Powder at 1s. per lb. in the Room of a like Number of lb. at 6d. per lb. will be a Number of Pence multiple of 6d. and consequently divisible by 6d. Difference in the Value of an Unit of the mean priced and lowest priced Powder.

We must therefore divide the Augmentation 228d. into two Parts, one of them divisible by the Difference 18d. in the Value of an Unit of the highest and lowest priced Powder, and the other divisible by the Difference 6d. in the Value of an Unit of the mean priced and lowest priced Powder.

The Parts of 228d. divisible by 18d. and the corresponding Parts of 228d. divisible by 6d. are

18, 36, 54, 72, 90, 108, 126, 144, 162, 180, 198, 216,
210, 192, 174, 156, 138, 120, 102, 84, 66, 48, 30, 12,

each lb. of Powder at 2s. substituted in the Room of a lb. at 6d. producing 18d. in the first Parts of 228d. and each lb. at 1s. substituted in the

ELEMENTS OF

Room of a *lb.* at *6d.* producing *6d.* in the second Parts of *228d.* it is manifest that if the first Parts be divided by 18, the Quotients will be the different Numbers of *lb.* of Powder at *2s.* that produce the first Parts, or that can compose the Mixture; and that by dividing the second Parts by *6d.* there will result the different Numbers of *lb.* of Powder at *1s.* per *lb.* that can compose the Mixture. Those Divisions being performed, the Number of *lb.* of Powder at *2s.* and at *1s.* per *lb.* will be as follows:

lb. at *2s.* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
lb. at *1s.* 35, 32, 29, 26, 23, 20, 17, 14, 11, 8, 5, 2,

As the Composition should consist but of *22lb.* it is manifest that the seven first Numbers of *lb.* at *2s.* per *lb.* which with the corresponding Numbers of *lb.* at *1s.* make more than *22lb.* should be rejected; and that the eighth Number *8lb.* at *2s.* which with the corresponding Number *14lb.* at *1s.* makes precisely *22lb.* should be also rejected with its Fellow *14lb.* since the Mixture should contain a Number of *lb.* at *6d.* whence there will remain but four different Numbers of *lb.* of Powder at *2s.* with four corresponding Numbers of *lb.* at *1s.* per *lb.* and as the whole Composition is limited to *22lb.* if from 22 be deducted each Sum made up of a Number of *lb.* at *2s.* together with the corresponding Number of *lb.* at *1s.* each Remainder will be the corresponding Number of *lb.* at *6d.* per *lb.* consequently the Particulars of the Mixture should be *9lb.* at *2s.* *11lb.* at *1s.* and *2lb.* at *6d.* per *lb.* Or, *10lb.* at *2s.* *8lb.* at *1s.* and *4lb.* at *6d.* per *lb.* Or, *11lb.* at *2s.* *5lb.* at *1s.* and *6lb.* at *6d.* per *lb.* Or, *12lb.* at *2s.* *2lb.* at *1s.* and *8lb.* at *6d.* per *lb.*

LX.

How the
Profit or
Loss made
upon any
Article in
Trade is
estimated.

In buying and selling Goods the Merchant should be able, by comparing both together, on the different Articles in which he deals, to make a true Estimate of his Trade, thereby to judge with Certainty what Articles turn out more or less to his Account. To effectuate which it was found necessary to have some common Standard, by which the Gain or Loss made, or proposed to be made upon any Commodity or Article in Trade, should be tried and expressed, and this by universal Consent seems to have been fixed to the *Centum* or Hundred; so that when it is said that the Gain is at *10 per Cent.* it is to be understood that when *100l.* *100s.* &c. have been laid out in purchasing Goods, *110l.* *110s.* &c. have been recovered by the Sales; and in the same Manner if *100l.* were laid out in the Purchase of Goods, and but *90l.* received back, it is said that *10l.* per Cent. was lost by such Goods.

LXI.

The buying and selling Prices being given, to find the Rate *per Cent.* *Case I.*
of the Gain or Loss. Say, as the buying Price is to the Difference be- The buying
tween the buying and selling Price, so is 100 to the Gain or Loss and selling
per Cent. Prices given
to find the
Rate.

A Merchant bought Cloth at 15s. 6d. per Yard, and sold it again for 18s. what did he gain per Cent.?

To solve this Question I say, if 15s. 6d. gained 2s. 6d. what will 100 Example.
gain? $\text{£ } 16 \frac{4}{3}$ *per Cent.*

A Merchant bought Cloth at 10s. per Yard, and sold it again for 9s. per Yard; what did he lose per Cent.? *Second Ex ample.*

I say, $10 : 1 = : 100 : \text{£ } 10$ *per Cent.*

A Merchant bought 5 Puncheons of Rum for 245l. 19s. 6d. and sold them for 295l. 19s. 6d. what did he gain per Cent.? *Third Ex ample.*

I say, if 245,975l. gain 50l. what will 100 gain? $\text{£ } 20 \frac{220}{239}$ *per Cent.*

A Merchant bought a Puncheon of Rum for 66l. 13s. 4d. it run 150 Gal- lons; which he retailed at 12s. 6d. per Gallon; whether did he gain or lose, how much, and at what per Cent.? *Fourth Ex ample.*

I say, as 66l. 13s. 4d. the buying Price, is to 27l. 1s. 8d. the Difference between the buying and selling Price, so is 100l. to $\text{£ } 40 \frac{1}{2}$, the Rate required *per Cent.*

LXII.

The buying Price, and the proposed Rate *per Cent.* Profit being given, *Case II.*
to find the advanced Price. Say, as 100 is to the Amount of 100 at the Rate given, so is the buying Price to the advanced Price required. The buying
Price and
the Rate per
Cent. Profit
given, to
find the ad-
vanced Price

A Merchant bought Cloth at 15s. per Yard; how may he charge it per Yard to gain 25 per Cent.?

I say, as 100 is to 125, so is 15s. to 18s. 9d. the advanced Price.

A Merchant bought 400 Spindles of Yarn for 41l. Carriage and other Charges came to 17s. how may he retail it per Spindle to clear 30 per Cent.? *Second Ex ample.*

I say $100 : 130 = 41,85l. : 54,405l.$ the advanced Price of the 400 Spindles, which is at the Rate of 2s. 8d. $\frac{1}{2}$ d. per Spindle.

A Merchant bought 80 Puncheons of Rum, containing 7548 Gallons, at 10s. 6d. per Gallon. Lighterage, Porterage, and other Charges came to 25l. 7s. There was of Leakage 25 Gallons. How may he sell the Remainder per Gallon to clear 12 $\frac{1}{2}$ per Cent. upon the whole? *Third Ex ample.*

To solve this Question, I first investigate the Price of the 7548 Gal- lons, at 10s. 6d. per Gallon, to which adding the Charges, I find the

first Cost to be 3988,05*l*. I then say, $100 : 112,5 = 3988,05*l* : 4485,43125*l*$. the advanced Price of the 7523 Gallons that remain after the Leakage is deducted, which is at the Rate of 11*s*. 11*d*. per Gallon.

Fifth Exam- ple. *A Merchant bought 17Cwt. 3qrs. of Sugar, at 60*s*. and 14Cwt. 2qrs. 14*lb*. at 70*s*. which he mixed together and proposes to sell it at 30 per Cent. advance, how must he value 1 Cwt. of the Mixture?*

To solve this Question, I first investigate the Prices of 17Cwt. 3qrs. of Sugar, at 60*s*. per Cwt. and of 14Cwt. 2qrs. 14*lb*. at 70*s*. per Cwt. and I find the first Cost of the Mixture 32,375Cwt. to be 104,4375*l*. I then say, $100 : 130 = 104,437 : 135,76875*l*$. the advanced Price of the Mixture, which is at the Rate of 4*l*. 3*s*. 10 $\frac{1}{4}$ *d*. per Cwt.

LXIII.

Case III.

The Rate given and another proportional to it required.
First Exam- ple.

If the Rate proportional to another upon a new advanced Price is required. Say, as the lowest advanced Price is to the Amount of 100, at the given Rate; so is the highest advanced Price to the Amount of 100 at the required Rate.

*A Merchant sold Tobacco at 7*d*. per *lb*. upon which he gained 10 per Cent. Markets rose to 8*d*. what did that advanced Price bring per Cent.?*

I say, $7 : 110 = 8*d*. : 125 \frac{1}{2}*d*. \text{ \text{or} } 25 \frac{1}{2} \text{ per Cent.}$

Second Exam- ple.

*A Merchant sold Cloth at 18*s*. per Yard, upon which he had 15 per Cent. How much per Cent. had his Neighbour, who sold his Cloth of the same Cost at 19*s*. 6*d*.? \text{ \text{or} } 24 \frac{1}{2} \text{ per Cent.}*

LXIV.

Case IV.

The advanced Price given, to find the prime Cost.
First Exam- ple.

If it be required to find the prime Cost. Say, as the Amount of 100 at the Rate given, is to 100; so is the advanced Price to the prime Cost required.

*A Merchant sold a Pipe of Wine for 43*l*. 15*s*. by which he had 20 per Cent. what was the prime Cost?*

I say, $120 : 100 = 43,75*l*. : \text{ \text{or} } 36*l*. 9*s*. 2*d*.$

Second Exam- ple.

*A Merchant sold 50 Pieces of Scotch Lawns, 10 Yards each, for 135*l*. by which he gained 15 per Cent. what did it cost him per Yard?*

I say, $115 : 100 = 135*l*. : 117,391*l*$. the prime Cost of the 50 Pieces, or 500 Yards of Lawn, which is at the Rate of 4*s*. 8 $\frac{1}{4}$ *d*. per Yard.

LXV.

What is meant by Barter.

Barter is the Commutation of one Commodity for another, and teacheth so to proportion the Quantities to be exchanged, according to the Conditions of the Barter, that neither Party may sustain Loss.

Method of solving Questions in Barter at one Operation.
First Exam- ple.

To solve Questions relating to Barter at one Operation. Say, as the Rate of the Quantity required per Yard, *lb*. Cwt. &c. is to the Rate of the assigned Quantity, so is the assigned Quantity to the Quantity required; as will appear by the following Example.

*How many Yards of Shalloon may a Merchant barter for 80 Yards of broad Cloth, rating the broad Cloth at 15*s*. 6*d*. per Yard, and the Shalloon at 2*s*.?*

I say, as 1 Yard of broad Cloth is to 15s. 6d. so is 80 Yards to x its Value. I then say, as 2s. is to 1 Yard of Shalloon, so is x the Value of the broad Cloth to y , the Number of Yards of Shalloon required. Consequently $1 \times 2 : 15\frac{1}{2} \times 1 = 80 Y \times x : y \times x$, or $2 : 15\frac{1}{2} = 80 Y : y$ equal 620 Y.

How many Yards of Irish Linen, at 2s. 3d. per Yard, may a Linen-Draper have in Barter for 80 Pieces of Holland, of 20 Yards each, at 3s. 6d. per Yard? Second Example.

I say, $2\frac{1}{4} : 80 \times 20 Y = 3\frac{1}{2} : x$ 2488 Yards.

How many Pieces of Indian Chints may a Merchant have in Barter for 86 Pieces of broad Cloth, rating the former at 25l. 10s. per Piece, and the latter at 15l. 15s. per Piece? Third Example.

I say, $25\frac{1}{2} : 86 \text{ Pieces} = 15\frac{3}{4} : x$ 53 $\frac{2}{7}$ Pieces.

How many lb. of Almonds may a Grocer have in Barter for 379lb. of Raisins, reckoning the Almonds at 7 $\frac{1}{2}$, and the Raisins at 3 $\frac{3}{4}$ per lb? Fourth Example.

I say $7\frac{1}{2} : 379 \text{ lb.} = 3\frac{3}{4} : x$ 189 $\frac{1}{2}$ lb.

LXVI.

Exchange is the Commutation of the Money of one Country for that of another, by Means of a Bill or Writ, commonly called a *Bill of Exchange*.

When the Purchases and Payments of one Country in another exactly balance the Purchases and Payments of the latter in the former, the Exchange on both Sides are at Par; that is, one who gives Money in one Country will receive as much in the other in Weight and Standard. From whence arises the Course of Exchange.

But if one Country supplies another with more than it takes from it, there will be a Balance against the latter, which it must necessarily pay; in order to which, the Demand for the Money, or Bills of Exchange of the former, becomes greater in the latter than the Quantity to supply that Demand, and consequently puts the Price of their Money above Par, and that of the other below Par, and this constitutes what is called the *Course or current Rate of Exchange*.

LXVII.

In Ireland, Accounts are kept in Pounds, Shillings, and Pence, Irish. Britain exchanges with Ireland upon the 100l. the Par of which in Irish Money is 108l. 6s. 8d. so that the Shilling Sterling is worth 13d. Irish. The Course of Exchange runs from 5 to 12 per Cent. according as the Balance of Trade is in Favour of Ireland, or against it. Manner of exchanging between England and Ireland.

If Irish Money is required; as, for Example, how much in Dublin for 357l. 18s. in London, when the Exchange is at 10 $\frac{1}{2}$ per Cent.?

Say, if 100l. English gives 110 $\frac{1}{2}$ l. Irish, according to the Course, what will 357l. 18s. English give? x 395l. 18s. 6 $\frac{3}{4}$ d.

If English Money is required; as, for Example, how much in London for 395l. 18s. 6 $\frac{3}{4}$ d. in Dublin, Exchange at 10 $\frac{1}{2}$ per Cent.?

Say, if 110 $\frac{1}{2}$ l. Irish, the Course of Exchange equivalent to 100l. English, gives 100l. English, what will 395l. 18s. 6 $\frac{3}{4}$ d. give?

x 357l. 18s. the English Money required.

Reduction of English Money to its equivalent Irish.

Reduction of Irish Money to its equivalent English.

Manner of
exchanging
with the
British Plan
tations.

In America and the West-Indies, as in other Parts of the British Dominions, Accounts are kept in Pounds, Shillings and Pence, divided as in Britain, and their Money, for Distinction's Sake, is called *Currency*. The Method of computing the Exchanges is the same as with Ireland.

LXVIII.

Exchange
applied to
Drawing
and Remit-
ting.

When Britain exchanges upon the 100*l*. the higher the Exchange is, the Advantage to Britain in remitting is the greater, and in drawing the less, as will appear by the following Examples.

*When the Exchange was at 12 per Cent. Britain remitted to Ireland 5000*l*. for how much Sterling ought Britain to draw for a Reimbursement, when the Exchange falls to 6 per Cent.?*

To solve this Question, I first say, if 100*l*. English amounts to 112*l*. Irish, what will 5000*l*. English amount in Ireland to? $\text{R } 5600\text{.}$ I next say, if 106*d*. Irish gives 100*l*. English, what will 5600*l*. Irish give? $\text{R } 5283\text{. os. } 4\frac{1}{2}\text{d.}$ English; whence Britain gains by this Transaction 283*l*. os. $4\frac{1}{2}\text{d.}$ which is above 5 per Cent.

*When the Exchange with Ireland rose to 12 per Cent. Ireland drew on London for 5000*l*. English, how much Irish must be remitted to London to discharge the Debt, when the Exchange falls to 6 per Cent.?*

To solve this Question, I first say, if 100*l*. English amount to 112*l*. Irish, what will 5000*l*. English amount to in Ireland? $\text{R } 5600\text{.}$ I afterwards say, if 100*l*. English amounts to 106*l*. Irish, what will 5000*l*. English amount to in Ireland? $\text{R } 5300\text{.}$ whence Ireland gains by this Transaction 300*l*. which is 6 per Cent.

LXIX.

Manner of
exchanging
between
England and
France.

In France Accounts are kept in Livres, Sols, and Deniers, divided as the British Pound. The Manner of exchanging between London and Paris, &c. is to give so many Pence Sterling for the Crown or Ecu of 3 Livres, or of 60 Sols Tournois. The Par of Exchange, according to the present Currency of the Coins in England and France, is 29,149 Pence English per Crown, or per 60 Sols Tournois. According to this Par, their Coins may be estimated as follows.

Estimation
of the
French
Coins.

1°. A Denier $\frac{1}{12}\text{d.}$ 2°. A Liard of three Deniers $\frac{1}{4}\text{d.}$ 3°. A Dardene of 2 Liards $\frac{1}{6}\text{d.}$ 4°. A Sol of 2 Dardenes $\frac{1}{3}\text{d.}$ 5°. A Frank of 20 Sols or 1 Livre $9\frac{1}{4}\text{d.}$ 6°. A Crown of Exchange or 60 Sols 2*s.* $5\frac{1}{4}\text{d.}$ 7°. A double Crown of 6 Livres 4*s.* 10*d.* 8°. A Louis d'or of 8 Crowns 19*s.* 6*d.* English Money.

Reduction
of English
Money to
its equivalent
French.

To compute the Exchanges with France. If French Money is required, as, for Example, how many Livres in Paris for 566*l.* 13*s.* 4*d.* in London, when the Exchange is at 31*d.* per Ecu?

Say, if 31*d.* the Course of Exchange, gives 3 Livres, what will 566*l.* 13*s.* 4*d.* English give? $\text{R } 1316\text{. Liv. } 5\text{Sol. } 9\frac{1}{2}\text{Den.}$

If English Money be required, as, for Example, how much in London for 13161 *Liv.* 5 *Sol.* 9 $\frac{2}{3}$ *Den.* in Paris, the Exchange being at 31*d.* per Ecu?

Say, if 3 *Livres* gives 31*d.* what will 13161 *Liv.* 5 *Sol.* 9 $\frac{2}{3}$ *Den.* give? $\text{R } 566\text{l. } 13\text{s. } 4\text{d.}$

LXX.

In Genoa Accounts are kept in Piaftres, or Pezzos, which are divided into Soldi and Denari, as the British Pound. Some of the Merchants keep their Accounts in Lires or Liras, Soldi and Denari, divided as before. This Money is only the $\frac{1}{2}$ of the Value of the other.

Manner of exchanging between England and Genoa.

The Manner of exchanging between London and Genoa, is to give so many Pence Sterling for the Pezzo or Piaftre, according to the Course of Exchange which runs between 45 and 50*d.* per Piaftre. The Par of the Piaftre is 54*d.* Sterling, and the Par of the Lire 10 $\frac{1}{2}$ *d.*

According to which their Coins may be estimated as follows: 1°. A Estimation Denari $\frac{1}{12}$ *d.* 2°. A Soldi or 12 Denari $\frac{1}{12}$ *d.* 3°. A Chevalet or 4 Soldi $\frac{1}{3}$ *d.* 4°. A Testoon or 30 Soldi 1*s.* 1 $\frac{1}{2}$ *d.* 5°. A Genouine or 6 Testoons 6*s.* 9*d.* 6°. A Pistole 15*s.* 7°. A Spanish Pistole 17*s.* 11*d.*

of the Coins of Genoa.

If Genoish Money is required, as, for Example, how much in Genoa for 1710*l.* 16*s.* 4*d.* in London, the Exchange at 47 $\frac{1}{2}$ *d.* Sterling per Pezzo?

Reduction of Sterling to its equivalent Money of Genoa.

Say, if 47 $\frac{1}{2}$ *d.* Sterling, the Rate of Exchange, gives 1 Pezzo, how many Pezzos will 1710*l.* 16*s.* 4*d.* give? $\text{R } 8644\text{Pez. } 2\text{Sol. } 6\text{Den.}$

If Sterling be required, as, for Example, how much in London for 8644*Pez.* 2*Sol.* 6*Den.* in Genoa, the Exchange at 47 $\frac{1}{2}$ *d.* per Pezzo?

Reduction of Money of Genoa to its equivalent Sterling.

Say, if 1 *Pez.* gives 47 $\frac{1}{2}$ *d.* what will 8644 *Pez.* 2*Sol.* 6 *Den.* give? $\text{R } 1710\text{l. } 16\text{s. } 4\text{d.}$ Sterling.

LXXI.

In Leghorn Accounts are kept in Piaftres, Soldi, and Denari, divided as at Genoa. Some likewise keep their Accounts in Liras or Lires, divided as the Piaftre; but this Money is only $\frac{1}{2}$ of the Money of Exchange. The Par with London is 4*s.* 4*d.* but the Course runs from 45 to 50*d.* Sterling only per Piaftre.

Manner of exchanging between London and Leghorn.

The Coins of Leghorn are estimated as follows. 1°. A Denari $\frac{1}{12}$ *d.* 2°. A Quatrini or 4 Denari $\frac{1}{3}$ *d.* 3°. A Soldi or 3 Quatrini $\frac{1}{3}$ *d.* 4°. A Craca, or Grain of 5 Quatrini $\frac{1}{3}$ *d.* 5°. A Julio, or Paulo of 8 Grains 5 $\frac{1}{2}$ *d.* 6°. A Piaftre of Exchange 4*s.* 4*d.* 7°. A Ducat of 150 Soldi 5*s.* 5*d.* 8°. A Pistole of 21 Lires 15*s.* 6*d.*

Estimation of the Coins of Leghorn.

When Money of Leghorn is required, as, for Example, how much in Leghorn for 465*l.* 19*s.* 6*d.* Sterling, the Exchange at 46*d.* per Piaftre?

Reduction of Sterling to its equivalent Money of Legh

Say, if 46*d.* gives 1 Piaſtre, what will 46*ſ.* 19*s.* 6*d.* Sterling give ?
 R 2431 Piaſt. 3*Sol.* 5*Den.*

Reduction
of Leghorn
Money to
its equiva-
lent Sterling

When Sterling is required, as, for Example, how much in London
for 2431 Piaſt. 3*Sol.* 5*Den.* Exchange at 46*d.* per Piaſtre ?

Say, if 1 Piaſt. gives 46*d.* what will 2431 Piaſt. 3*Sol.* 5*Den.* give ?
 R 465*l.* 19*s.* 6*d.*

LXXII.

Manner of
exchanging
between
England and
Venice.

The Accounts of the Bank of Venice are kept in Livres, Sols and Deniers Gros. The Livre is equal to 10 Ducats Bank, or 240 Groſs, (the Ducat being equal to 24 Groſs) 100 whereof make 120 Ducats current Money, ſo that the Difference betwixt Bank and current Money is an Agio of 20 per Cent.

London exchanges with Venice, by giving an uncertain Number of Pence Sterling for the Ducat Banco. The Par of a Ducat Banco is 4*s.* 4*d.* Sterling, and the Courſe between 45 and 50*d.*

Estimation
of the Venetian
Coins.

The Venetian Coins are as follow : 1°. A Picoli 1*ſ.* 1*d.* 2°. A Soldi or 12 Picoli 6*ſ.* 3°. A Jule or 18 Soldi 5*ſ.* 4°. A Teſtoon or 3 Jules 1*ſ.* 5*ſ.* 5°. A Ducat current or 124 Soldi 3*s.* 4*d.* 6°. A Chequin or 17 Lires 9*s.* 2*d.* Lire Money is divided as the Britiſh Pound, and 1 Ducat Banco is worth 7 $\frac{2}{3}$ Lires.

Reduction
of Sterling
to its equi-
valent Ven.
Money.

When Venetian Money is required, as, for Example, how much in Venice for 541*l.* 18*s.* Sterling in London, the Exchange at 45 $\frac{1}{2}$ *d.* per Ducat Banco ?

Say, if 45 $\frac{1}{2}$ *d.* the Rate of Exchange, gives 1 Ducat Banco, how many Ducats will 541*l.* 18*s.* Sterling give ? R 2850 Duc. 10*Sol.* 10 $\frac{3}{4}$ *Den.*

Reduction
of Ven. Mo-
ney to its
equivalent
Sterling.

When Sterling is required, as, for Example, how much in London for 2850 Duc. 10*Sol.* 10 $\frac{3}{4}$ *Den.* in Venice, Exchange at 45 $\frac{1}{2}$ *d.* per Ducat ?

Say, if 1 Ducat gives 45 $\frac{1}{2}$ *d.* what will 2850 Duc. 10*Sol.* 10 $\frac{3}{4}$ *Den.* give ? R 541*l.* 18*s.* Sterling.

LXXIII.

Manner of
exchanging
between
England and
Portugal.

In Liſbon, and in general throughout the Portugueſe Dominions, Accounts are kept in Milreas and Reas, reckoning 1000 of the latter to one of the former. The Exchange betwixt London and Portugal is rated at an uncertain Number of Pence Sterling for a Milrea, according to the Courſe of Exchange, which runs betwixt 5*s.* and 5*s.* 8*d.* per Milrea. The Par of a Milrea is 5*s.* 7 $\frac{1}{2}$ *d.* according to which the Gold Monies of Portugal are eſtimated as follow :

Estimation
of the Coins
of Portugal.

1°. The Piece of 25 $\cup$ 600 double Joannes 7*l.* 4*s.* 2°. Of 24 ditto 6*l.* 15*s.* 3°. Ditto of 12 $\cup$ 800 ſingle Joannes 3*l.* 12*s.* 4°. Of 12 ditto 3*l.* 7*s.* 6*d.* 5°. Ditto of 6 $\cup$ 400 half Johannes 1*l.* 16*s.* 6°. Ditto of 4 $\cup$ 800 Moldore ſtamped 1*l.* 7*s.* 7°. Ditto of 3 $\cup$ 200 quarter

Joannes 18s. 8^o. Ditto of 2 $\cup$ 400 half Moidore 13s. 6d. 9^o. Ditto of 1 $\cup$ 600 $\frac{1}{2}$ Joannes 9s. 10^o. Ditto of 1 $\cup$ 200 quarter Moidore 6s. 9d. 11^o. Ditto of 1 $\cup$ 800 $\frac{1}{8}$ Joannes or Testoon Piece 4s. 6d.

The Silver Monies are as follow: 1^o. The Crusado of 400 Reas not stamped 2s. 3d. 2^o. Ditto of 480 Reas stamped in 1643, 2s. 8 $\frac{1}{2}$ d. 3^o. The 12 Vintin Piece of 240 Reas 1s. 6d. 4^o. The 5 ditto of 100 Reas 9d. 5^o. The 2 $\frac{1}{2}$ Vintin, ditto of 50 Reas 4 $\frac{1}{2}$ d.

The Copper Coins as follow: The Vintin Piece of 20 Reas 1 $\frac{1}{2}$ d. the half and quarter ditto.

When Portugueze Money is required, as, for Example, how much in Lisbon for 578l. 19s. 6d. in London, Exchange at 5s. 3d. per Milrea?

Say, if 5s. 3d. gives 1 Milrea, how many Milreas will 578l. 19s. 6d. give? $\text{R } 2205 \text{ Mil. } 619 \text{ Reas.}$

When Sterling is required, as, for Example, how much in London for 2205 Mil. 619 Reas in Lisbon, Exchange at 5s. 3d. per Milrea?

Say, if 1 Milrea gives 5s. 3d. what will 2205 Mil. 619 Reas give? $\text{R } 578 \text{ l. } 19 \text{ s. } 6 \text{ d.}$

Reduction of Sterling to its equivalent Portugise Money.

Reduction of Portugise Money to its equivalent Sterling.

LXXIV.

In Spain the foreign Bankers, or Remitters, keep their Accounts in Piaftres, Rials, and Maravedies, old Plate, reckoning 34 Maravedies to a Rial, and 8 Rials to a Piaftre.

The Shopkeepers of Madrid, the Customhouse, and other Dealers within the Kingdom, keep their Accounts in Rials and Maravedies, Vellon or Copper Money. Some Merchants, particularly in Valentia, Alicant, &c. keep their Accounts in Piaftres, Sols and Denier, divided as the French Livre. The Exchange betwixt London and Spain is rated at an uncertain Number of Pence Sterling per Piaftre, or Piece of 8, according to the Course of Exchange, which varies from 35 to 40 Pence per Piaftre. The Par of the Piaftre is 3s. 7d. Sterling, according to which the Spanish Silver and Copper Coins are estimated as follows:

Manner of exchanging between England and Spain.

1^o. A Maravedie $\frac{1}{34}$ d. 2^o. A Quartil equal 2 Maravedies $\frac{2}{34}$ d. 3^o. A Rial Plate equal 17 Quartils or 34 Maravedies $5 \frac{1}{2}$ d. 4^o. A Pistrone equal 2 Rials Plate 10s. $\frac{1}{2}$ d. 5^o. A Dollar old Plate of Seville equal 10 Rials 4s. 6d. 6^o. A Dollar of new Plate equal 8 Rials Plate 3s. 7d. 7^o. Mexico ditto 4s. 6d. 8^o. Pillar ditto 4s. 6 $\frac{1}{2}$ d. 9^o. Peru ditto old Plate 4s. 5d. 10^o. A cross Dollar 4s. 4 $\frac{1}{2}$ d.

Estimation of the Spanish Coins.

The Gold Coins are Pistoles, Half Pistoles, &c. Double Pistoles, Quadruples. The Pistole is worth 4 Dollars or 17s. 11d. Sterling.

When Spanish Money is required, as, for Example, how much in Cadiz for 576l. 12s. 2 $\frac{1}{2}$ d. in London, Exchange at 37 $\frac{1}{2}$ d. per Piaftre?

Say, if 37 $\frac{1}{2}$ d. gives 1 Piaftre, how many Piaftres will 576l. 12s. 2 $\frac{1}{2}$ d. give? $\text{R } 3653 \text{ Piaft. } 6 \text{ Ri. } 7 \text{ Mer.}$

Reduction of Sterling to its equivalent Spanish Money.

Reduction
of Spanish
Money to
its equivalent
Sterling

When English Money is required, as, for Example, how much in London for 3653 *Piaſt.* 6 *Ri.* 7 *Mer.* Exchange at 37 $\frac{1}{2}$ *d.* per *Piaſtre*?

Say, if 1 *Piaſtre* gives 37 $\frac{1}{2}$ *d.* what will 3653 *Piaſt.* 6 *Ri.* 7 *Mer.* give?
£ 576*l.* 12*s.* 2 $\frac{1}{2}$ *d.*

LXXV.

Manner of
exchanging
between
England and
Holland.

In Holland Accounts are kept in Pounds, Shillings, and Pence Flemish, divided as the British Pound; but more generally in Guilders or Florins, Stivers, and Phinnings, reckoning 16 Phinnings to a Stiver, and 20 Stivers to a Guilder or Florin, and 6 Guilders or Florins to a Pound Flemish.

Britain exchanges with Holland upon the Pound Sterling, for which the latter gives an uncertain Number of Shillings and Pence, or Grotes Flemish, according to the Course of Exchange, which runs from 30 to 40*s.* Flemish per 20*s.* Sterling. The Par of a Pound Sterling is 1*l.* 16*s.* 6*d.* Flemish; but a Guinea passes in Holland for 12 Guilders. According to this Estimation their Coins may be reckoned, as follows:

Estimation
of the Dutch
Coins.

1°. A Duke 1 $\frac{1}{2}$ *d.* 2°. A Stiver 1 $\frac{1}{20}$ *d.* 3°. A Schilling 6 $\frac{1}{10}$ *d.* 4°. A Guilder 1*s.* 9*d.* 5°. A Zeland Dollar 2*s.* 7 $\frac{1}{2}$ *d.* 6°. A Rix-Dollar 4*s.* 4 $\frac{1}{2}$ *d.* 7°. A Dry Guilder 5*s.* 3*d.* 8°. A Ducat 9*s.* 2 $\frac{1}{2}$ *d.*

Reduction
of Sterling
to its equivalent
Dutch Money.

If Dutch Money is required, as, for Example, how much in Amsterdam for 270*l.* 8*s.* 2*d.* Sterling, when the Exchange is at 35*s.* 6*d.* Flemish per Pound Sterling?

Say, if 1*l.* gives 35*s.* 6*d.* Flemish, what will 270*l.* 8*s.* 2*d.* give?
£ 479*l.* 19*s.* 5 $\frac{1}{2}$ *d.* Flemish.

Reduction
of Dutch
Money to
its equivalent
Sterling

If Sterling is required, as, for Example, how much in London for 479*l.* 19*s.* 6*d.* Flemish, the Exchange being at 35*s.* 6*d.* per Pound Sterling?

Say, if 1*l.* 15*s.* 6*d.* Flemish gives 1*l.* Sterling, what will 479*l.* 19*s.* 6*d.* Flemish give? £ 270*l.* 8*s.* 2*d.* Sterling.

LXXVI.

Manner of
exchanging
between
England and
Hambourg.

In Hambourg Accounts are kept in Rix-Dollars, Sols, and Deniers Lubs, or in Marks, Sols, and Deniers Lubs. The Rix-Dollar is worth 3 Marks or 48 Sols Lubs, the Livre Gros or Pound Flemish is equal to 7 $\frac{1}{2}$ Marks Lubs, or 20 Sols Gros, or 120 Sols Lubs. The Mark Lubs are divided sometimes into 32 Gros, but more generally into 16 Schillings Lubs, and each of these into 12 Phennings.

The Exchanges betwixt London and Hambourg are rated at an uncertain Number of Schillings and Grotes Flemish per Pound Sterling, according to the Course of Exchange. The Par of their Rix-Dollar is reckoned at 4*s.* 6*d.* Sterling; so that the Par of 1*l.* Sterling is 13 Marks 5 Schillings Lubs, 35*s.* 6 $\frac{1}{2}$ *d.* Flemish; according to which Estimation their Coins may be estimated, as follows:

1°. A Tryling $\frac{1}{4}$ of a Phenning $\frac{1}{12}d.$ 2°. A Sexling $\frac{1}{2}$ of a Phenning $\frac{3}{4}d.$ 3°. A Phenning $\frac{1}{12}$ of a Schilling Lubs $\frac{3}{4}d.$ 4°. One Schilling Lubs $\frac{1}{8}$ of a Mark $1\frac{1}{2}d.$ 5°. The Dollar equal 2 Marks $3s.$ 6°. The Rix-Dollar equal 3 Marks $4s. 6d.$ 7°. The Ducat of $6\frac{1}{4}$ Marks $9s. 4\frac{1}{2}d.$

Estimation
of the Coins
of Ham-
bourg.

When Hambourg Money is required, as, for Example, how much in Hambourg for 500*l.* Sterling, the Exchange at 35*s.* 6*d.* Flem. Banco per Pound Sterling?

Reduction
of Sterling
to its equi-
valent Ham-
bourg Mo-
ney.

Say, if 1*l.* gives 35*s.* 6*d.* Flemish, what will 500*l.* give?
R 6656 Marks, 4 Schillings Lubs.

How much in London for 6656 Marks, 4 Schill. Lubs. in Hambourg, the Exchange at 35*s.* 6*d.* Flem. Banco per Pound Sterling?

Reduction
of Hamb.
Money to
its equiva-
lent Ster-
ling.

Say, if 35*s.* 6*d.* Flem. gives 1*l.* Sterling, what will 6656 Marks, 4 Schillings give? R 500*l.* Sterling.

The foregoing are the most remarkable Places of Exchange in Europe with which Britain hath Occasion to negotiate; and it is presumed that those Examples are sufficient, to shew how Sterling is reduced to its equivalent Value in the Money of Account in any other Country, and the contrary, the Course of Exchange being given.

LXXVII.

Being commissioned to remit a certain Sum of Money to a Place, at a certain Rate of Exchange, and at the same Time to draw for the Reimbursement upon some other Place, at a certain Rate of Exchange; to determine whether the Advantage in performing the one Part of the Commission will be sufficient to compensate for the Loss that may arise from the other.

Method of
arbitrating
Exchanges,
explained
by an Ex-
ample.

Say, as the Rate of Exchange assigned for remitting, is to the Rate of Exchange assigned for drawing, so is the real Rate at which the Remittance can be made, to the Rate at which the Draught ought to be made to be at Par with the Rate of the Remittance; which being compared with the Rate of Bills upon that Place where the Draught is to be made, will shew whether the Order is to be obeyed or not, as will appear by the following Example.

LXXVIII.

A Factor at London receives an Order to remit to Venice 1000 Ducats, at 4*s.* Sterling per Ducat, and, for this Purpose, to put himself in Cash by drawing on Spain at 3*s.* 2*d.* per Piastre. When this Order came to Hand, Bills for Venice were at 50*d.* at what Price must London draw upon Spain, to compensate the Advance on the Remittance to Venice by the Rise of the Exchange?

To solve this Question, let the Number of Piaſtres, at 38*d.* per Piaſtre, that the Factor is to draw from Spain, to put himſelf in Caſh to make the Remittance of 1000 Ducats, at 48*d.* per Ducat, be expreſſed by x ; then $1000 \times 48d.$ equal $x \times 38d.$ or $48 : x = 38 : 1000$. But ſince he is obliged to remit at 50*d.* per Ducat, and yet not draw on Spain for a greater Number of Piaſtres, the Piaſtres muſt be rated higher than at 38*d.* to answer this Remittance. Let this Rate be expreſſed by y , then $1000 \times 50d. = x \times y$, or $x : 50 = 1000 : y$; wherefore $48 \times x : 50 \times x = 38 \times 1000 : y \times 1000$, or $48 : 50 = 38d. y$ equal $39 \frac{1}{2}d.$

LXXIX.

What is
meant by
Stock.

By the Word Stock is meant any Sum of Money lent to Government, or to publick Companies, on Condition of receiving a certain Interest till the Money is repaid. Every Stock, or Fund of a Company, being raised for a particular Purpose, and limited to a certain Sum, when the Fund is compleated, no Stock can be bought of the Company, though Shares already purchased may be transferred from one Person to another; whence there is frequently a great Diſproportion between the original Value of the Shares, and what is given for them when transferred; for if there are more Buyers than Sellers, a Person who is indifferent about ſelling, will not part with his Share without a conſiderable Profit; and on the contrary, if many are diſpoſed to ſell, and few Purchaſes appear, the Value of Stocks will naturally fall in Proportion to the Impatience of thoſe who want to turn their Stock into Specie.

The differ-
ent Govern-
ment Funds,
and Stocks
of Compa-
nies.

The preſent Government Funds are, 1°. Three per Cent. reduced Bank Annuities. 2°. Three per Cent. conſolidated ditto. 3°. Three per Cent. ditto, 1726. 4°. Ditto, 1751. 5°. Three and a Half per Cent. ditto, 1756. 6°. Ditto, ditto, 1758. 7°. Four per Cent. ditto, 1760. 8°. Long Annuities, 1761. 9°. Four per Cent. Subscription, 1762. 10°. Long Annuities, 1762.

Stocks of Companies are, 1°. Bank Stock. 2°. South Sea Stock. 3°. India Stock. 4°. South Sea Annuities. 5°. India Annuities. 6°. India Bonds.

LXXX.

Computa-
tions in
Stock-job-
bing.

The Price of a Quantity of Stock bought or ſold is obtained by multiplying the Quantity by the Rate per Cent. and dividing the Product by 100; as will appear by the following Examples.

What muſt be paid for 135 Annuities $3 \frac{1}{2}$, at $87 \frac{1}{2}$ per Cent.?

To ſolve this Question I ſay, if 100 is reduced to $87 \frac{1}{2}$, what will 135 be reduced to? $\text{R } 118l. 2s. 6d.$

*What is the Value of 97*l.* 10*s.* Bank Stock, at 120 per Cent.?*

To ſolve this Question I ſay, if 100 becomes 120, what will 97*l.* 10*s.* become? $\text{R } 117l.$

3 per C.	
at 60	
61 $\frac{1}{2}$	
63	
64 $\frac{1}{2}$	
66	
67 $\frac{1}{2}$	
69	
70 $\frac{1}{2}$	
72	
73 $\frac{1}{2}$	
75	
76 $\frac{1}{2}$	
78	
79 $\frac{1}{2}$	
81	

LXXXI.

A TABLE, exhibiting at one View the intrinsic Value per Cent. of the several publick Funds, and the Proportion they bear to each other, by which any Person may know which it will be most advantageous to purchase, and what Proportion such Purchases bear to the Value of land-Ed Estates, and Life Annuities.

Table, exhibiting the intrinsic Value of the different publick Funds.

3 per C.	3½	4	4½	5	5½	6	Years Pur.	Ann. Inter. 5 per C.	3 per C.	3½	4	4½	5	5½	6	Years Pur.	Ann. Inter. 5 per C.
at 60	70	80	90	100	110	120	20	5 0 0	at 82½	96½	110	123½	137½	151½	165	27½	3 12 0
61½	71½	82	92½	102½	112½	123	20½	4 17 6	84	98	112	126	140	154	168	28	3 11 4
63	73½	84	94½	105	115½	126	21	4 15 2	85½	99½	114	128½	142½	156½	171	28½	3 10 2
64½	75½	86	96½	107½	118½	129	21½	4 13 0	87	101½	116	130½	145	159½	174	29	3 9 0
66	77	88	99	100	121	132	22	4 10 10	88½	103½	118	132½	147½	162½	177	29½	3 7 9
67½	78½	90	101½	112½	123½	135	22½	4 8 10	90	105	120	135	150	165	180	30	3 6 8
69	80½	92	103½	115	126½	138	23	4 6 11	91½	106½	122	137½	152½	167½	183	30½	3 5 7
70½	82½	94	104½	117½	120½	141	23½	4 5 1	93	108½	124	139½	155	170½	186	31	3 4 7
72	84	96	108	120	132	144	24	4 3 4	94½	110½	126	141½	157½	173½	189	31½	3 3 5
73½	85½	98	110½	120½	134½	147	24½	4 1 7	96	112	128	144	160	176	192	32	3 2 6
75	87½	100	112½	125	137½	150	25	4 0 0	97½	113½	130	146½	162½	178½	195	32½	3 1 6
76½	89½	102	114½	127½	140½	153	25½	3 18 5	99	115½	132	148½	165	181½	198	33	3 0 7
78	91	104	117	130	143	156	26	3 16 11	100½	117½	134	150½	167½	184½	201	33½	2 19 8
79½	92½	106	119½	132½	145½	159	26½	3 15 5	102	119	135	153	170	187	204	34	2 18 10
81	94½	108	121½	135	148½	162	27	3 14 0	103½	120½	138	155½	172½	189½	207	34½	2 18 0

To illustrate the foregoing Table by an Example, let it be supposed that 3 per Cent. Annuities may be purchased at 84 per Cent. and India Stock at 140. In the Column of 3 per Cent. find 84, and opposite there- to, in the Column of Interest, it will be found to produce 3*l*. 11*s*. 4*d*. In the Column of 6 per Cent. the Dividend on India Stock stands 168, which shews, that when 3 per Cent. give 84 per Cent. India Stock ought to give 168*l*. so that to purchase India Stock is, in this Case, by 28 per Cent. a better Bargain than 3 per Cent. Annuities.

The Use of the foregoing Table is explained by an Example.

LXXXII.

When it is proposed to divide a given Number into Parts proportional to those of another Number divided any how, the Operation to be performed to solve this Question is called a *Rule of Company*; because it is usually employed to divide Gains or Losses among Merchants in Company.

Rule of Company.

General
Rule for
performing
a Rule of
Company.

It appears from this Definition, that each of the Parts of the Number to be divided will be found by the following Operation of the Rule of Three.

As the Number already divided, is to one of its Parts; so is the Number to be divided, to one of its Parts corresponding to the Part that has been taken for second Term.

A Rule of Company may be either simple or compound. It is simple when the Rules of Three on which its Solution depends are simple, and is compound when those Rules are compound. As the different Rules of Three have been sufficiently explained, a few Examples will be sufficient to shew how every Rule of Company, either simple or compound, is to be performed.

LXXXIII.

Application
of the fore
going Rule
to an Exam-
ple.

Three Merchants, A, B, C, were concerned in a Store of Corn, whereof A had 3000 Sacks, B 2500 Sacks, and C 4500 Sacks. The Store was sold off for 12700l. what is the Dividend of each?

It is manifest that in order to solve this Question, we must divide 12700l. into Parts proportional to the Shares 3000 Sacks, 2500 Sacks, and 4500 Sacks of the Merchants in the Store; and that the whole Number of Sacks is to their Produce, as each particular Number of Sacks is to the Dividend of each Merchant.

I therefore add together the three Numbers of Sacks of Corn that compose the Store, and there results 10000 Sacks of Corn, which is worth 12700l. I then say,

1°. *If 10000 Sacks are worth 12700l. what will 3000 Sacks be worth?*
R 3810l.

2°. *If 10000 Sacks are worth 12700l. what will 2500 Sacks be worth?*
R 3175l.

3°. *If 10000 Sacks are worth 12700l. what will 4500l. Sacks be worth?*
R 5715l.

We might have found the Dividend of the third Merchant C without employing the Rule of Three; for adding together the Dividends of the two first Merchants, 3810l. and 3175l. and subtracting their Sum 6985l. from the Price of the whole Store 12700l. the Remainder 5715l. will be the Dividend of the third Merchant.

LXXXIV.

A more
commodi-
ous Method
of finding
the propor-
tional Parts.

When it is required to find a great Number of proportional Parts, it will be more commodious to investigate the Part of the Number to be divided that corresponds to an Unit of the Number already divided, and then multiply by this Part each Part of the Number already divided; by this Means all the proportional Parts required will be obtained, without employing as many Rules of Three as there are Numbers to be found.

LXXXV.

Suppose a Bankrupt's Effects should amount to 1739l. 13s. 8½d. what Dividend thereof will fall to each of the following Creditors in Proportion to their respective Sums? He owes to A 313l. 7s. 3d. to B 290l. 4s. 6d. to C 700l. to D 486l. 13s. 8d. to E 600l. to F 500l. to G 381l. 10s. and to H 418l.

To solve this Question, I first find the Dividend for 1l. by the following Rule of Three.

As the total Debt 3689l. 15s. 5d. is to the whole Subject 1739l. 13s. 8½d. so is 1l. to 0,4714875, the Dividend for 1l.

Whence the Dividend of A will be 313l. 7s. 3d. $\times$,4714875, that is, 147l. 14s. 11½d. The Dividend of B will be 290l. 4s. 6d. $\times$,4714875, that is, 136l. 16s. 9¼d. The Dividend of C will be 700l. $\times$,4714875, that is, 330l. 0s. 10d. The Dividend of D will be 486l. 13s. 8d. $\times$,4714875, that is, 229l. 9s. 3¼d. The Dividend of E will be 600l. $\times$,4714875, that is, 282l. 17s. 10¼d. The Dividend of F will be 500l. $\times$,4714875, that is, 235l. 14s. 10½d. The Dividend of G will be 381l. 10s. $\times$,4714875, that is, 179l. 17s. 5½d. And the Dividend of H will be 418l. $\times$,4714875, that is, 197l. 1s. 8d.

LXXXVI.

A Merchant put 40000l. into Trade. At the End of 6 Months a second Merchant takes a Share of 2500l. in the 40000l. and at the End of two other Months the first Merchant made over to a third a Part 3000l. of his Share that remained in the Capital 40000l. and at the End of 6 other Months there was a nett Gain of 18000l. Now it was agreed by the Parties, that each should have a Share of the Gain proportional to their Shares in the Capital, and the Time of their Continuance in Trade; the Dividend of each Merchant is required.

The compound Rule of Company, explained by an Example.

Since the Capital 40000l. has been 14 Months in Trade, and consequently has gained during those 14 Months as much as 14 times 40000l. would gain in one Month, 560000l. may be considered as the Cause of the whole Gain.

The second Merchant having, at the End of 6 Months, taken a Share of 2500l. in the 40000l. the 2500l. of this Merchant have been 8 Months in Trade, and consequently will have gained during 8 Months as much as 8 times 2500l. or 20000l. would have gained in a Month; whence 20000l. may be considered as the Cause of the Gain of the second Merchant.

The third Merchant having taken, at the End of two other Months, a Share of 3000l. in the 40000l. the 3000l. of this Merchant have been 6 Months in Trade, and consequently will have gained during 6 Months as much as 6 times 3000l. or 18000l. in one Month; whence 18000l. may be considered as the Cause of his Gain. But the Causes of

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The third Merchant having taken, at the End of two other Months, a Share of 3000l. in the 40000l. the 3000l. of this Merchant have been 6 Months in Trade, and consequently will have gained during 6 Months as much as 6 times 3000l. or 18000l. in one Month; whence 18000l. may be considered as the Cause of his Gain. But the Causes of

ELEMENTS OF

the Gains are proportional to the Gains; whence the Gain of the second Merchant will be found by the following Rule of Three.

As 560000l. reduced Cause of the whole Gain, is to the whole Gain 180000l. so is 200000l. reduced Cause of the Gain of the second Merchant, is to the Gain of this second Merchant; which will be found to be 7017l. 17s. 0½d.

The Gain of the third Merchant, who had 30000l. in Trade during 6 Months, will be found by the following Proportion.

As 560000l. reduced Cause of the total Gain, is to the total Gain 180000l. so is 180000l. reduced Cause of the Gain of the third Merchant, to the Gain of this Merchant; which will be found to be 578l. 11s. 5d.

To obtain the Gain of the first Merchant, there is no Rule of Three required; because, the second having gained 642l. 17s. 1¾d. and the third 578l. 11s. 5½d. deducting their Sum 1221l. 8s. 6¾d. from the total Gain, the Remainder will be the Gain of the first Merchant 16778l. 11s. 5½d.

LXXXVII.

The different Species of Rules of false Position.

The Rule of false Position resembles the Rule of Company, being employed to divide a given Number, or a Part of a given Number, into Parts proportional to those of another Number taken at Will, as the Nature of the Question may require.

The Computists distinguish two Species of Rules of false Position, viz. the Rule of one false Position, and the Rule of two false Positions.

In the Rule of one false Position there is only one Supposition made of Parts, proportional to those into which the proposed Number is to be divided; but in the Rule of two false Suppositions there are two Suppositions made, both false, whereby the true Parts of the proposed Number to be divided are discovered.

LXXXVIII.

In what the Rule of single false Position consists.

The Rule of one false Position consisting in assuming Parts proportional to those into which the proposed Number is to be divided, those supposed Parts form a false Position, as not being equal to the Parts into which the proposed Number should be divided; but as those supposed Parts are proportional to those required, the Totality of those supposed Parts is to each of them in particular, as the Number given to be divided is to each of its Parts required.

LXXXIX.

The Rule of single false Position explained by an Example.

A Father dying, left his Wife with Child, and an Estate worth 200000l. to whom he bequeathed, if she had a Son, $\frac{1}{3}$ of his Estate, and $\frac{2}{3}$ to the Son; but if she had a Daughter, $\frac{1}{3}$ to her, and $\frac{2}{3}$ to her Mother. It happened that she had both a Son and a Daughter; how shall the Estate be divided to answer the Father's Intention?

As the Father plainly designed the Son to have double of the Mother's Part, and the Mother double of the Daughter's Part, therefore for every 1 the Daughter got the Mother must have 2, and the Son 4; whence the Question is reduced to divide 20000*l.* into Parts proportional to 4, 2 and 1, saying, the Totality 7 of those Parts is to each of them in particular, as the Number 20000*l.* is to each of its Parts required. The Operation being performed, the Dividend of the Son will be found to be 11428 $\frac{4}{7}$ *l.* the Dividend of the Mother 5714 $\frac{2}{7}$ *l.* and the Dividend of the Daughter 2857 $\frac{1}{7}$ *l.*

XC.

An Army employed three Days to pass thro' a Town: The $\frac{1}{2}$ of the Army passed the first Day, the $\frac{1}{3}$ the second Day, and 6000 Men the third Day. The Number of Men the Army consisted of is required.

Let the whole Army be expressed by 1, the Part that passed thro' the Town the first Day will be expressed by $\frac{1}{2}$, and the Part that passed the second Day will be expressed by $\frac{1}{3}$. Those two Parts, reduced to the same Denomination, will be $\frac{2}{3}$ and $\frac{1}{3}$, whose Sum is $\frac{1}{2}$, which subtracted from the whole Army 1, leaves $\frac{1}{2}$ for the Part of the Army that passed thro' the Town the third Day. Now as the Parts are proportional to the wholes, I say, as $\frac{1}{2}$, the Number expressing the Part of the Army that passed thro' the Town the third Day, is to 1, the supposed Number expressing the whole Army; so is 6000 Men, the real Part of the Army that passed thro' the Town the third Day, to 36000, the real Number of Men that the Army consisted of.

Application
of the Rule
of single
false Posi-
tion to ano-
ther Exam-
ple.

XCI.

In a Rule of two false Positions, it is Question of dividing a Number into two Parts, and to divide again one of those Parts into Parts proportional to other supposed Parts. Those two Divisions require two false Suppositions, as will appear by the following Example.

*Let it be proposed to divide 120*l.* between three Persons, so that the first may have twice as much as the second and 3*l.* over, and the third as much as the two first and 4*l.* over.*

If the Share of the first was supposed to be 1*l.* the Share of the second would be 5*l.* and the Share of the third 10*l.* consequently the Totality of the Shares would be 16*l.* Here is the first Supposition, which is false, not only because the supposed Shares are not the real ones, but also because those Shares are not proportional to those into which 120*l.* is to be divided; for the two last supposed Shares include each two other Parts, one of which is relative to the first Share 1*l.* and the other is a determined Quantity.

The Rule
of double
false Posi-
tion ex-
plained by
an Example.

The second Share 5*l.* for Example, is composed of two Parts, 2*l.* and 3*l.*

one of which should be double of the first supposed Share, and will change its Value proportionally to the Variations made in the first Share 1*l*. whilst the second Part 3*l*. is a determined Magnitude, which will not be altered when the Value of the first Share 1*l*. is changed.

Each Share being thus considered as composed of two Parts, one of which is relative to the first supposed Share, and the other is a determined Quantity, we must examine which is the Portion of the Number 120*l*. containing the Parts of the Shares proportional to the first supposed Share, and which is the Portion of the same Number 120*l*. that contains the determined Parts of those Shares, and when this last Portion of 120*l*. is found, subtract it from 120*l*. in order to obtain the former.

To determine this second Portion of 120*l*. I make a second Supposition, in which the determined Parts 3*l*. and 4*l*. do not enter, that is, I consider the Question as if it was proposed thus :

*To divide 120*l*. among three Persons, so that the second shall have twice as much as the first, and the third as much as the two others.*

I suppose as before the Share of the first to be 1*l*. consequently that of the second will be 2*l*. and that of the third will be 3*l*.

This second Supposition is also false, not only because the supposed Shares are not the real ones, but likewise because they are not proportional to the Shares required.

As the Shares taken in this second Supposition do not contain the determined Parts 3*l*. and 4*l*. their Totality 6*l*. will not contain the Result of those two determined Parts, as the Totality 16*l*. of the Shares of the first false Position does.

Wherefore, if the Sum 6*l*. of the three Shares of the second Supposition be deducted from 16*l*. the Sum of the three Shares of the first Supposition, the Remainder 10*l*. will be the Portion for which the determined Parts 3*l*. and 4*l*. enter into the Sum 120*l*. to be divided; whence deducting 10*l*. from 120*l*. the Remainder 110*l*. will be the Portion that contains the Parts of the Shares relative to the first Share; whence to find the first Dividend,

I say, as the Sum 6*l*. of the three supposed Shares, is to the first Share 1*l*. so is 110*l*. to the first Dividend 18*l*. 6*s*. 8*d*.

The second Dividend being equal to twice the first, and 3*l*. over, will be 39*l*. 13*s*. 4*d*. and the third Dividend being equal to the two first, and 4*l*. over, will be 62*l*.

END OF NUMERAL ARITHMETICK.

E L E M E N T S O F S P E C I O U S A R I T H M E T I C K.

C H A P. I.

Of the analytick Method of expressing Problems by Equations, and of the Resolution of Equations of the first Degree.

AMONGST the different Problems which employed the first Mathematicians called Analysts, I chuse the following, as the most proper to shew how they formed the Science styled specious Arithmetick.

I.

To divide a Sum, for Example, £890 between three Persons, in such a Manner that the first may have £180 more than the second, and the second £115 more than the third.

It is thus I imagine a Person would have argued, who, without the least Tincture of specious Arithmetick, attempted to solve this Problem.

It is manifest that if one of the three Parts was known, the other two would be immediately discovered; let us suppose, for Example, the third which is the least to be known, we must add £115 to it, and this Sum will be the Value of the second; to obtain afterwards the first Part, we must add £180 to this second, which comes to the same as if we added £180 more £115 or £295 to the third.

Let therefore this third Part be what it will, we know that this Part, more itself together with £115, more itself again together with £295 should make a Sum equal to £890.

From whence it follows that the Triple of the least Part, more £115 more £295, or more £410, is equal to £890.

But, if the Triple of the Part sought more £410 be equal to £890, this Triple of the Part sought must be less than £890 by £410, therefore it is equal to £480, therefore the least Part is equal to £160. The second will consequently be £275, and the first or greatest £450.

It is probable the first Analysts argued in this Manner when they proposed to themselves Questions of this Nature, without doubt in proportion as they advanced in the Solution of a Problem, they burthened their Memories with all the Arguments which had conducted them to

Example of a Problem similar to those which the first Analysts might have proposed to themselves.

Solution of this problem such as might be found without specious Arithmetic.

ELEMENTS OF

the Point they had arrived at, and when the Problems were not more complicated than the foregoing, it was no difficult Matter; but as soon as their Researches presented a greater Number of Ideas to be retained, they were under the Necessity of having recourse to a more concise Method of expressing themselves, and of employing some simple Symbols, by Means of which however advanced they were in the Solution of a Problem, they might perceive at one View what they had done and what remained for them to do. Now the Kind of Language they imagined for this Purpose, is called specious Arithmetick.

H.

Algebraic
Method of
expressing
the forego-
ing Problem

To explain the Principles of this Science more clearly, we will resume the same Question, write down in Words the Arguments which the Analyst employs to solve his Problem, and in analytick Symbols what is requisite to assist his Memory.

The least or third Part, be it what it will, I denote by one Letter, for Example by x .

The Sign +
denotes ad-
dition.

The second consequently will be x more 115, which I denote thus, $x + 115$ employing the Sign + which signifies *more* to express the Addition of the two Quantities between which it is placed.

As to the first Part or greatest, since it exceeds the second by 180, it will be expressed by $x + 115 + 180$.

Adding those three Parts we will have $3x + 115 + 115 + 180$, or when reduced $3x + 410$.

The Sign =
designs E-
quality.

But this Sum of the three Parts should be equal to 890, which I express thus $3x + 410 = 890$, employing the Symbol = which signifies *equal* to denote the Equality of the two Quantities between which it is placed.

An equation
is the equa-
lity of two
quantities.

To solve an
equation is
to find the
value of the
unknown
quantity it
includes.

Resolution
of the equa-
tion which
expresses the
foregoing
Problem.

The symbol
— denotes
subtraction.

The Question therefore by this Computation is changed into another, where it is required to find a Quantity, the Triple of which being added to 410 makes 890. To find the Resolution of similar Questions, is what is understood by solving an Equation, the Equation in the present Case is $3x + 410 = 890$ it is so called, because it indicates the Equality of two Quantities, to solve this Equation, is to find the Value of the unknown Quantity x from this Condition that its Triple more 410 makes 890.

III.

To solve this Equation the Analyst argues and writes down his Arguments as follows. The Equation to be solved $3x + 410 = 890$, teaches us that we are to add 410 to $3x$ to make up the Sum 890, wherefore $3x$ are less than 890 by 410, which I express thus $3x = 890 - 410$ employing the Sign — which signifies *less* to denote that the Quantity which it precedes should be subtracted from that which it follows.

From this new Equation $3x = 890 - 410$, we deduce, by subtracting in effect 410 from 890, this other Equation $3x = 480$.

SPECIOUS ARITHMETICK.

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But if three x be equal to 480, one x will be the third Part of 480 or 160, which I write down thus, $x = \frac{480}{3} = 160$, and the Question is solved, since it suffices to know one of the Parts to discover the rest.

IV.

If we were desirous of solving the Question by investigating the greatest Part first, we might effect it in like Manner.

Another solution of the foregoing Problem.

Let this first Part be expressed by y , the second having 180 less, will be $y - 180$, and the third having 115 less than the second, will be $y - 180 - 115$.

Now the Sum of those three Quantities is $3y - 180 - 180 - 115$, that is, $3y - 475$.

But this Sum should be equal to 890.

We have therefore the Equation $3y - 475 = 890$, by which we learn that $3y$ exceeds 890 by 475, since we must subtract 475 from $3y$ to obtain 890. Wherefore $3y = 890 + 475$, or $3y = 1365$. Wherefore y or the greatest Part = 455, as found above.

V.

If in the Problem it was proposed to divide a Sum greater or less than the one employed, and that the Differences were Numbers different from those made use of, it is manifest it would be solved after the same Manner. Let us suppose, for Example, the Problem to have been expressed thus.

To divide 9600 between four Persons, in such a Manner that the first may have 300 more than the second, and the second 250 more than the third, and the third 200 more than the fourth.

Another example of the foregoing Problem.

We would have argued after the following Manner: Let the fourth Part be x , the third will be $x + 200$, the second $x + 200 + 250$, and the first $x + 200 + 250 + 300$.

But the Sum of all those Parts should be equal to 9600, we have therefore the Equation $4x + 1400 = 9600$.

To solve this Equation, I observe, as in the foregoing, that if to $4x$ 1400 is to be added to make it equal to 9600, it must be equal to what remains of 9600 when 1400 has been subducted from it, which is wrote down thus, $4x = 9600 - 1400$, or $4x = 8200$.

But if four x be equal to 8200, one x will be equal to the fourth Part of 8200, that is, $x = \frac{8200}{4} = 2050$, the least Part x being known, the other Parts will be immediately discovered, the third = 2250, the second = 2500, and the first 2800.

VI.

The Problem may be still varied, and yet still depend upon the same Principles; Let us suppose, for Example, that it was expressed thus.

To divide 5500 into two Parts, so that the first may have a third more than the second, together with 180.

Third example of the foregoing Problem.

ELEMENTS OF

It would be solved in the following Manner: Let the second Part be x , the first will be $x + \frac{1}{3}x + 180$. Now as their Sum should be equal to 5500, we will have the Equation $2x + \frac{1}{3}x + 180 = 5500$.

To solve this Equation, I first add $2x$ & $\frac{1}{3}x$ together, and their Sum will be $\frac{7}{3}x$, because two Integers is equal to six thirds, and consequently those two Integers with one third make up seven thirds. Wherefore the foregoing Equation is reduced to $\frac{7}{3}x + 180 = 5500$, which by the same Reasoning employed in the foregoing Examples will become $\frac{7}{3}x = 5500 - 180$, or $\frac{7}{3}x = 5320$.

The sign $\times$
denotes mul-
tiplication.

Now if the third Part of $7x$ be 5320, the whole $7x$ will be three Times that Sum, which I express thus, $7x = 5320 \times 3$ employing the Sign $\times$ to denote the Multiplication of the two Quantities it separates.

Afterwards, instead of $7x = 5320 \times 3$, it suffices to write $7x = 15960$, which results on multiplying in Effect 5320 by 3.

And by the Means of this new Equation we have $x = \frac{15960}{7} = 2280$, Value of the second Part.

The first Part will be readily found, since we have only to add to this Quantity 2280 its third Part 760 and 180 more, agreeable to the Conditions, and we will have 3220 for this first Part.

Beginners may exercise themselves in varying still more the Expressions of the foregoing Problem, and solving it in the different Cases they may imagine, the Readiness they will acquire will recompence them sufficiently for their Trouble. For their further Assistance, I shall propose another Problem, in several Respects similar to the former.

VII.

A new pro-
blem of the
same nature
with the
foregoing.

Three Merchants form a Society, the first furnishes £17000, the second £13000, the third £10000; but as they have Need of a Person to transact the Business attending their Commerce, he who furnished £10000, is satisfied to take on himself this Trouble, on Condition of being allowed 3 per Cent on the whole Profit that will arise more than the rest: It happens that this Profit amounts to £100000, it is required to determine the Share of each.

Let the Part of the first be x .

The second having furnished less in the Ratio of 13 to 17 should have a Sum less in this same Ratio, that is, only $\frac{13}{17}x$.

The third, supposing he received only a Sum proportioned to what he furnished, would have the ten 17ths of the first, but as he is to have besides 3 per Cent upon the whole Profit, that is £3000 his Share will be $\frac{10}{17}x + 3000$.

And as the Sum of those three Parts should be £100000, we will have $x + \frac{13}{17}x + \frac{10}{17}x + 3000 = 100000$, or $x + \frac{13}{17}x + \frac{10}{17}x = 97000$.

To disengage the unknown Quantity in this Equation, I observe, that $x + \frac{13}{17}x + \frac{10}{17}x$, or $\frac{17}{17}x + \frac{13}{17}x + \frac{10}{17}x$ is the same as $\frac{40}{17}x$;

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wherefore $\frac{40}{17}x = 97000$, or $40x = 97000 \times 17$, or $40x = 1649000$,
or $x = \frac{1649000}{40} = 41225$.

The Share of the first being found, that of the second expressed by $\frac{13}{17}x$, will be $\frac{13}{17} \times 41225$, that is 31525, that of the third expressed by $\frac{10}{17}x + 3000$ will be $\frac{10}{17} \times 41225 + 3000 = 27250$.

VIII.

By those two Problems the Readers may form some Notion of Specious Arithmetick, they learn that in general the Solution of a Problem consists of two Parts, in the first the Analyst expresses by a Letter as x or y , &c. the unknown Quantity sought, or one of those which when known, serves to determine the rest, he afterwards endeavours to arrive at an Equation including the unknown Quantity, which is effected by expressing the same Quantity two different Ways.

The analytic solution of a problem consists of two parts. In the first the problem is expressed by an equation.

In the second it is Question of bringing the unknown Quantity to one Side of the Equation, leaving only known Quantities on the other.

In the second the equation is solved.

The first of those two Parts is not easily reduced to precepts intelligible to Beginners, and perhaps can be learned only by Examples.

As to the second Part, it may be explained with much greater Ease, after a general Manner.

IX.

In the Questions which we have already solved, we arrived at Equations, in which the unknown Quantity was affected only by the Multiplication or Division of known Numbers; these Kinds of Equations are called Equations of the first Degree, such are $2x - 10 = 56$, $\frac{2}{3}x + 15 = x - \frac{1}{3}x + 30$, &c. and the Problems which lead to those Equations, are called Problems of the first Degree.

Equations of the first degree are those in which the unknown quantity is only multiplied or divided by known quantities.

They are so called to distinguish them from those in which the unknown Quantity is raised to the Square, Cube, &c. which are said to be of the second Degree if the unknown Quantity be squared, &c.

Let it be proposed, for Example, to find a Number, the Triple of which being added to its Square, will be 65, the Problem to be solved in this Case is of the second Degree, and the Equation $3x + xx = 65$, (in which xx denotes the Square of x) which expresses the Conditions of this Problem is an Equation of the second Degree.

The first Analysts could not have attained to the Resolution of those Equations till they had been a long Time exercised in solving Equations of the first Degree. We shall therefore proceed to investigate the Rules which those require.

X.

To find them, let us first resume the Equation $4x + 1400 = 9600$ treated Art. v. which is composed of three Terms, $4x$, 1400, 9600 (thus are called all the Parts of an Equation separated from each other by the Signs $+$ or $-$) and let us observe, that by the same Reasoning, by which we deduced $4x = 9600 - 1400$, we may in all Kinds of

The terms of an equation are its parts separated by the

signs + or — Equations take any Term whatsoever preceded by the Sign + and carry it over to the other Side of the Sign = by giving it the Sign —. If we had for Example, $50 + \frac{10}{3}x = 5x + 30$, we may carry the Term $\frac{10}{3}x$ over to the other Side, giving it the Sign — and write down the Equation thus, $50 = 5x + 30 - \frac{10}{3}x$; for we may say, as in Art. v. that since $\frac{10}{3}x$ is to be added to 50 to make it equal to the Quantity $5x + 30$, therefore 50 must be less than $5x + 30$ by the Quantity $\frac{10}{3}x$, that is, it must be equal to $5x + 30 - \frac{10}{3}x$.

We may in like Manner as we found, Art. III. that the Equation $3y - 475 = 890$ was changed into $3y = 890 + 475$, we may, I say, perceive that in general, the Terms which have the Sign — on one Side of the Sign of Equality, may be carried over to the other, by giving them the Sign +. If we had, for Example, $32 - 6x = 9x + 119$, we would conclude that $32 = 6x + 9x + 119$. For if 32 should be diminished by $6x$ to equal $9x + 119$, it must exceed this Quantity by $6x$, that is, it is equal to $6x + 9x + 119$.

XI.

Any term may be carried over from one side of the equation to the other on changing its sign. The members of an equation are its two parts separated by the sign =

Hence there results this general Principle for all Equations, that we may carry over what Terms we please from one Side of the Equation to the other, provided we change their Signs. Now this Principle is of infinite Use, in as much as it saves a great deal of Reasoning.

XII.

By its Means we may change any Equation into another, in which there will be on one Side of the Sign =, that is, in one of the Members of the Equation, the Terms affected by x , and on the other Side of the Sign =, that is, in the other Member of the Equation the Terms which are entirely known.

Let, for Example, the Equation $8x + 30 = \frac{5}{3}x + 250$ be given; I deduce from it $8x - \frac{5}{3}x = 250 - 30$; in like Manner from $60 - \frac{5}{3}x = 250 - \frac{2}{3}x$, I deduce $\frac{7}{3}x - \frac{5}{3}x = 250 - 60$, and so of any other Equation.

XIII.

When after the necessary Transpositions we have carried over all the Terms affected by x to one Side of the Equation, and the known Terms to the other; what most naturally occurs, is to reduce each of the two Members of the Equation to its most simple Expression. Let, for Example, the Equation $8x - \frac{5}{3}x = 250 - 30$ be given; I reduce it to $\frac{19}{3}x = 220$, by taking away in effect 30 from 250, and $\frac{5}{3}x$ from $8x$, or from $\frac{24}{3}x$ which is equal to it. The Equation $\frac{7}{3}x - \frac{5}{3}x = 250 - 60$ will be reduced to $\frac{2}{3}x = 190$, because $\frac{7}{3}x$ and $\frac{5}{3}x$ brought to the same Denomination become $\frac{7}{3}x$ and $\frac{5}{3}x$ whose Difference is $\frac{2}{3}x$, and taking away 60 from 250 there remains 190.

XIV.

By similar Reductions all Equations of the first Degree, however com-

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posed, will be changed into others, consisting only of two Terms, one of which composed of a certain Number of x , either integral or fractional, and the other a Term entirely known; such are the Equations $4x = 8200$, $\frac{7}{3}x = 5320$ and solved in the Articles v and vi.

Let us now resume what we have said concerning those Equations, in order to deduce from thence general Principles applicable to all other Equations.

From the Equation $4x = 8200$ we deduced $x = \frac{8200}{4}$ because $4x$ being equal to 8200, it followed that one x must be the fourth Part of that Sum, from this Reasoning, and from those which may be framed in like Manner for other Numbers of x , we deduce this general Principle, that the Multiplier which affects the unknown Quantity in one of the Members of the Equation may be taken away, by making it serve as a Divisor to the other Member.

Method of making the multiplicator which affects the unknown quantity disappear.

xv.

From the Equation $\frac{7}{3}x = 5320$ we deduced $7x = 3 \times 5320$, observing that if the third Part of $7x$ was equal to 5320, the whole $7x$ must be equal to three Times that Sum. From whence we deduce this general Principle, that the Divisor which affects the unknown Quantity in a Member of an Equation may be taken away, by making it serve as a Multiplier to the other Member.

Method of making the divisor which affects the unknown quantity disappear.

xvi.

With those Rules we are enabled to solve all Kinds of Equations of the first Degree. To exercise Beginners here are some Examples.

$\frac{6}{5}x - 90 + \frac{2}{3}x = \frac{4}{3}x - 82$ is changed by Transposition into $\frac{6}{5}x + \frac{2}{3}x - \frac{4}{3}x = 90 - 82$, and by Reduction becomes $\frac{6}{5}x - \frac{2}{3}x = 8$, or $\frac{18}{15}x - \frac{10}{15}x = 8$, or $\frac{8}{15}x = 8$, or $8x = 8 \times 15$, or infine $x = 15$.

Examples of equations of the first degree solved by the foregoing principles.

In like Manner $\frac{2}{7}x + 9 = \frac{1}{3}x - 10$ by Transposition is changed into $\frac{2}{7}x - \frac{1}{3}x = -10 - 9$, or $\frac{2}{7}x - \frac{1}{3}x = -19$, or $\frac{2}{21}x = -19$, or $x = 399$.

Likewise $\frac{2}{5}x + 40 - \frac{1}{4}x = 60 - \frac{2}{3}x$, by Transposition becomes $\frac{2}{5}x - \frac{1}{4}x + \frac{2}{3}x = 100$, which by reducing first $\frac{2}{5}$ & $\frac{1}{4}$ to the same Denominator, becomes $\frac{7}{20}x - \frac{1}{40}x = 100$, and by reducing $\frac{7}{20}$ & $\frac{1}{40}$ to the same Denominator, becomes $\frac{14}{40}x - \frac{1}{40}x = 100$, or $x = \frac{18000}{13}$.

xvii.

Instead of reducing all the Fractions to the same Denominator, all the Divisors of the Equation may be made to vanish one after the other, by the following Method, which must have soon occurred to those who first handled those Kinds of Equations.

Let the foregoing Example $\frac{2}{5}x - \frac{1}{4}x + \frac{2}{3}x = 100$ be resumed, it is manifest that if the two Members of this Equation be multiplied by 9, the two Products will be the same; for equal Quantities multiplied by the same Number should give the same Product, we will have by this Multiplication $\frac{18}{5}x - \frac{9}{4}x + \frac{6}{3}x = 900$, which on Account of $\frac{18}{5}x = 2x$,

Method of making the fractions of an equation disappear.

ELEMENTS OF

is reduced to $2x - \frac{2}{3}x + \frac{61}{3}x = 900$, in which the Divisor 9 has vanished, and it is easy to perceive that it should necessarily do so; for $\frac{2}{3}$ of any Quantity whatsoever multiplied by 9 should give 2 Integers of this Quantity. To make 4 vanish in the same Manner, I multiply all the Terms of the Equation by 4, only observing that with Respect to the Term $\frac{2}{3}x$, the Multiplication by 4 is performed by taking away the Denominator 4. Hence there results $8x - 9x + \frac{252}{3}x = 3600$, or $\frac{252}{3}x - x = 3600$, which by multiplying the two Members by 5, will become $252x - 5x = 28000$, or $247x = 18000$, or $x = \frac{18000}{247}$.

The general Principle which results from hence, is that to make a Divisor of a Term vanish, we have only to multiply all the other Terms by the Divisor, and take it away from the Term it affects.

XVIII.

Another method whereby they are made to disappear all at once.

We may find another Method for taking away all the Divisors at once, by observing that if we multiply all the Terms by the same Number divisible by each of those Divisors, each Term will be reduced. If we multiply, for Example, the Equation $\frac{2}{3}x - \frac{1}{4}x + \frac{7}{5}x = 100$ by 180, which is divisible by 9, by 4, and by 5, we will have $\frac{360}{9}x - \frac{180}{4}x + \frac{1260}{5}x = 18000$, or $40x - 45x + 252x = 18000$, or $247x = 18000$.

Now to find this Number which may be divisible by all the Divisors, we have only to multiply successively those Divisors into each other. Let the Divisors of the Equation $\frac{2}{3}x + \frac{1}{5}x = 160 - \frac{2}{7}x$ be proposed to be taken away. I first multiply 3 by 5, and afterwards their Product 15 by 7, which gives 105 for the Number divisible by 3, 5, 7; this Number being found, I multiply the whole Equation by it, which gives me $\frac{235}{3}x + \frac{105}{5}x = 16800 - \frac{210}{7}x$, or $245x + 21x = 16800 - 30x$.

To render this Operation more commodious, instead of forming the Product 105 of the three Divisors, it suffices to write it down thus, $3 \times 5 \times 7$, and then we will have $\frac{7 \times 3 \times 5 \times 7}{3}x + \frac{7 \times 3 \times 5}{5}x =$

$160 \times 3 \times 5 \times 7 - \frac{7 \times 5 \times 3 \times 2}{7}x$, where we see by Inspection,

that the Number 3 should vanish in the Numerator of the first Fraction, because the Division by 3 should be destroyed by the Multiplication by 3, it is the same with Respect to 5 and 7, which are at the same Time in the Numerators and Denominators of the other Fractions.

By this Means we arrive at the Equation $7 \times 5 \times 7x + 7 \times 3x = 160 \times 3 \times 5 \times 7 - 2 \times 3 \times 5x$, which after performing the Multiplications indicated by the Signs $\times$, becomes $245x + 21x = 16800 - 30x$ delivered from Fractions.

XIX.

To follow in the most likely Manner possible the Order of the Inventors, we shall dwell no longer on the Method of disengaging the un-

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known Quantity, but return to the Manner of reducing Problems to Equations. The Resolution of Equations might, independent of the Problems to which they relate, employ the first Analysts when this Science had been advanced to a certain Point, but it is to be presumed that those who layed the Foundations of it, only examined Equations as they related to Problems, these being the Conclusions to which Problems were to be brought. Besides, there sometimes occur Complications in Equations which had never been thought of, if the Nature of the Problems sought had not pointed them out.

We cannot explain more fully, the Manner of reducing Problems to Equations, than has been already done, Art. VIII. but will proceed to give several Examples in order to render this Research familiar to Beginners.

To pay a certain Number of Workmen at the Rate of £3 each, £8 is wanting to the Person who employed them, but on giving them £2 each he has £3 left, it is required to determine how much Money he had. Third Problem.

Let x express the Number of Pounds which this Man possesses, wherefore $x + 8$ will be the Sum sufficient to pay all the Workmen at the Rate of £3 each; and since the Number of Workmen should be three Times less than that which is expressed by this Sum, it will be expressed

by the third Part of $x + 8$, which is denoted thus, $\frac{x+8}{3}$; for in specious as in numeral Arithmetick, a Line placed between two Quantities indicates the Division of the superior Quantity by the inferior.

In specious as in numeral arithmetick a bar is employed to indicate division.

Moreover, since there Remains £3 after £2 has been given to each Man, $x - 3$ will be the Sum sufficient to pay all the Workmen at this rate. Wherefore $\frac{x-3}{2}$ will denote the Number of Workmen, but

since we have two Values of the same Number, they must be equal, the Problem is therefore reduced to the Resolution of the Equation $\frac{x-3}{2} = \frac{x+8}{3}$.

To solve this Equation, I first make the Divisor 2 of the Member $\frac{x-3}{2}$ disappear, by multiplying the other Member by this same Number 2, which will change the Equation into $x - 3 = \frac{2x+16}{3}$; for it

is manifest that the Double of $\frac{x-3}{2}$ is $x - 3$, and that the Double of $\frac{x+8}{3}$ will be $\frac{2x+16}{3}$ for the same Reason that $2x + 16$ is double of $x + 8$. I afterwards make the Divisor 3 vanish in the Equation

3 B

$\frac{2x+16}{3} = x-3$, by multiplying the second Member by 3 and taking it away from the first, which will give $2x+16=3x-9$, or $x=25$.

If we would know how many Workmen there were, we must take one of the two Expressions $\frac{x-3}{2}$ or $\frac{x+8}{3}$ found for this Number, $\frac{x-3}{2}$ for Example. Now since $x=25$, $x-3$ will be 22, and consequently $\frac{x-3}{2}$ will be $\frac{22}{2}=11$ Number of Workmen sought.

XX.

It is to be observed with Respect to the Equation $\frac{x-3}{2} = \frac{x+8}{3}$ that it is not permitted in order to apply the Rule of Art. XI. to change the Side and Sign of the Quantities -3 and $+8$, and write the Equation thus $\frac{x-8}{2} = \frac{x+3}{3}$, because the Number -3 is not, properly speaking, a Term of the first Member, but only a Term of its Dividend $x-3$; the Quantity $\frac{x-3}{2}$ being in Reality only one Term of the Equation, as also $\frac{x+8}{3}$. To apply therefore the Rule of Art. XI. we must first take, as indicated by the Number 2 which is under the first Bar, the Half of $x-3$ which will be $\frac{1}{2}x - \frac{3}{2}$; afterwards we must take, on Account of 3 which is under the other Bar, the third Part of $x+8$ which will be $\frac{1}{3}x + \frac{8}{3}$: Then equalling those two Quantities we will have the Equation $\frac{1}{2}x - \frac{3}{2} = \frac{1}{3}x + \frac{8}{3}$ in which we may make what Transpositions we please.

XXI.

Another solution of the same problem.

The foregoing Problem may be also solved after the following Manner. Let y express the Number of Workmen, $3y$ will be the Sum of Money which should be distributed among them at the Rate of £3 each. But £8 is wanting to satisfy them at this Price: Wherefore $3y-8$ is the Sum of Money which their Employer possesses.

On the other Hand $2y$ will be the Sum sufficient to pay the Workmen at the Rate of £2 each, and in this Case there will remain £3 Wherefore $2y+3$ is another Expression of the Sum which their Employer possesses.

We must therefore equal the two Quantities $2y+3$ and $3y-8$, or what amounts to the same, we must solve the Equation $2y+3=3y-8$ to obtain the Value of y . Which being done by the foregoing Principles, 11 will be found for y , that is, for the Number of Workmen required.

XXII.

A Messenger being departed 9 Hours from a certain Place, travelling at the Rate of 5 Miles in 2 Hours, another Messenger is sent after him, who travels at the Rate of 11 Miles in 3 Hours; it is required to find what Number of Miles the second Messenger will ride before he overtakes the first. Fourth problem.

Let x be this Number of Miles, it is manifest that this Distance should be equal to the Number of Miles which the first Messenger made during his 9 Hours of Advance, together with what he made whilst the second Messenger is on Rout. I investigate first the Number of Miles travelled by the first Messenger in 9 Hours, by the following Proportion or Rule of Three.

As 2 Hours are to 5 Miles, so 9 Hours are to a fourth Term, which, according to the known Rules of Arithmetick, is obtained by multiplying the second Term 5 of the Proportion by the third 9, and dividing their Product by the first 2; and which consequently will be $\frac{45}{2}$ Number of Miles made by the first Messenger in 9 Hours.

But as the Analysts always endeavour to express their Operations in the most concise Manner; they denote this Proportion thus; - -

Hours.	Miles.	Hours.	Miles.
2	: 5	=	9 : $\frac{45}{2}$

the Signs: Serving, one to compare 2 to 5 and the other 9 to $\frac{45}{2}$, and the Sign = serving to denote the Equality which subsists between the Ratios of 2 to 5 and that of 9 to $\frac{45}{2}$.

Manner of
expressing
proportions
in specious
arithmetick

To find afterwards the Distance rode by the same Messenger during the Time the second Messenger travels the Number of Miles x , I investigate, first, the Time which the second Messenger takes to travel the Distance x , by Means of the following Proportion; - - - - -

Miles.	Hours.	Miles.	Hours.
11	: 3	=	x : $\frac{3}{11}x$

Whereby, without embarrassing ourselves about the Number of Miles expressed by x , we learn that it suffices to multiply this Number by 3 and divide it by 11, to obtain the Number of Hours employed by the second Messenger in travelling the Number of Miles x .

Now without minding whether the Number of Hours expressed by $\frac{3}{11}x$ be known, or unknown, I make the following Proportion;

Hours.	Miles.	Hours.	Miles.
2	: 5	=	$\frac{3}{11}x$: $\frac{15}{11}x$

The fourth Term of which $\frac{15}{11}x$ expresses the Number of Miles travelled by the first Messenger in the Time $\frac{3}{11}x$, that is, before he has been overtaken.

By this Means we have the same Quantity expressed two different Ways; for the Distance rode by the second Messenger is first expressed by x , secondly it is the Sum of $\frac{45}{2}$ Miles which the first Messenger had travelled before the other had set out, and of $\frac{15}{11}x$ Miles which this same Messenger must have travelled, before he was overtaken.

ELEMENTS OF

Equalling therefore those two Expressions we will have the Equation $x = \frac{45}{2} + \frac{1}{2}x$ which by the foregoing Rule gives $x = 70 + \frac{1}{2}$.

XXIII.

If the first Messenger, besides the Advantage he has of having set out sooner, has also that of having set out from a Place more advanced, the Problem though more complicated, will be easily reduced to the same Principles.

Let the first Messenger, for Example, going to *Londonderry*, set out from *Cashel* on *Monday* at 8 in the Afternoon, travelling at the Rate of 7 Miles in 3 Hours, and the second Messenger pursuing the first set out on *Tuesday* at 10 in the Morning from *Cork*, supposed to be 34 Miles distant from *Cashel*, travelling at the Rate of 13 Miles in 4 Hours, it is required to find what Number of Miles the second Messenger will ride before he overtakes the first.

To solve this Problem, we must take the Difference between 8 in the Afternoon and 10 in the Morning, which is 14 Hours; and as the first travels 7 Miles in 3 Hours, we will have by the following Rule of three;

Hours.	Miles.	Hours.	Miles.
3	: 7	= 14	: $\frac{98}{3}$

which being added to 34 Miles of Advance will give $34 + \frac{98}{3}$ or $\frac{400}{3}$ Miles for the Distance of the first Messenger from *Cork*, when the second set out. I afterwards make as

Miles.	Hours.	Miles.	Hours.
13	: 4	= x	: $\frac{4}{13}x$

above the following Proportion; 13 : 4 = x : $\frac{4}{13}x$ Number of Hours in which the second Messenger travels the Distance x.

But during this same Number of Hours, the first Messenger would have travelled a Number of Miles with will be found thus

Hours.	Miles.	Hours.	Miles.
3	: 7	= $\frac{4}{13}x$	: $\frac{28}{13}x$

We will have therefore the following Equation, $x = \frac{28}{13}x + \frac{400}{3}$, from whence is deduced, by the Rules explained above $x = 236 + \frac{4}{11}$, Number of Miles travelled by the second Messenger before he overtakes the first.

XXIV.

As soon as the first Analysts had found out the Solution of any interesting Problem, they did not fail of making several Applications by varying the Numbers given in those Problems. For Example, they would have repeated several Times the foregoing Question, by varying the Ratio of the Velocities of the Messengers, and the Distances of the Places from each other they set out from. In those different Applications, they perceived that a Part of the Operation was repeated in each particular Example of the same Problem, which might be performed once for all by investigating a Solution which was not restrained to any particular Number, but applicable to every Number given. To explain what they imagined in this Respect we will resume the foregoing Problem, and treat it after as general a Manner as possibly we can.

Let the Distance of the Messengers from each other be expressed by a , when the Question is brought to a Conclusion, we may make this Letter represent any Number of Miles we please.

General solution of the foregoing problem

Let the Number of Hours which the Departure of the first Messenger precedes that of the second be expressed by the Letter b .

Let the Velocity of the first Messenger be such that he travels the Number of Miles c in the Number of Hours d .

Let the Velocity of the second Messenger be such that he travels the Number of Miles e in the Number of Hours f .

Finally, let the Number of Miles which the second Messenger travels before he overtakes the first be expressed by x .

It is usual in Specious Arithmetick to express the known Quantities by the first Letters of the Alphabet a, b, c , &c. and the Quantities sought by the last s, t, u, x , &c.

The first letters of the alphabet serve to express the known quantities, and the last letters the unknown ones.

To find now, according to the Method pursued in the foregoing Example, the Number of Miles travelled by the first Messenger in the Number of Hours b , we must investigate the fourth Term of a Proportion, of which the first Term is the Number of Hours d , the second the Number of Miles c , the third the Number of Hours b , and it is manifest that this Operation will be performed, as in all other Rules of three, by multiplying the second and third Term, one by the other, and dividing their Product by the first Term.

As to the Manner of expressing the Product of those Terms which are no more as heretofore Numbers, but Letters adapted to express any Numbers, what has appeared the most simple, is to place the Letters to be multiplied, beside each other; with Respect to Division, we have seen already that in Specious as in numeral Arithmetick, a Bar is placed between the Quantities to be divided.

Letters which follow one another without having any sign between them are to be considered as multiplied by each other.

By this Means the foregoing Proportion is wrote down thus,

$$d : c :: b : \frac{bc}{d}.$$

Having therefore $\frac{bc}{d}$ to express the Distance which the first Messenger travelled before the second set out, if we add to it the Distance a of the Places from whence they set out, we will have for the Number of Miles which the first Messenger is advanced at the Time of the Departure of the second $a + \frac{bc}{d}$.

To find afterwards the Distance travelled by the first Messenger whilst the other pursues him and rides x Miles, I first investigate, as above, the Time in which the second Messenger rides the Distance x , by Means of the Proportion $e : f :: x : \frac{fx}{e}$ whose first Term is the

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Number of Miles e , the second, the Number of Hours f , the third the Number of Miles x , and the fourth $\frac{fx}{e}$ the Time sought.

Now, whatever be the Number of Hours $\frac{fx}{e}$ which the second Messenger employed to overtake the first, it is obvious if a Proportion be made of which the three first Terms are, 1^o. the Number of Hours d ; 2^o. the Number of Miles c ; 3^o. the Number $\frac{fx}{e}$, the fourth Term will be the Distance travelled by the first Messenger in the same Time that the second travelled the Distance x .

I write down this Proportion thus $d : c = \frac{fx}{e} : c \times \frac{fx}{e}$ Number of Miles travelled by the first Messenger in the Time the second rides x Miles.

But the Number of Miles travelled by the first Messenger together with the Number of Miles $a + \frac{bc}{d}$ which he had advance, should be equal to the Number of Miles travelled by the second.

We have therefore the Equation $x = a + \frac{bc}{d} + c \times \frac{fx}{e}$. Now recollecting the Operations on Fractions, I find that to multiply a Fraction as $\frac{6}{3}$ by 4, we are to multiply the Numerator by this Number and write down $\frac{6 \times 4}{3}$ or $\frac{24}{3}$. in like Manner to multiply $\frac{fx}{e}$ by c we must multiply c by fx which gives $\frac{cfx}{e}$ for $c \times \frac{fx}{e}$. It also appears that to divide a Fraction as $\frac{6}{3}$ by any Number as 6, we are to multiply the Denominator 3 by this Number 6, and write down $\frac{6}{3 \times 6}$ or $\frac{6}{18}$. In like Manner to divide the Fraction $\frac{cfx}{e}$ by d , we must write down $\frac{cfx}{ed}$.

Having thus changed the foregoing Expression $c \times \frac{fx}{e}$ into $\frac{cfx}{ed}$ the Equation to be solved will be $x = a + \frac{bc}{d} + \frac{cfx}{ed}$. Operation which requires we should begin as has been taught Art. XVIII. by multiplying all the Terms, except the last, by the Divisor ed in order to take it away from this Term.

By this Operation we shall have $edx = ade + \frac{bcde}{d} + cfx$,

or $dex = ade + bce + cfx$, because $\frac{bcd e}{d}$ is the same as bce , since the Quantity bce remains the same when multiplied and divided by d .

Carrying the Term cfx over to the first Member, we will have $dex - cfx = ade + bce$.

In order to find x in this Equation, I observe that if the Numbers de and cf which express how often x is contained in the Terms dex and cfx were known, we would subtract the second from the first, and the Remainder which would express the Quantity of x contained in the first Member of the Equation, would serve as a Divisor to the second Member, to obtain the Value of x . But without knowing the Numbers de and cf , it is manifest that $de - cf$ expresses their Difference, and consequently the Quantity of x which is contained in the first Member of the Equation $dex - cfx = ade + bce$. Therefore the Value of x will be what results by dividing the second Member by this Number $de - cf$. wherefore $x = \frac{ade + bce}{de - cf}$, and this is the general Solution of the foregoing Problem; for when the Quantities a, b, c, d, e, f are known, we have no more to do than to make the Use of them indicated by this general Value of x , that is, to multiply successively a, d, e , one by the other: To add to this Product that arising from the Multiplication of the Quantities b, c, e , one by the other, and to divide the Sum of those Products, by the Number which is the Difference of the Product of c into f compared to the Product of d into e , and we will have by this Operation any particular Solution we please.

xxv.

Let us suppose, as in Art. xxiii the Distance of the two Messengers from each other to be 34 Miles, that the first Messenger sets out 14 Hours sooner than the second, and travels 7 Miles in 3 Hours, and that the second rides 13 Miles in 4 Hours, we will have $a = 34, b = 14, c = 7, d = 3, e = 13, f = 4$, which will give $ade = 34 \times 3 \times 13$, that is, $= 102 \times 13 = 1326, bce = 14 \times 7 \times 13 = 1274$, and consequently $ade + bce = 2600, de = 39, cf = 28$, and of Course $de - cf = 11$.

Example of
the foregoing
solution
in numbers.

From whence we deduce $x = \frac{ade + bce}{de - cf} = \frac{2600}{11} = 236 + \frac{4}{11}$, the same as found in Art. xxiii.

From this general Solution we may also deduce the first Case calculated in Art. xxii. in which the two Messengers were supposed to set out from the same Place, the First 9 Hours sooner than the Latter, and travelling at the Rate of 5 Miles in 2 Hours the second travelling at the Rate of 11 Miles in 3 Hours, in this Case $a = 0, b = 9, c = 5$,

ELEMENTS OF

$d = 2, e = 11, f = 3$, and substituting those Values in the general Formula or Value of x , we will have $x = \frac{9 \times 5 \times 11}{2 \times 11 - 5 \times 3} = \frac{495}{7} = 70 + \frac{5}{7}$, as found in Art. xxii, in the same Manner we may make as many Applications as we please.

XXVI.

No sooner had the first Analysts found out the Method of rendering the Solution of a Problem general by making Use of Letters instead of Numbers, but probably they ever after considered Problems in their greatest Generality; it is therefore proper to accustom Beginners to treat them in this Manner. With this View we shall solve the following Problem.

Fifth problem.

A Workman can do a Piece of Work expressed by a in a Time expressed by b ; a second can do a Piece of Work c in a Time d ; a third can do a Piece of Work e in a Time f , it is required to determine in what Time those three Workmen together can do a Piece of Work g .

Let x be the Time sought, the Work done by the first in this Time will be found by the following Proportion: $b : a = x : \frac{ax}{b}$.

The Work done by the second in the same Time will be found by the following Proportion: $d : c = x : \frac{cx}{d}$.

Finally we will have the Work done in the same Time by the third Workman, by Means of the following Proportion; $f : e = x : \frac{ex}{f}$.

Wherefore $\frac{ex}{f} + \frac{cx}{d} + \frac{ax}{b}$ is the Work done by the three Workmen together in the Time sought, but this Work should be equal g , we have therefore the Equation $\frac{ex}{f} + \frac{cx}{d} + \frac{ax}{b} = g$.

To solve it, I multiply according to the Rules of Art. xviii. the whole Equation by the Product fdb of the Divisors, and I have $\frac{edbf x}{f} + \frac{cdbf x}{d} + \frac{axfdb}{b} = bdfg$, which is reduced to $edbx + fcbx + adfx = bdfg$, in which observing that $edbx + fcbx + adfx$ expresses the Number of x contained in the second Member, I conclude that $x = \frac{bdfg}{bde + bcf + adf}$.

XXVII.

Example in Numbers.

To make some Application of this Problem, let us suppose that a Mason can do 7 Feet square of a Wall in 5 Days, that a second Mason can do 10 Feet in 3 Days, and a third 11 Feet in 4 Days, it is required

in what Time those three Mafons together, would do 150 feet square of the same wall.

According to those Suppositions we will have $a = 7$; $b = 5$; $c = 10$; $d = 3$; $e = 11$; $f = 4$; $g = 150$; and consequently $b d f g = 5 \times 3 \times 4 \times 150 = 9000$, $b d e = 5 \times 3 \times 11 = 165$; $b c f = 5 \times 10 \times 4 = 200$, $a d f = 7 \times 3 \times 4 = 84$; wherefore $x = \frac{9000}{449} = 20 + \frac{20}{449}$ number of Days in which the proposed Work will be finished.

XXVIII.

Let us suppose it be required in what Time a Cistern of 200 cubick Feet will be filled by three Pipes, the first of which can fill 9 cubick Feet in $2\frac{1}{2}$ Days, the second 15 cubick Feet in $3\frac{1}{3}$ Days, and the third 19 cubick Feet in $5\frac{1}{4}$ Days; we will have $a = 9$, $b = 2\frac{1}{2}$, or $\frac{5}{2}$; $c = 15$; $d = 3\frac{1}{3}$, or $\frac{10}{3}$; $e = 19$, $f = 5\frac{1}{4}$, or $\frac{21}{4}$; $g = 200$.

Another Example.

$$\text{Wherefore, } x = \frac{\frac{5}{2} \times \frac{10}{3} \times \frac{21}{4} \times 200}{\frac{5}{2} \times \frac{10}{3} \times 19 + \frac{5}{2} \times 15 \times \frac{21}{4} + 9 \times \frac{10}{3} \times \frac{21}{4}}$$

$$\frac{210000}{2 \times 3 \times 4}$$

which becomes

$$\frac{950}{2 \times 3} + \frac{1575}{2 \times 4} + \frac{1890}{3 \times 4}$$

To reduce this Quantity I multiply the Numerator and Denominator of the first Fraction of the Divisor by 4; the Numerator and Denominator of the second by 3; and the Numerator and Denominator of the

third by 2, which gives

$$\frac{210000}{2 \times 3 \times 4} + \frac{3800}{2 \times 3 \times 4} + \frac{4725}{2 \times 3 \times 4} + \frac{3780}{2 \times 3 \times 4}$$

or $\frac{210000}{12305}$, or $\frac{210000}{12305}$, or $17 + \frac{163}{2461}$. Number of Days

fought in which the three Pipes running at the same Time will fill the proposed Cistern.

XXIX.

It appears from the two foregoing Problems that the Rules delivered in Art. x. &c. for solving numeral Equations of the first Degree are equally applicable to literal Equations; but it appears at the same Time that Beginners require to be directed in the Manner of employing them, we think ourself so much the more obliged to assist them by a great Number of those Applications, as it is probable that it is to them we

The application of those rules has produced several operations of Specious arithmetick

are indebted for a great Number of useful Operations of Specious Arithmetick, which we shall discover as it were accidentally.

First example of the resolution of literal equations.

Let it be proposed to solve the Equation $2ac + ab - ax = 3ac + 2ax - 5ab - dx$.

I first carry the Terms $3ac$ and $-5ab$ over to the other Member of the Equation having changed their Signs, and there results $2ac + ab - ax - 3ac + 5ab = 2ax - dx$. I carry likewise the Term $-ax$ over to the other Side, observing also to change its Sign, which gives me $2ac + ab - 3ac + 5ab = 2ax - dx + ax$. I afterwards reduce this Equation, 1^o. by adding ab to $5ab$, which gives me $6ab$; 2^o. by substituting $-ac$ for the Terms $2ac$ and $-3ac$; 3^o. by substituting $3ax$ for $2ax + ax$; the proposed Equation therefore becomes $6ab - ac = 3ax - dx$ which gives $x = \frac{6ab - ac}{3a - d}$.

XXX.

Second example of the resolution of literal equations.

Let $5ab + 2ax - 3bd = 2ab - 5ax + 7bd - ac - dx$; the Terms $5ab - 3bd$ will become $-5ab + 3bd$ when carried over to the second Member, and the Terms $-5ax - dx$ will become $+5ax + dx$ when carried over to the first; we will have therefore $2ax + 5ax + dx = 2ab + 7bd - ac - 5ab + 3bd$, which is reduced to $7ax + dx = 10bd - 3ab - ac$ by substituting $7ax$ for $2ax + 5ax$, $10bd$ for $7bd + 3bd$, and $-3ab$ for $2ab - 5ab$. Now disengaging x in this Equation we will have $x = \frac{10bd - ac - 3ab}{7a + d}$.

XXXI.

Reduction of quantities to their most simple expression.

In the Resolution of the two foregoing Equations we had Occasion to reduce to a more simple Expression different Terms of the same Kind, such as $2ac$ and $-3ac$; $5ab$ and ab , &c. As this Operation is frequently necessary in Equations which are to be solved and in the other Parts of Specious Arithmetick, Beginners should render it familiar to themselves, to assist them thereto; Here follow some Examples.

Let $15abc - 13bcd - 7abc + 29bcd - 5abf + 9abc + 6cbi$ be proposed to be reduced.

Affirmative terms are those which are preceded by +. Negative those which are preceded by -.

I first take the Terms $15abc$, $-7abc$ and $9abc$ which are of the same Kind, I add the two Terms $15abc$ and $9abc$, which are both one and the other Affirmative, that is, affected with the Sign +; I afterwards subtract from their Sum which is $24abc$, the Term $7abc$ because it is negativ or preceded by the Sign -, by this Means the three Terms $15abc - 7abc + 9abc$ are reduced to $17abc$. In like Manner, instead of $29bcd - 13bcd$ I substitute $16bcd$. As to the Terms $-5abf$ and $6cbi$ which are alone of their Kinds, I write them down as they are, hence the Quantity reduced will be $17abc + 16bcd - 5abf + 6cbi$.

The Quantity $\frac{3}{2}ab - \frac{1}{2}ac + \frac{3}{2}ax - ad + 7ab + \frac{1}{2}ax$, will be reduced to $\frac{23}{2}ab + \frac{1}{2}ax - \frac{1}{2}ac - ad$.

The Quantity $2acd - 5acb - 3acd + 3acb - 6bfi$ will become after being reduced $-acd - 2acb - 6bfi$, which being entirely negative, shews that the Quantity which was to be reduced includes more negative than affirmative Parts.

XXXII.

It is proper to observe here that the Reduction which has been employed in the foregoing Examples, is precisely the same Rule as that which is called Addition; for when two Quantities are proposed to be added, it suffices to write them down one after the other, and afterwards to reduce them to their most simple Expression: If it was required, for Example, to add the Quantity $6ab - 2ac - 3ad$ to $3ab + ac - 2ad + bf$, there is no more to be done than to reduce the Quantity $6ab - 2ac - 3ad + 3ab + ac - 2ad + bf$, which will consequently give $9ab - ac - 5ad + bf$ for the Sum of the two proposed Quantities.

The algebraic addition is the same operation as the foregoing.

If it was proposed to add the two Quantities $2ac - 3ad + af$ and $ad - 5ac - 2af$, no more is required than to reduce the Quantity $2ac - 3ad + af + ad - 5ac - 2af$. The Reduction being performed there will result $-3ac - 2ad - af$. It may appear at first View surprizing that the Result of an Addition should be a negative Quantity; but this Difficulty will soon be solved when it is observed that the two Quantities $2ac - 3ad + af$ and $ad - 5ac - 2af$ must be both Negative, or at least one of the two negative and greater than the other.

This will more readily appear by some Examples in Numbers. Let us first suppose $a = 2$, $c = 3$, $d = 4$, $f = 5$, in this Case, instead of $2ac - 3ad + af$ we will have $12 - 24 + 10$ or simply -2 , and instead of $ad - 5ac - 2af$ we will have $8 - 30 - 20 = -42$ and their Sum will be -44 , and it is no way surprizing that the Sum of two negativ Quantities should be negativ.

Let us now suppose $a = 6$, $c = 5$, $d = 3$, $f = 2$, we will have $2ac - 3ad + af = 18$ and $ad - 5ac - 2af = -156$. Wherefore the second Quantity being negative, and greater than the first the Sum should be negative.

XXXIII.

It may perhaps be asked, if a negative Quantity can be added to an affirmative one, or rather can a negative Quantity be said to be added. To which I reply that this Expression is exact when the Words to add and to augment are not confounded. Let two Men, for Example, join their Fortunes, be them what they will, I would say they add their Fortunes; let one have Debts and real Effects, if his Debts exceed his Ef-

In what sense a negative quantity can be said to be added.

fects, he will possess but a negative Fortune, and the Junction of his Fortune with that of the first would diminish that of the first; so that the Sum will be found, either less than what the first possesses, or even entirely negative.

XXXIV.

From the foregoing operation the algebraick subtraction is deduced.

The Reduction taught in the foregoing Articles, has given rise to another Rule of Specious Arithmetick namely Subtraction; for when, for Example, in the Equation $2ac + ab - ax = 3ac + 2ax - 5ab - dx$ (Art. xxix.) we carried over the Terms $3ac - 5ab$ to the other Side, after having changed their Signs, and we arrived at the Equation $2ac + ab - ax - 3ac + 5ab = 2ax - dx$, or $-ac + 6ab - ax = 2ax - dx$, I say that the Quantity $3ac - 5ab$ was subtracted from the Quantity $2ac + ab - ax$ and the Remainder is $-ac + 6ab - ax$. For by making $3ac - 5ab$ disappear in the second Member of the Equation, a Subtraction of this Quantity was performed, but to preserve the Equality, a similar Subtraction must have been made in the first Member; therefore $2ac + ab - ax - 3ac + 5ab$ or $-ac + 6ab - ax$ is what remains of the Quantity $2ac + ab - ax$ after $3ac - 5ab$ has been subtracted from it.

Process of subtraction.

Hence when there are two Quantities of which one is to be subtracted from the other, we must change the Signs of the Quantity to be subtracted, and write it down after the other, then reduce the Quantities of the same Kind, which independant of what has been said may be demonstrated after the following Manner.

Let $2ac + ab - ax$ be the Quantity from which it is proposed to subtract the Quantity $3ac - 5ab$. It is manifest that if $3ac$ was only to be subtracted from it, we should write down $2ac + ab - ax - 3ac$, but by subtracting the Quantity $3ac$ instead of $3ac - 5ab$ a Quantity too great by $5ab$ has been subtracted: Wherefore we must add the $5ab$ which has been taken away too much by subtracting $3ac$. Therefore we must write down $2ac + ab - ax - 3ac + 5ab$ for the Remainder of $2ac + ab - ax$ after $3ac - 5ab$ has been subtracted from it.

To exercise Beginners in this Rule which it is obvious should frequently occur I have added the following Examples.

If from $5ab + 10fg - 3ac + 2de$ we subtract $2ab - 5fg + 6ac + de$, there will remain $5ab + 10fg - 3ac + 2de - 2ab + 5fg - 6ac - de$, or $3ab + 15fg - 9ac + de$.

If from the Quantity $6aeb + 3agb - 10bcd$ we subtract $abc - 10aeb - 8agb$, there will remain $16aeb - abc - 10bcd + 11agb$.

If from the Quantity $3ac + ab + be$ we subtract $-ac - 3ab$, there will remain $4ac + 4ab + be$.

XXXV.

If it appears surprizing that in this subtraction the Remainder - - - $4ac + 4ab + be$ is greater than the Quantity $3ac + ab + be$ from which it was proposed to subtract $-ac - 3ab$, it can only arise from not making a proper Distinction between subtracting and diminishing; for if we consider that to subtract any Quantity, a , for Example, from another b , is to determine how much b exceeds a , it will appear very possible that a Quantity may be increased by a Subtraction. Let it be required to determine, for Example, how much one Man is richer than another, if the latter has only Debts, it is manifest that the Excess of the Riches of the first will be what he possesses more a Sum equal to the Debts of the other.

A quantity is increased when a negative quantity is subtracted from it.

XXXVI.

Let it be proposed to solve the Equation $\frac{cx}{2a} - \frac{ac}{2b} = x - \frac{4ad}{3c}$, to make first the Divisor $2a$ disappear, I make it serve according to Art. xv. as a Multiplier to all the Terms of the Equation, and we will have $cx - \frac{ac \times 2a}{2b} = 2a \times x - \frac{4ad \times 2a}{3c}$, but it is ma-

Third example of the resolution of literal equations.

nifest that instead of $ac \times 2a$ we may substitute $2aac$, because the Product of $2a$ into ac should be double of the Product of a into ac , and the Product of a into ac is aac . Likewise $2a \times x$ will be $2ax$ and $4ad \times 2a$ will be $8aad$; for the Product of ad into a is aad and the Product of $4ad$ into $2a$ should be octuple of that of ad into a .

The Equation is therefore changed into $cx - \frac{2aac}{2b} = 2ax - \frac{8aad}{3c}$ or $cx - \frac{aac}{b} = 2ax - \frac{8aad}{3c}$, because $\frac{2aac}{2b}$ or $\frac{aac}{b}$ are one and the same. Multiplying the Terms of this Equation by b , it will become $b \times cx - aac = 2ax \times b - \frac{8aad}{3c} \times b$ or

$b cx - aac = 2abx - \frac{8aabd}{3c}$ which will be again changed into $b cx \times 3c - aac \times 3c = 2abx \times 3c - 8aabd$, or $3bccx - 3aac = 6abcx - 8aabd$, for the Products of c into $b cx$, aac and $2abx$, will be $bccx$, $aacc$, $2abcx$, and consequently the Products of $3c$ by the same Quantities should be triple of the foregoing, that is, $3bccx$, $3aacc$, $6abcx$; now transposing we will have $3bccx - 6abcx = 3aacc - 8aabd$ which at Length gives $x = \frac{3aacc - 8aabd}{3bcc - 6abc}$.

XXXVII.

In the foregoing Example, the Multiplication of some Quantities

A number placed at the top of a letter to the right denotes how often it is repeated by multiplication, and in this case the letter is said to be raised to the power expressed by this number which is called exponent.

The numbers which are to the left and on the same line are called coefficients.

which contained the same Letters, produced the Repetition of those Letters in the Products: Now as the Analysts always endeavour to express themselves in the most concise Manner, instead of repeating a Letter several Times one after another they write it down but once, placing at the Top of it on the right Hand a Figure denoting the Number of Times that this Letter should be repeated. Hence, instead of the foregoing

Expression $x = \frac{3 a a c c - 8 a a b d}{3 b c c - 6 a b c}$ they write - - -

$$x = \frac{3 a^2 c^2 - 8 a^2 b d}{3 b c^2 - 6 a b c}.$$

When in an Operation the Analyst has Occasion for $a a a$, that is, for the Product of $a a$ by a or for a multiplied by itself twice, he puts down simply a^3 . In like Manner for $c c c c$, he writes c^4 . When a Letter is thus repeated or rather considered as repeated by Means of a Number, it is said to be raised to the Power expressed by this Number, and that this Number is its Exponent, thus c^4 or $c c c c$ which is the Product of c multiplied three Times by itself is said to be raised to the fourth Power, and 4 is its Exponent. Care is to be taken not to confound the Numbers which serve as Exponents with those which are on the left Hand of the Letters and on the same Line, which are called Coefficients; in $4 a^2 c$, for Example, 4 is the Coefficient of the Term, 2 is the Exponent of a .

XXXVIII.

Fourth example of the resolution of literal equations.

Let the Equation $\frac{2 a b^2 x}{3 c^2 d} + \frac{5 a c^2}{b^2} = \frac{6 c d^2}{a^2} - 3 x$, be proposed to be solved, multiplying all its Terms by the Divisor $3 c^2 d$, we will have

$$2 a b^2 x + \frac{5 a c^2 \times 3 c^2 d}{b^2} = \frac{6 c d^2 \times 3 c^2 d}{a^2} - 3 x \times 3 c^2 d.$$

To perform the Multiplications indicated by the Signs $\times$, I observe first that $a c^2$ multiplied by $c^2 d$ should give for Product $a c^4 d$, for if instead of $a c^2$ and of $c^2 d$ we write $a c c$ and $c c d$, as may be done, their Product will be $a c c c c d$, that is, according to the foregoing Art. $a c^4 d$. Having therefore $a c^4 d$ for the Product of $a c^2$ into $c^2 d$, it is manifest that $15 a c^4 d$ will be the Product of $5 a c^2$ into $3 c^2 d$.

In like Manner, the Product of $6 c d^2$ into $3 c^2 d$ will be found to be $18 c^3 d^3$ and the Product of $3 x$ into $3 c^2 d$ to be $9 c^2 d x$. Wherefore the foregoing Equation will be changed into - - - - -

$$2 a b^2 x + \frac{15 a c^4 d}{b^2} = \frac{18 c^3 d^3}{a^2} - 9 c^2 d x.$$

Multiplying afterwards this new Equation by b^2 it will become

$$2 a b^4 x + 15 a c^4 d = \frac{18 b^2 c^3 d^3}{a^2} - 9 b^2 c^2 d x, \text{ and this last by } a^2,$$

there will result $2 a^3 b^4 x + 15 a^3 c^4 d = 18 b^2 c^3 d^3 - 9 b^2 a^2 c^2 d x,$

and by transposition $2 a^3 b^4 x + 9 b^2 a^2 c^2 d x = 18 b^2 c^3 d^3 - 15 a^3 c^4 d$,
 from whence at length is deduced $x = \frac{18 b^2 c^3 d^3 - 15 a^3 c^4 d}{2 a^3 b^4 + 9 a^2 b^2 c^2 d}$.

XXXIX.

In the foregoing Examples, we had Occasion to multiply Quantities expressed by a simple Term such as $4 a d$, $9 c^2 d$, &c. which are commonly called *incomplex Quantities* or *Monomes*, and we found at the same Time how this Operation was to be performed. The general Method which results from the Reasoning employed in those particular Examples, is first to multiply the Coefficients; to add afterwards the Exponents of the same Letters, and to write those which are different one after another. Hence according to this Rule, $3 a^4 b^3 d \times 7 a^2 b d^2 = 21 a^6 b^4 d^3$; $\frac{2}{3} a^2 c d \times \frac{2}{3} a c^3 b d = \frac{4}{9} a^3 c^4 b d^2 = \frac{4}{9} a^3 c^4 b d^2$; $\frac{2}{3} a c^2 d e \times 9 a^4 f g = 6 a^5 c^2 d e f g$.

XL.

Let it be proposed to solve the Equation $\frac{a^2 c}{2 b^2} + \frac{4 c x}{3 a} = \frac{5 a b}{c} - 3 a$, Fifth example of the resolution of literal equations.
 multiplying it by $2 b^2$ I have $a^2 c + \frac{8 b^2 c x}{3 a} = \frac{10 a b^5}{c} - 6 a b^2$,
 then multiplying by $3 a$, I have $3 a^3 c + 8 b^2 c x = \frac{30 a^2 b^3}{c} - 18 a^2 b^2$,
 and performing again the same Operation to make the Divisor c vanish, there results $3 a^3 c^2 + 8 b^2 c^2 x = 30 a^2 b^3 - 18 a^2 b^2 c$, from whence is deduced $x = \frac{30 a^2 b^3 - 18 a^2 b^2 c - 3 a^3 c^2}{8 b^2 c^2}$, which may also be wrote thus $x = \frac{30 a^2 b^3}{8 b^2 c^2} - \frac{18 a^2 b^2 c}{8 b^2 c^2} - \frac{3 a^3 c^2}{8 b^2 c^2}$; because $8 b^2 c^2$ dividing the whole Quantity $30 a^2 b^3 - 18 a^2 b^2 c - 3 a^3 c^2$ divides each of its Parts.

Now the Value of x thus wrote may be more simply expressed by reducing each Term. For 1°. instead of $\frac{30 a^2 b^3}{8 b^2 c^2}$ may be put down $\frac{15 a^2 b}{4 c^2}$, because the Numerator may be considered, as the Product of $2 b^2$ into $15 a^2 b$, and the Denominator as the Product of the same Quantity $2 b^2$ into $4 c^2$; dividing therefore both one and the other by the same Quantity $2 b^2$, there results $\frac{15 a^2 b}{4 c^2}$; 2°. instead of $\frac{18 a^2 b^2 c}{8 b^2 c^2}$ we may put down $\frac{9 a^2}{4 c}$; for the Numerator is the Product of $2 b^2 c$ into $9 a^2$, and the Denominator is the Product of the same Quantity $2 b^2 c$ into $4 c$. Instead of $\frac{3 a^3 c^2}{8 b^2 c^2}$ we may put down $\frac{3 a^3}{8 b^2}$. Therefore the

A number placed at the top of a letter to the right denotes how often it is repeated by multiplication, and in this case the letter is said to be raised to the power expressed by this number which is called exponent.

The numbers which are to the left and on the same line are called coefficients.

which contained the same Letters, produced the Repetition of those Letters in the Products: Now as the Analysts always endeavour to express themselves in the most concise Manner, instead of repeating a Letter several Times one after another they write it down but once, placing at the Top of it on the right Hand a Figure denoting the Number of Times that this Letter should be repeated. Hence, instead of the foregoing

Expression $x = \frac{3 a a c c - 8 a a b d}{3 b c c - 6 a b c}$ they write - - -

$$x = \frac{3 a^2 c^2 - 8 a^2 b d}{3 b c^2 - 6 a b c}.$$

When in an Operation the Analyst has Occasion for $a a a$, that is, for the Product of $a a$ by a or for a multiplied by itself twice, he puts down simply a^3 . In like Manner for $c c c c$, he writes c^4 . When a Letter is thus repeated or rather considered as repeated by Means of a Number, it is said to be raised to the Power expressed by this Number, and that this Number is its Exponent, thus c^4 or $c c c c$ which is the Product of c multiplied three Times by itself is said to be raised to the fourth Power, and 4 is its Exponent. Care is to be taken not to confound the Numbers which serve as Exponents with those which are on the left Hand of the Letters and on the same Line, which are called Coefficients; in $4 a^2 c$, for Example, 4 is the Coefficient of the Term, 2 is the Exponent of a .

XXXVIII.

Fourth example of the resolution of literal equations.

Let the Equation $\frac{2 a b^2 x}{3 c^2 d} + \frac{5 a c^2}{b^2} = \frac{6 c d^2}{a^2} - 3 x$, be proposed to be solved, multiplying all its Terms by the Divisor $3 c^2 d$, we will have

$$2 a b^2 x + \frac{5 a c^2 \times 3 c^2 d}{b^2} = \frac{6 c d^2 \times 3 c^2 d}{a^2} - 3 x \times 3 c^2 d.$$

To perform the Multiplications indicated by the Signs $\times$, I observe first that $a c^2$ multiplied by $c^2 d$ should give for Product $a c^4 d$, for if instead of $a c^2$ and of $c^2 d$ we write $a c c$ and $c c d$, as may be done, their Product will be $a c c c c d$, that is, according to the foregoing Art. $a c^4 d$. Having therefore $a c^4 d$ for the Product of $a c^2$ into $c^2 d$, it is manifest that $15 a c^4 d$ will be the Product of $5 a c^2$ into $3 c^2 d$.

In like Manner, the Product of $6 c d^2$ into $3 c^2 d$ will be found to be $18 c^3 d^3$ and the Product of $3 x$ into $3 c^2 d$ to be $9 c^2 d x$. Wherefore the foregoing Equation will be changed into - - - - -

$$2 a b^2 x + \frac{15 a c^4 d}{b^2} = \frac{18 c^3 d^3}{a^2} - 9 c^2 d x.$$

Multiplying afterwards this new Equation by b^2 it will become

$$2 a b^4 x + 15 a c^4 d = \frac{18 b^2 c^3 d^3}{a^2} - 9 b^2 c^2 d x, \text{ and this last by } a^2,$$

there will result $2 a^3 b^4 x + 15 a^3 c^4 d = 18 b^2 c^3 d^3 - 9 b^2 a^2 c^2 d x,$

and by transposition $2 a^3 b^4 x + 9 b^2 a^2 c^2 d x = 18 b^2 c^3 d^3 - 15 a^3 c^4 d$,
 from whence at length is deduced $x = \frac{18 b^2 c^3 d^3 - 15 a^3 c^4 d}{2 a^3 b^4 + 9 a^2 b^2 c^2 d}$.

XXXIX.

In the foregoing Examples, we had Occasion to multiply Quantities expressed by a simple Term such as $4 a d$, $9 c^2 d$, &c. which are commonly called *incomplex Quantities* or *Monomes*, and we found at the same Time how this Operation was to be performed. The general Method which results from the Reasoning employed in those particular Examples, is first to multiply the Coefficients; to add afterwards the Exponents of the same Letters, and to write those which are different one after another. Hence according to this Rule, $3 a^4 b^3 d \times 7 a^2 b d^2 = 21 a^6 b^4 d^3$; $\frac{2}{3} a^2 c d \times \frac{2}{3} a c^3 b d = \frac{4}{9} a^3 c^4 b d^2 = \frac{4}{9} a^3 c^4 b d^2$; $\frac{2}{3} a c^2 d e \times 9 a^4 f g = 6 a^5 c^2 d e f g$.

Incomplex quantities are those which consist only of one term. Their multiplication deduced from the foregoing examples.

XL.

Let it be proposed to solve the Equation $\frac{a^2 c}{2 b^2} + \frac{4 c x}{3 a} = \frac{5 a b}{c} - 3 a$, Fifth example of the resolution of literal equations.
 multiplying it by $2 b^2$ I have $a^2 c + \frac{8 b^2 c x}{3 a} = \frac{10 a b^3}{c} - 6 a b^2$,
 then multiplying by $3 a$, I have $3 a^3 c + 8 b^2 c x = \frac{30 a^2 b^3}{c} - 18 a^2 b^2$,
 and performing again the same Operation to make the Divisor c vanish, there results $3 a^3 c^2 + 8 b^2 c^2 x = 30 a^2 b^3 - 18 a^2 b^2 c$, from whence is deduced $x = \frac{30 a^2 b^3 - 18 a^2 b^2 c - 3 a^3 c^2}{8 b^2 c^2}$, which may also be wrote thus $x = \frac{30 a^2 b^3}{8 b^2 c^2} - \frac{18 a^2 b^2 c}{8 b^2 c^2} - \frac{3 a^3 c^2}{8 b^2 c^2}$; because $8 b^2 c^2$ dividing the whole Quantity $30 a^2 b^3 - 18 a^2 b^2 c - 3 a^3 c^2$ divides each of its Parts.

Now the Value of x thus wrote may be more simply expressed by reducing each Term. For 1^o. instead of $\frac{30 a^2 b^3}{8 b^2 c^2}$ may be put down $\frac{15 a^2 b}{4 c^2}$, because the Numerator may be considered, as the Product of $2 b^2$ into $15 a^2 b$, and the Denominator as the Product of the same Quantity $2 b^2$ into $4 c^2$; dividing therefore both one and the other by the same Quantity $2 b^2$, there results $\frac{15 a^2 b}{4 c^2}$; 2^o. instead of $\frac{18 a^2 b^2 c}{8 b^2 c^2}$ we may put down $\frac{9 a^2}{4 c}$; for the Numerator is the Product of $2 b^2 c$ into $9 a^2$, and the Denominator is the Product of the same Quantity $2 b^2 c$ into $4 c$. Instead of $\frac{3 a^3 c^2}{8 b^2 c^2}$ we may put down $\frac{3 a^3}{8 b^2}$. Therefore the

Value of x reduced is $\frac{15 a^2 b}{4 c^2} - \frac{9 a^2}{4 c} - \frac{3 a^3}{8 b^2}$.

XLI.

Division of
incomplex
quantities
deduced
from this
example.

The Method to be pursued in general in all Operations of the same Nature as the foregoing, that is, in Divisions of incomplex Quantities is easily deduced from what has been said, particularly after having seen the Multiplication of incomplex Quantities. This Method may be expressed thus.

Divide first the Coefficients if the Division be possible, take away the Letters which have the same Exponents in the Numerators and Denominators, divide afterwards the Letters which have different Exponents in the Denominator and Numerator, by subtracting the least Exponents from the greatest, and writing the Remainders in the Place of the greatest Exponents. As to different Letters there is no more to be done than to copy them.

As this Operation frequently occurs, to render it familiar to Beginners here follow some Examples.

$$\frac{9 a^5 d^2 b^2}{3 a^4 c^2 d^2} = \frac{3 a b^2}{c^2}, \quad \frac{18 a^4 b c d}{14 a b^2} = \frac{9 a^3 c d}{7 b}, \quad \frac{27 a^3 b^2 c^5}{3 a^2 b c^2} = 9 a b c^3, \quad \frac{5 a^2 b^4 c^2}{15 a b^3} = \frac{a b c^2}{3}.$$

XLII.

Sixth exam-
ple of the
resolution
of literal
equations.

Use of Pa-
rentheses in
Specious a-
rithmetick.

Let it be proposed to solve the Equation $\frac{a^2 x}{b-c} + d c = b x - a c$.

To make the Divisor $b-c$ vanish, I multiply all the Terms by this Divisor, which gives $a^2 x + (b-c) \times d c = (b x - a c) \times (b-c)$, where I observed 1^o. to put $b-c$ in the first Member between Parentheses lest it might be imagined that only c was to be multiplied by $d c$; 2^o to put $b x - a c$ and $b-c$ in the second Member between Parentheses in Order that it may appear that those two entire Quantities are to be multiplied by each other.

It is now question to perform the Multiplications indicated by the Signs $\times$. Let it be proposed first to multiply $d c$ by $b-c$, it is manifest that we must multiply $d c$ by b and from their Product subtract the Product of $d c$ into c ; for $b-c$ being less than b by the Quantity c , the Product of $b-c$ into $d c$ should be less than that of b into $d c$, by the Quantity $c \times d c$. Therefore the Product of $b-c$ into $d c$ is $b d c - c c d$.

To find the Product of $b x - a c$ into $b-c$; I observe that if the two Terms $b x - a c$ be considered as a single Quantity, the Product of this Quantity into $b-c$ should be, the Quantity which the Product of $b x - a c$ into b exceeds the Product of $b x - a c$ into c . The Ques-

tion is therefore reduced to two Multiplications similar to the foregoing one and to a Subtraction.

The first of those two Multiplications, that of $b x - a c$ by b , will give $b^2 x - a b c$; the second that of $b x - a c$ by c , will give $b c x - a c^2$. It remains therefore to subtract this last Quantity from the first, which will give, according to Art. xxxiv. $b^2 x - a b c - b c x + a c^2$, and this is the Product of $b x - a c$ into $b - c$.

So that the Equation $\frac{a^2 x}{b-c} + c d = b x - a c$, or $a^2 x + (b-c) \times c d = (b x - a c) \times (b-c)$ is become $a^2 x + b c d - c^2 d = b^2 x - a b c - b c x + a c^2$, which, by the ordinary Transpositions, will give $b c d - c^2 d + a b c - a c^2 = b^2 x - b c x - a^2 x$, and at length

$$x = \frac{b c d - c^2 d + a b c - a c^2}{b^2 - b c - a^2}.$$

XLIII.

In this Example, we had Occasion to form a Rule of specious Arithmetick, which we had not as yet employed, and which on several Occasions may be of Use. This Rule is called the Multiplication of Multinomials. A Multinomial or compound Quantity signifies in general a Quantity composed of several Terms. When the Number of Terms of a Quantity are specified, if it consists of two it is called a Binomial; if it consists of three it is called a Trinomial, &c.

Multiplication of compound quantities or multinomials deduced from the foregoing article.

To exercise Beginners in the Multiplication of those Kinds of Quantities, here follow some Examples: Let first the Product of $2 a^3 c^2 - 5 a^4 b + 6 a^5$ into $3 a b^2 - 4 b c d$ be required.

Example of the multiplication of multinomials.

By reasoning in the same Manner as in the foregoing Article, it will appear, that since the Quantity $3 a b^2 - 4 b c d$ is less than $3 a b^2$ by $4 b c d$, the Product of this Quantity into $2 a^3 c^2 - 5 a^4 b + 6 a^5$ should be less than the Product of $3 a b^2$ into $2 a^3 c^2 - 5 a^4 b + 6 a^5$ by the Product of $4 b c d$ into $2 a^3 c^2 - 5 a^4 b + 6 a^5$. In Consequence, I write the Product thus $(2 a^3 c^2 - 5 a^4 b + 6 a^5) \times 3 a b^2 - (2 a^3 c^2 - 5 a^4 b + 6 a^5) \times 4 b c d$.

Performing now the two Multiplications indicated by the Signs $\times$, after the same Manner as those of simple Quantities, we will have $6 a^4 b^2 c^2 - 15 a^5 b^3 + 18 a^6 b^2$ for the first Product $(2 a^3 c^2 - 5 a^4 b + 6 a^5) \times 3 a b^2$. In like Manner we will have $8 a^3 b c^3 d - 20 a^4 b^2 c d + 24 a^5 b c d$ for the second Product $(2 a^3 c^2 - 5 a^4 b + 6 a^5) \times 4 b c d$. Then subtracting the second from the first, as indicated in the foregoing Expression, we will have $6 a^4 b^2 c^2 - 15 a^5 b^3 + 18 a^6 b^2 - 8 a^3 b c^3 d + 20 a^4 b^2 c d - 24 a^5 b c d$ for the Product of the two proposed Quantities.

XLIV.

If the Multiplier of the foregoing Quantity, contained besides the two Terms $3ab^2 - 4bcd$, another $5abc$, for Example, it is manifest that to obtain the whole Product, we must subtract from the foregoing Quantity, the Product of $2a^3c^2 - 5a^4b + 6a^5$ into $5abc$. For the Multiplier $3ab^2 - 4bcd - 5abc$ being less than the Multiplier $3ab^2 - 4bcd$, by $5abc$, the Product of it into $2a^3c^2 - 5a^4b + 6a^5$ should be less than the Product of $3ab^2 - 4bcd$ into $2a^3c^2 - 5a^4b + 6a^5$ by $5abc \times (2a^3c^2 - 5a^4b + 6a^5)$. In like Manner if there was another Term in the Multiplier, $3ac^2$ for Example, with the Sign $+$ the Product $3ac^2 \times (2a^3c^2 - 5a^4b + 6a^5)$ should be added to the foregoing Products.

Fundamental principle of multiplication.

It appears in general that any Quantity being given to be multiplied, with the Quantity which is to serve as Multiplier, we are to form all the Products of the Multiplicand into each of the Terms of the Multiplier, and add or subtract those Products, according as the Terms of the Multiplier are affected with the Sign $+$ or $-$.

To perform this Operation after a regular Manner, proceed thus.

XLV.

Method to be pursued in multiplication.

First write down the Multiplier under the Multiplicand, and draw a Bar under the Multiplier. To form afterwards the first Line of the Product which is to be set down under this Bar, multiply the first Term of the Multiplier by each of the Terms of the Multiplicand, observing to prefix to each of those Products, the Sign of the Term of the Multiplicand, if the first Term of the Multiplier has no Sign, and consequently is considered as having the Sign $+$.

To form afterwards the second Line which should be wrote under the first, multiply the second Term of the Multiplier by all the Terms of the Multiplicand, and if this second Term of the Multiplier has also the Sign $+$, the Operation is absolutely the same as for the first Line, but if it has the Sign $-$, to each of the Products which compose this Line, prefix a Sign contrary to that which affects the Term of the Multiplicand to which it relates. All the other Lines of the Product being formed after the same Manner, by Means of the other Terms of the Multiplier multiplied by all those of the Multiplicand, draw a Bar, and add or reduce all those particular Products; the Quantity which results will be the Product required.

We supposed that the first Term of the Multiplier had the Sign $+$, if however it had the Sign $-$, it is easy to perceive that with Respect to this Term, as with Respect to the others affected by the Sign $-$, Signs contrary to those of the Terms of the Multiplicand, are to be prefixed to the Product of those Terms.

SPECIOUS ARITHMETICK.

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$$\begin{array}{r}
 2ab - 4ac + ad \\
 3ab - 5ac + 2ad \\
 \hline
 \left\{ \begin{array}{l} 6a^2b^2 - 12a^2bc + 3a^2bd \\ - 10a^2bc + 20a^2c^2 - 5a^2cd \\ + 4a^2bd - 8a^2cd + 2a^2d^2 \end{array} \right\} \\
 \hline
 6a^2b^2 - 22a^2bc + 7a^2bd + 20a^2c^2 - 13a^2cd + 2a^2d^2
 \end{array}$$

Mult.
by

Product.

Sum.

To explain this Method, we shall apply it to an Example: Let it be proposed to multiply the two Quantities $2ab - 4ac + ad$ and $3ab - 5ac + 2ad$. The first being chosen for Multiplicand, and the second for Multiplier, I write down this last under the other, as above.

Application
of the fore-
going me-
thod to an
example.

Which being done, I remark that the first Term of the Multiplier is affirmative, and consequently all the Signs of the Terms of the first Line of the Product should be the same as those of the Multiplicand. In Consequence of this Remark, I write down $6a^2b^2$ the Product of $3ab$ into $2ab$, without prefixing any Sign to it, which comes to the same as if it had been affected by the Sign +.

I afterwards put down — for the Sign of the second Term of the same Line, because it is the Sign of the second Term of the Multiplicand, and after this — I write $12a^2bc$ Product of $4ac$ into $3ab$. I prefix in like Manner the Sign + of the third Term of the Multiplier to the third Term of the first Line of the Product, which is $3a^2bd$ Product of ad into $3ab$. The first Line of the Product being thus finished, I remark that the second Term of the Multiplier has the Sign —, and consequently that the Signs of the Multiplicand must be changed to form the Terms of the second Line of the Product. Wherefore the first Term of this second Line should have the Sign —, which I prefix to the Product $10a^2bc$ of the two Terms $2ab, 5ac$.

The second Term of the same Line should have the Sign +, since the second Term of the Multiplicand has the Sign —, I therefore prefix the Sign + to the Product $20a^2c^2$ of the two Terms $4ac, 5ac$.

The third Term ad of the Multiplier being preceded by the Sign +, the third Term of the second Line will be affected by the Sign — which I prefix to the Product $5a^2cd$ of the two Terms $ad, 5ac$.

As to the third Line of the Product sought, because the third Term of the Multiplier has the Sign +, all the Signs of the Multiplicand are to be preserved, consequently the first Term, that is the Product of $2ab$ into $2ad$ will be $4a^2bd$ preceded by the Sign +, the second, that is, the Product of $4ac$ into $2ad$ will be $8a^2cd$ preceded by the Sign —, and the third, that is, the Product of $2ad$ into ad will be $2a^2d^2$ preceded by the Sign +.

To render this Operation familiar to Beginners, here follow more Examples.

Multiplicand.	$5a^3b - 2ab^3 + 4a^2c^2$
Multiplier.	$2a^3b - ab^3 + 3a^2c^2$
Product.	$\left\{ \begin{array}{l} 10a^6b^2 - 4a^4b^4 + 8a^5bc^2 \\ - 5a^4b^4 + 2a^2b^6 - 4a^3b^3c^2 \\ + 15a^5bc^2 - 6a^3b^3c^2 + 12a^4c^4 \end{array} \right\}$
Sum.	$10a^6b^2 - 9a^4b^4 + 23a^5bc^2 + 2a^2b^6 - 10a^3b^3c^2 + 12a^4c^4$

Multiplicand.	$2a^3x^2 - 3b^3y^2$	$5ab + 3ac - c^2$
Multiplier.	$2a^3x^2 + 3b^3y^2$	$-5ab + 3ac - c^2$
Product.	$\left\{ \begin{array}{l} 4a^6x^4 - 6a^3b^3y^2x^2 \\ + 6a^3b^3x^2y^2 - 9b^6y^4 \end{array} \right\}$	$\left\{ \begin{array}{l} -25a^2b^2 - 15a^2bc + 5ab^2c^2 \\ + 15a^2bc + 9a^2c^2 - 3ac^3 \\ - 5ab^2c^2 - 3ac^3 + c^4 \end{array} \right\}$
Sum.	$4a^6x^4 - 9b^6y^4$	$-25a^2b^2 + 9a^2c^2 - 6ac^3 + c^4$

Multiplicand.	$yy + 2ay - \frac{1}{2}aa$	$xx - ax$
Multiplier.	$yy - 2ay + aa$	$x + a$
Product.	$\left\{ \begin{array}{l} aayy + 2a^3y - \frac{1}{2}a^4 \\ - 2ay^3 - 4aayy + a^3y \\ y^4 + 2ay^3 - \frac{1}{2}aayy \end{array} \right\}$	$\left\{ \begin{array}{l} x^3 - axx \\ + axx - aax \\ x^3 - aax \end{array} \right\}$
Sum.	$y^4 - 3\frac{1}{2}aayy + 3a^3y - \frac{1}{2}a^4$	

Multiplicand.	$aa + 2ac - bc$	$2a - 4b$
Multiplier.	$a - b$	$2a + 4b$
Product.	$\left\{ \begin{array}{l} -aab - 2abc + bbc \\ a^3 + 2aac - abc \end{array} \right\}$	$\left\{ \begin{array}{l} 4aa - 8ab \\ + 8ab - 16bb \end{array} \right\}$
Sum.	$a^3 + 2aac - aab - 3abc + bbc$	$4aa - 16bb$

Multiplicand.	$2abx - bxy + a^2x + 3a^2y$
Multiplier.	$2ax - 3ay$
Product.	$\left\{ \begin{array}{l} 4a^2bx^2 - 2abx^2y + 2a^3x^2 + 6a^3xy \\ - 6a^2bxy + 3abxy^2 - 3a^3xy - 9a^3y^2 \end{array} \right\}$
Sum.	$4a^2bx^2 - 2abx^2y + 2a^3x^2 + 6a^3xy - 6a^2bxy + 3abxy^2 - 3a^3xy - 9a^3y^2$

XLVII.

Let it be proposed to solve the Equation $\frac{ab^2 + abd - abx}{d - c} =$ Sixth exam-
 $ax - ac$, I first make the Divisor $d - c$ disappear by multiplying ple of the re-
 $ax - ac$ by $d - c$; which gives $ab^2 + abd - abx = (ax - ac) \times (d - c)$, solution of
 $\times (d - c)$, or $ab^2 + abd - abx = adx - acd - acx + ac^2$, literal equa-
 which, by carrying over all the Terms affected by x to one side, and tions.
 the known ones to the other, will become $ab^2 + abd + acd - ac^2$
 $= abx + adx - acx$, from whence is deduced - - - - -
 $x = \frac{ab^2 + abd + acd - ac^2}{ab + ad - ac}$.

A certain Relation which is perceived to subsist between the Terms of the Dividend and Divisor in this Expression, might induce us to think that the Division may be performed exactly, and consequently invites us to attempt this Operation, which should appear easy to execute, after having seen that of Multiplication of which it is the Converse.

To discover in effect if $ab + ad - ac$ can divide exactly $ab^2 + abd + acd - ac^2$. I divide first one of the Terms of this last Quantity by one of those of the first, for Example, ab^2 by ab , and write down the Quotient b , I afterwards multiply this Quotient b , or rather this first Part of the Quotient sought, by the whole Divisor $ab + ad - ac$, and subtract the Product $ab^2 + abd + abc$ from the Dividend, the Remainder $ab^2 + abd + acd - ac^2 - ab^2 - abd + abc$, or $acd - ac^2 + abc$, is still to be divided by the same Divisor, and its Quotient added to the foregoing b to form the whole Quotient sought.

Manner of performing the division indicated in this example.

To perform this Division I divide one of the Terms of the Quantity $acd - ac^2 + abc$ which remains to be divided, by one of those of the Divisor. For Example, acd by ad , which gives c for the Quotient; I multiply this new Quotient by the whole Divisor $ab + ad - ac$ and I subtract the Product $abc + acd - ac^2$ from the remaining Dividend $acd - ac^2 + abc$; and as the two Quantities are the same, there remains nothing to be divided, from whence I perceive that $b + c$ is the exact Quotient of $ab^2 + abd + acd - ac^2$ divided by $ab + ad - ac$ and consequently the Value of x .

XLVIII.

After having performed the foregoing Division, it is easy to perceive what Method is to be pursued in other Examples. To reduce this Operation to a regular form, the Analysts write down the Divisor at the right Hand of the Dividend, separating them by a Line, as in Division of Numbers, chusing afterwards a Term in the Dividend divisible by one of those of the Divisor, they write the Quotient of those two Terms.

General method for dividing compound quantities.

under the Divisor, prefixing to it the Sign $+$, if the two Terms which are divided one by the other have the same Signs, but prefixing to it the Sign $-$, if those two Terms have different Signs. This being done, they multiply the Quotient by all the Terms of the Divisor, and write the Product under the Dividend. But as the Use of this Product should be to subtract it from the Dividend, they observe when writing it under this Dividend to prefix to each Term a Sign contrary to that which results from the Multiplication.

The Product being thus wrote down, they draw a Line and perform the Reduction with the Dividend, and the Quantity which remains, is to be divided a-new by the same Divisor. They chuse as before a Term divisible by one of those of the Divisor, and write the Term which results for Quotient beside the first, observing to prefix to it the Sign $+$, or the Sign $-$, according as the two Terms that have been divided one by the other have the same or different Signs. They multiply afterwards this Term by all those of the Divisor, and write the Product under the Quantity to be divided, observing as before to change the Signs which result from the Multiplication. Then drawing a Line and reducing, if all the Terms do not destroy each other they write the Remainder under this Line, and continue the Operation after the same Manner until all the Terms of the Dividend have vanished.

Manner of
avoiding
working by
conjecture
in division.

As in this Operation, it may be some Times embarrassing to chuse among the Terms of the Dividend and the Divisor, those which should serve to form the Terms of the Quotient. To remove all uncertainty in this Choice: Here is what the Analysts have imagined.

What is un-
derstood by
ordering a
quantity ac-
cording to a
letter

They first chuse at will a Letter which is found in the Dividend and Divisor, and they dispose the Terms of those two Quantities in such a Manner that the Terms may stand first in which this Letter has the greatest Exponents, and those next in which this Letter has the next greatest Exponent and so on. Having therefore ordered the two proposed Quantities according to the same Letter (it is thus this Operation is called) the Terms to be divided are no more determined by conjecture, it is always the first Terms of the Dividend and Divisor which are to be chosen.

When the first Term of the Quotient is formed by those two Terms of the Divisor and Dividend, and the Product is wrote with different Signs under the Dividend, if this Operation introduces Terms of a different Kind from any of those in the Dividend; Care is to be taken in writing the Quantity which results after the Reduction to place the Terms, so that the Quantity which remains to be divided be always ordered according to the same Letter as the Divisor.

XLIX.

$$\begin{array}{r|l}
 2a^4 - 13ba^3 + 31b^2a^2 - 38b^3a + 24b^4 & 2a^2 - 3ba + 4b^2 \\
 - 2a^4 + 3ba^3 - 4b^2a^2 & a^2 - 5ba + 6b^2 \\
 \hline
 - 10ba^3 + 27b^2a^2 - 38b^3a + 24b^4 & \\
 + 10ba^3 - 15b^2a^2 + 20b^3a & \\
 \hline
 + 12b^2a^2 - 18b^3a + 24b^4 & \\
 - 12b^2a^2 + 18b^3a - 24b^4 & \\
 \hline
 0 &
 \end{array}$$

To illustrate this Method, we will apply it to some Examples. Let it be first proposed to divide the Quantity $31a^2b^2 + 2a^4 + 24b^4 - 38ab^3 - 13a^3b$ by the Quantity $-3ab + 2a^2 + 4b^2$. Application of the foregoing method to an example.

Having wrote down those Quantities in the Manner above, where they are ordered according to the Letter a .

I divide the first Term $2a^4$ of the Dividend by the first Term $2a^2$ of the Divisor, and I write the Quotient a^2 under the Divisor, without prefixing any Sign to it, that is, I mak it affirmative, because the Terms $2a^4$ and $2a^2$ are preceded by the same Signs. The Quotient a^2 being set down I multiply it by all the Terms of the Divisor, and as this Multiplication should give me for first Term $2a^4$ Product of a^2 into $2a^2$ with the Sign $+$, I set down this Term under the Dividend with the Sign $-$, because it is to be subtracted.

In like Manner, because the second Term $3ba^3$ Product of a^2 into $3ba$ should have the Sign $-$ by the Multiplication, I write under the Dividend $+3ba^3$ since it is to be subtracted, finally because the third Term $4b^2a^2$ Product of a^2 into $4b^2$ should have by the Multiplication the Sign $+$ I write it under the Dividend with the Sign $-$.

This being done, I draw a Line and reduce; the Quantity which remains is $-10ba^3 + 27b^2a^2 - 38b^3a + 24b^4$ which is to be divided by the same Divisor $2a^2 - 3ba + 4b^2$. To perform this Division, I divide the first Term $-10ba^3$ of the Quantity to be divided by the first Term $2a^2$ of the Divisor, I write the Quotient $-5ba$ beside a^2 , and multiply this new Term of the Quotient by the Divisor, and write the Terms of the Product with contrary Signs under the Quantity to be divided. After the Reduction there remains $12b^2a^2 - 18b^3a + 24b^4$ to be divided.

The first Term $12b^3a^2$ of this Quantity being divided by $2a^2$ of the Divisor, gives $+6b^2$ for third Term of the Quotient, and as the Product of this third Term into the Divisor destroys all those of the Quantity to be divided, I conclude that the Division is ended, and that $a^2 - 5ba + 6b^2$ is the Quotient sought.

$$\begin{array}{r|l}
 \text{L.} & \\
 4c^4 - 9b^2c^2 + 6b^3c - b^4 & 2c^2 - 3bc + b^2 \\
 - 4c^4 + 6b^2c^2 - 2b^2c^2 & 2c^2 + 3bc - b^2 \\
 \hline
 6b^3c - 11b^2c^2 + 6b^3c - b^4 & \\
 - 6b^3c + 9b^2c^2 - 3b^3c & \\
 \hline
 - 2b^2c^2 + 3b^3c - b^4 & \\
 + 2b^2c^2 - 3b^3c + b^4 & \\
 \hline
 0 &
 \end{array}$$

Another example.

Let it be proposed to divide $6b^3c - b^4 - 9c^2b^2 + 4c^4$ by $-3cb + b^2 + 2c^2$, I write those two Quantities in the Manner above after having ordered them according to the Letter c .

Dividing afterwards the two first Terms, I have $2c^2$ for the first Term of the Quotient, which being multiplied by the Divisor, gives, after changing the Signs, the Quantity $-4c^4 + 6b^3c - 2b^2c^2$, which being placed under the Dividend, gives for Remainder $6b^3c - 11b^2c^2 + 6b^3c - b^4$, in which I observed to place first the Term $6b^3c$ affected by c^3 introduced by the Multiplication, in order that the Quantity may remain ordered according to c . Dividing afterwards this first Term $6b^3c$ by $2c^2$, I have $3bc$ for Quotient with the Sign $+$. I multiply this new Term of the Quotient by the Divisor, and write the Terms of the Product under the Dividend, after having changed their Signs. After the Reduction there remains $-2b^2c^2 + 3b^3c - b^4$ to be divided. The first Term of this Quantity being divided by that of the Divisor, gives for third Term of the Quotient b^2 affected with the Sign $-$, because the Terms $2b^2c^2$ and $2c^2$ have different Signs, and as the Product of this third Term into the Divisor, destroys all those of the Quantity to be divided, I conclude that the Division is ended, and that $2c^2 + 3bc - b^2$ is the Quotient sought.

LI.

If in ordering the Dividend and the Divisor according to the same Letter, there occur several Terms in which this Letter has the same Exponent, we would fall into the same Inconveniency which we proposed to avoid; unless those Terms be ordered again according to another Letter, common to the two Quantities.

Let us suppose, for Example, that the Dividend being ordered according to the Letter d , the first Terms are, $3ac^2d^3 - c^3d^3 - 3a^2cd^3 + a^3d^3$, and those of the Divisor, ordered according to the same Letter $a^2d^2 + c^2d^2 - 2acd^2$. Arranging these two Quantities thus $a^3d^3 - 3c^2ad^3 + 3c^2ad^3 - c^3d^3, a^2d^2 - 2cad^2 + c^2d^2$, that is, ordering them according to the Letter a , the Division will be performed without being liable to be embarrassed in chusing in the Dividend and Divisor the Terms which should serve to form those of the Quotient.

SPECIOUS ARITHMETICK.

33

To accustom Beginners to be attentive to those Particularities in Division, here follow more Examples.

	<i>Dividend.</i>	<i>Divisor.</i>
	$y^4 \bullet - 3\frac{1}{2}a^2y^2 + 3a^3y - \frac{1}{2}a^4$	$y^2 - 2ay + a^2$
Product.	$-y^4 \quad + 2ay^3 - a^2y^2$	$y^2 + 2ay - \frac{1}{2}a^2$
Reduction. 1.	$+ 2ay^3 - 4\frac{1}{2}a^2y^2 + 3a^3y - \frac{1}{2}a^4$	
Product.	$- 2ay^3 + 4a^2y^2 - 2a^3y$	
Reduction. 2.	$- \frac{1}{2}a^2y^2 + a^3y - \frac{1}{2}a^4$	
Product.	$+ \frac{1}{2}a^2y^2 - a^3y + \frac{1}{2}a^4$	
Reduction. 3.	0	

	<i>Dividend.</i>	<i>Divisor.</i>
	$\left\{ \begin{array}{l} y^6 + aay^4 + b^4yy - a^6 \\ - 2bb^2y^4 - a^4yy - 2a^4bb \\ - aab^4 \end{array} \right\}$	$\left\{ \begin{array}{l} yy - aa - bb \\ y^4 + 2aayy + a^4 \\ - bb^2yy + aabb \end{array} \right\}$
Product.	$\left\{ \begin{array}{l} -y^6 + aay^4 \\ - bb^2y^4 \end{array} \right\}$	
Reduction. 1.	$\left\{ \begin{array}{l} + 2aay^4 + b^4yy - a^6 \\ - bb^2y^4 - a^4yy - 2a^4bb \\ - aab^4 \end{array} \right\}$	
Product.	$\left\{ \begin{array}{l} - 2aay^4 + 2a^4yy \\ + 2a^2b^2yy \end{array} \right\}$	
Reduction. 2.	$\left\{ \begin{array}{l} - bb^2y^4 + b^4yy - a^6 \\ + a^4yy - 2a^4bb \\ + 2aabb^2yy - aab^4 \end{array} \right\}$	
Product.	$\left\{ \begin{array}{l} + bb^2y^4 - aabb^2yy \\ - b^4yy \end{array} \right\}$	
Reduction. 3.	$\left\{ \begin{array}{l} + a^4yy - a^6 \\ + aabb^2yy - 2a^4bb \\ - aab^4 \end{array} \right\}$	
Product.	$\left\{ \begin{array}{l} - a^4yy + a^6 \\ + a^4bb \end{array} \right\}$	
Reduction. 4.	$\left\{ \begin{array}{l} + aabb^2yy - a^4bb \\ - aab^4 \end{array} \right\}$	
Product.	$\left\{ \begin{array}{l} - aabb^2yy + a^4bb \\ + aab^4 \end{array} \right\}$	
Reduction. 5.	0	

3 E

In the Solution of the foregoing Problems, we had occasion to employ only one unknown Quantity, because in those Problems there was, properly speaking, only one Quantity to be investigated. But as we advance in the Science of specious Arithmetick, there occur Problems, to solve which, it is requisite to employ several unknown Quantities, we shall therefore proceed to explain how they are to be managed.

Problem in which two unknown quantities are employed.

There being given the specifick Gravities of two Substances which are mixed together, the Bulk and Weight of the Mixture, to find the Quantity of each of those Substances which compose the Mixture.

Let the Number of cubick Inches contained in this Mixture, or in general, its Bulk after whatever Manner it is measured, be expressed by a , and its Weight by b .

Let the Quantity of the first Substance contained in the Mixture, for Example, the Number of cubick Inches of this Substance be expressed by x .

Let the Weight of a cubick Inch of this Substance, or in general, its specifick Gravity be expressed by c .

Let the Quantity of the second Substance be expressed by y , and its specifick Gravity by d .

The Weight of the Quantity of the first Substance contained in the Mixture will be expressed by cx .

For if x expresses the Number of cubick Inches of this Substance, and c the Weight of each cubick Inch, their whole Weight will be the Product of those two Numbers. In like Manner the Weight of the Quantity of the second Substance will be expressed by dy .

But those two Weights added together is equal to the Weight of the Mixture, hence we have the Equation $cx + dy = b$.

But this Equation is not sufficient to solve the Problem; for disengaging one of the unknown Quantities, for Example, x , we find

$x = \frac{b - dy}{c}$, whereby x cannot be determined unless we suppose y

to be known. There is therefore some other Operation to be performed to discover y . To arrive at which we must examine whether all the Particulars expressed in the Question have been attended to, or to speak in the Language of the Analysts, if all the Conditions of the Problem have been fulfilled; upon Examination it will appear that only one of the Conditions has been expressed, and the Condition implying that the Quantity of the first Substance, added to the Quantity of the second, should be equal to the Bulk of the Mixture has not been employed. Hence this second Condition furnishes the Equation $x + y = a$, which, as the first, determines the Value of x only by Means of that of y , by giving $x = a - y$.

But if we cannot by either of those two Equations, taken seperately, find x independent of y , by employing them both together we are enabled to determine y . For since each of those two Equations gives a Value of x , we may equal these two Values, which gives the Equation $\frac{b-dy}{c} = a-y$, from which is deduced by the foregoing Methods

$$b-dy = ac - cy, \text{ or } ac - b = cy - dy, \text{ or finally } y = \frac{ac-b}{c-d}.$$

y being known it easily appears that x which is expressed by $a-y$, or $\frac{b-dy}{c}$ is also known. We have therefore only to substitute in which of those two Quantities we please, in the first $a-y$, for Example, instead of y , $\frac{ac-b}{c-d}$, and we will have $a - \frac{ac-b}{c-d}$ for the Value of x .

On examining the foregoing Value $a - \frac{ac-b}{c-d}$ it will easily be perceived that it can be reduced; for if we put a to the same Denominator as the Fraction $\frac{ac-b}{c-d}$, we must multiply it by $c-d$, which gives $\frac{ac-ad}{c-d}$ instead of a ; we have no more to do than to subtract from this Fraction the second $\frac{ac-b}{c-d}$. To this End subtracting their Numerators, and dividing the Remainder by the common Denominator, we will have $\frac{ac-ad-ac+b}{c-d}$ or $\frac{b-ad}{c-d}$ for the reduced Value of x .

The Quantities required therefore of the first and second Substance which compose the Mixture are expressed, one by $\frac{ac-b}{c-d}$ and the other by $\frac{b-ad}{c-d}$, consequently the Problem is solved.

LIII.

If instead of substituting the Value $\frac{ac-b}{c-d}$ of y in $a-y$, we substituted it in $\frac{b-dy}{c}$, which also expresses x . There would result $\frac{b-d \times (\frac{ac-b}{c-d})}{c}$ which at first View seems to be a different Value from $\frac{b-ad}{c-d}$. But as we know that the two Values $a-y$,

ELEMENTS OF

and $\frac{b - dy}{c}$ of x are equal, those two Values of x expressed in known Quantities must coincide. The Manner of reducing one to the other is as follows.

I first give the Denominator $c - d$ to the Letter b , by multiplying it by $c - d$, that is, by substituting $\frac{bc - bd}{c - d}$ for b , and then the

foregoing Quantity $\frac{b - \frac{d \times (ac - b)}{c - d}}{c}$ will be transformed into $\frac{bc - bd - d \times (ac - b)}{c^2 - dc}$, or $\frac{bc - bd - d \times (ac - b)}{c^2 - dc}$;

but instead of $d \times (ac - b)$ we may write $acd - bd$, and as this Quantity should be subtracted from $bc - bd$, the foregoing Quantity $\frac{bc - bd - d \times (ac - b)}{c^2 - dc}$ when reduced is $\frac{bc - dca}{c^2 - dc}$, which by dividing the Numerator and Denominator by the same Quantity c at length becomes $\frac{b - da}{c - d}$ the same Value as before.

Application
of the fore-
going solu-
tion to an
example.

LIV.

To apply this general Solution to an Example; let the Mixture be supposed to be composed of Gold and Silver, its Weight to be 30 Ounces, its Bulk 3 cubick Inches, the Weight of a cubick Inch of Gold $12\frac{2}{3}$ Ounces, and that of a cubick Inch of Silver $6\frac{2}{3}$ Ounces.

We will have $a = 3$, $b = 30$, $c = 12\frac{2}{3}$, $d = 6\frac{2}{3}$. Substituting those Values in the two general Formulas $x = \frac{b - da}{c - d}$ and $y = \frac{ac - b}{c - d}$, they will become $x = \frac{21}{13}$ and $y = \frac{18}{13}$, that is, the Mixture will contain $\frac{21}{13}$ cubick Inches of Gold, and $\frac{18}{13}$ cubick Inches of Silver.

LV.

It is easy to perceive after what we have seen in the foregoing Problem, that as often as two unknown Quantities are employed in a Question, there must be two Equations to disengage them, and when there are two Quantities required in a Problem, there must be two Conditions given to determine them, in order to deduce from those two Conditions two Equations. To explain the Method of employing those Conditions we have added the following Problem.

LVI.

Problem in
which two
unknown

Two Pipes, each of which run uniformly together, filled a Cistern a, the one running during the Time b, the other during the Time c; the two same

Pipes filled another Cistern d, the first running during the Time e, the second during the Time f. The Discharge of each Pipe is required. quantities are employed.

Let x and y express those Discharges, that is, for Example, the Number of Hogheads that each of those two Pipes furnish in a Day, supposing the Cisterns a and d to be measured in Hogheads, whilst the Times b, c, e, f are counted in Days.

The Quantity of Water furnished by the first Pipe, during the Time b , will be expressed by bx ; and the Quantity of Water furnished by the second Pipe in the Time c , will be expressed by cy . But those two Quantities of Water by the first Condition of the Problem should be equal to the Cistern a , hence we have $bx + cy = a$.

In like Manner the Quantities of Water furnished by the same Pipes during the Times e, f will be expressed by ex, fy , and consequently the second Condition will give $ex + fy = d$.

We have no more to do now than to deduce from those two Equations the Values of x and of y , which will be performed, as in the foregoing Problem, by deducing a Value of x from each of those two Equations, and then putting these two Values equal to each other.

The first will be $\frac{a - cy}{b}$, the second $\frac{d - fy}{e}$. Putting these two Values equal to each other, there will arise $\frac{a - cy}{b} = \frac{d - fy}{e}$ or $ae - cey = bd - bfy$, or $bfy - cey = bd - ae$, or finally $y = \frac{bd - ae}{bf - ce}$. Substituting this Value of y in one of the two Values of x , in $\frac{a - cy}{b}$, for Example, there will result $x =$

$$\frac{a - c \times \left(\frac{bd - ae}{bf - ce} \right)}{b} \text{ or } x = \frac{a \times (bf - ce) - c \times (bd - ae)}{b \times (bf - ce)}$$

by giving the first Term a the same Denominator as the second, and by multiplying the two Denominators, one by the other.

Performing afterwards the Multiplications indicated in this Value, and reducing, there will arise $x = \frac{af - cd}{bf - ce}$.

Wherefore when the particular Values of a, b, c, d, e, f , are ascertained, we have only to substitute them in those two general Values of x and of y , to obtain any particular Solution we please.

Instead of disengaging x in the two foregoing Equations, and putting the two Values which arise equal to each other, in order to obtain y , it is manifest that if we disengage y , and afterwards put the two Values which arise equal to each other, to obtain x , the Result would be the same.

LVII.

Example of
the forego-
ing problem
in numbers.

To make Some Application of this Problem, let us suppose that the first Pipe running five Days, and the second four, filled a Cistern which contains 330 Hogsheads. That afterwards the first Pipe running two Days, and the second three, filled a Reservoir containing 195 Hogsheads.

In this Case $a = 330$, $b = 5$, $c = 4$, $d = 195$, $e = 2$, $f = 3$, and consequently $af - dc = 210$, $bf - ce = 7$, $bd - ae = 315$, consequently $x = \frac{af - dc}{bf - ce} = \frac{210}{7} = 30$, and $y = \frac{bd - ae}{bf - ce} = \frac{315}{7} = 45$. Hence the first Pipe in this Example furnishes 30 Hogsheads a Day, and the second 45.

LVIII.

Another ex-
ample.

Let us now suppose that the first Pipe running during 3 Days, and the second during seven, filled a Cistern containing 190 Hogsheads, that afterwards the first running 4 Days, and the second 6, they filled a Cistern containing 120 Hogsheads.

In this Case $a = 190$, $b = 3$, $c = 7$, $d = 120$, $e = 4$, $f = 6$, and consequently $af - dc = 300$, $bf - ce = -10$, $bd - ae = -400$, which will give $x = \frac{af - dc}{bf - ce} = \frac{300}{-10}$, and $y = \frac{bd - ae}{bf - ce} = \frac{-400}{-10}$.

Singularity
of the ex-
pressions ar-
rived at in
this exam-
ple.

The first Time the Analysts found similar Values, that is, negative Quantities divided by negative Quantities, and positive Quantities divided by negative ones; without Doubt they were embarrassed to know what they meant, and probably resumed the Question, in order to avoid those Kinds of Division, and in the present Case would have proceeded thus.

Method of
discovering
their mean-
ing.

They would have resumed the two general Equations $bx + cy = a$, and $ex + fy = d$, and substituting in those Equations for a, b, c, d, e, f , the Values which those Letters have in this Example, there would result $3x + 7y = 190$ and $4x + 6y = 120$, deducing from those two Equations $x = \frac{190}{3} - \frac{7y}{3}$, and $x = \frac{120}{4} - \frac{3y}{2}$, they would have put them equal to each other, which would have given $\frac{190}{3} - \frac{7y}{3} = 30 - \frac{3y}{2}$, or $\frac{7y}{3} - \frac{3y}{2} = \frac{190}{3} - 30$, or $y = 40$.

Substituting afterwards this Value of y in $30 - \frac{3y}{2}$ Value of x , they would have found $x = 30 - 60$, that is, $x = -30$. In this Manner they would have assured themselves that the Quotient arising from -400 divided by -10 is $+40$, and that arising from 300 divided by -10 is -30 .

LIX.

They soon after established as general Principles that $+$ divided by $+$, produces $+$, that $+$ divided by $-$ produces $-$, that $-$ divided by $+$ produces $-$, that $-$ divided by $-$ produces $+$, and the same for Multiplication.

General Theorems concerning the signs of quotients or products.

Those Principles were the more easily discovered as they resulted from the Observations which must necessarily have been made on the Signs found for the Terms of the Products and Quotients, in applying the Rules given for the Multiplication and Division of compound Quantities, but it is probable that the first Analysts did not adopt them until after they had verified them in several Examples.

EX.

To assure ourselves that the Multiplication of $-$ by $-$ should produce $+$ in the Product, let us see what Assistance we can draw from the general Method of Multiplications delivered Art. XLV. According to this Method, we see plainly that the Product of a Quantity, such as $a - b$ into another $c - d$ should be $ac - bc - ad + bd$, and consequently we see at the same Time that the Term bd which arises from the Multiplication of b into d has the Sign $+$, whilst its Factors b and d have the Sign $-$. It remains therefore to know if when two negative Quantities, such as $-b$ and $-d$ are not preceded by any positive Quantities, their Product will be still $+bd$, the Truth of which will easily appear by observing that the Method by which we found the Product of $a - b$ into $c - d$ to be $ac - bc - ad + bd$ was not restrained to any particular Value of a or c , and consequently took Place when those Quantities were equal to nothing. But in this Case, the Product $ac - bc - ad + bd$ is reduced to $+bd$, wherefore $-b \times -d = +bd$.

Demonstration that $-b$ into $-d$ is $+bd$ when those quantities are not preceded by others.

LXI.

As to the other Cases, that is, with Respect to the Multiplication and Division of $+$ by $-$, they will be proved after the same Manner.

The other cases are demonstrated in like manner.

LXII.

To return now to our last Application of the foregoing Problem, it is to be observed that after having found $x = -30$ and $y = +40$ we should be still at a Loss to know what this Value of x meant. The Method the first Analysts probably would have taken to satisfy themselves in this Point, would be to return back to the Conditions of the Problem or which comes to the same, they would resume the two Equations $3x + 7y = 190$, and $4x + 6y = 120$, and try how the Values -30 and $+40$ of x and of y , answered those Equations, we find first, that in this Case $3x = -90$ and $7y = 280$, consequently $3x + 7y = -90 + 280$, which in effect is equal to 190. In like Manner $4x + 6y$ is found to be $-120 + 240$ which is reduced to 120.

How the negative value found, solves the problem

Having therefore discovered how the Values -30 and $+40$ of x and of y answer the Equations $3x + 7y = 190$ and $4x + 6y = 120$, we perceive at the same Time how they answer the Conditions of the Problem; for since the Use that have been made of the Quantities $3x$ and $4x$, which express the Quantities of Water discharged by the first Pipe in the first and second Operation, was to subtract them from $7y$ and from $6y$, which express the Quantities of Water furnished in the same Operations by the second Pipe. The first Pipe must be considered in this Case as depriving the Cisterns of Water instead of furnishing any, as it did in the other Example, and as it was supposed in expressing the Conditions of the Problem.

We have here an Example of the Generality of the analytick Art, whereby we find in a Question Cases which we did not foresee could be included in it.

LXIII.

Unknown quantities becoming negative are of an opposite kind from what they were supposed in the expression of the problem, as also known quantities.

In almost every Question solved after a general Manner, the Analysts found Cases similar to the foregoing, and they always concluded, that when the Value of the unknown Quantity became negative, the Quantity expressed by it should be considered as being of an opposite Kind from what it was supposed in expressing the Conditions of the Problem.

What has been said with Respect to unknown Quantities, is equally applicable to known Quantities, that is, when a general Solution is applied to any particular Case, if any of the given Quantities a , b , &c. in the Problem are negative.

LXIV.

Example of the use of known quantities made negative.

Let it be proposed, for Example, to find what should be in the foregoing Problem the Discharges of two Pipes that the first furnishing Water during 3 Days, and the second during four Days, may fill a Cistern containing 320 Hogsheads, and that the second Pipe afterwards furnishing Water during six Days whilst the first discharges it during 3 Days, may fill a Cistern containing 180 Hogsheads.

We have only to put in the general Solution $a = 320$, $b = 3$, $c = 4$, $d = 180$, $e = -3$, $f = 6$.

And there will result $dc = 720$, $af = 1920$, $ce = 18$, $bf = -12$, $ae = -960$, $db = 540$, and consequently $af - dc = 1200$, $bf - ce = 30$, $db - ae = 1500$, which gives $x = \frac{af - dc}{bf - ce} = 40$,

and $y = \frac{db - ae}{bf - ce} = 50$.

From whence it appears that the Discharge of the first Pipe is 40 Hogsheads a Day either to carry away the Water as in the second Operation, or to furnish it as in the first, and the Discharge of the second 50 Hogsheads a Day, which it furnishes in both Operations.

SPECIOUS ARITHMETICK.

LXV.

To accustom Beginners to the Manner of extending the Solutions of Problems to those Cases in which the given Quantities are of an opposite Kind from what they have been supposed to be at first, I shall give another Example taken from the Problem of Art. xxiv. in which it is required to know where two Messengers will meet, and will endeavour to deduce from the general Solution, the Solution of the following Case.

Two Messengers at 50 Miles Distance from each other, one being, for Example, at *Newry*, the other at *Dublin*. The first sets out from *Newry* for *Dublin* at 8 in the Evening, running at the Rate of 4 Miles an Hour, the second sets out the same Day from *Dublin* for *Newry*, at 11 in the Morning, running 3 Miles an Hour, it is required to know at what Distance from *Dublin* they will meet.

Another example in which the known quantities are made negative.

Comparing this Case with the general Expression of the Problem, it appears first, that the Letter *c* which denoted the Number of Miles run by the first Messenger in a given Time, should be negative, since in the general Solution the first Messenger was supposed to recede from, and, in this Case, he goes to meet the second. It also appears that the Letter *b*, which expresses the Number of Hours that the first Messenger set out sooner than the second, should be also negative, because in this Case he set out later than the second.

Hence we have only to put in the general Formula - - -

$$x = \frac{ade + bce}{de - cf}, \quad a = 50, \quad b = -9, \quad c = -4, \quad d = 1, \quad e = 3,$$

$$f = 1, \text{ and we will have } x = \frac{50 \times 1 \times 3 - 9 \times -4 \times 3}{1 \times 3 + 4 \times 1}, =$$

$$\frac{150 + 108}{3 + 4} = \frac{258}{7} = 36 \frac{6}{7}, \text{ by which we learn that when the}$$

Messenger from *Dublin* has run $36 \frac{6}{7}$ Miles, he will meet the Messenger from *Newry*.

LXVI.

One of the principal Advantages of Specious Arithmetick, and which shews the Utility of assuming at Will the Signs of the given Quantities in the general Solution of Problems, is the reducing to the Solution of Equations expressed indefinitely all those in which the unknown Quantities are disposed after the same Manner, for Example, by Means of the two Equations $bx + cy = a$, and $ex + fy = d$ solved Article LVI. any two Equations whatsoever of the first Degree may be solved, provided they include only two unknown Quantities.

Two equations whatsoever of the first degree, including two unknown quantities may be reduced to the foregoing ones.

Let, for Example, the two Equations $mnx = p^2y - bbg$, and $mny = p^3 - nnx$ be proposed to be solved. To compare them with the two first I write them thus $mnx = p^2y - bbg$, and

Example.

$nnx + nmy = p^3$, then comparing them with the two Equations $bx + cy = a$ and $ex + fy = d$. The first with the first, and the second with the second, we will have $b = mn$, $c = -p^2$, $a = -bbg$, $e = n^2$, $f = mn$, $d = p^3$.

Which will give $cd = -p^5$, $af = -mn b^2 g$, $ce = -p^2 n^2$, $bf = m^2 n^2$, $ae = -b^2 n^2 g$, $bd = mnp^3$, and consequently $af - cd = -mn b^2 g + p^5$, $bf - ce = m^2 n^2 + p^2 n^2$, $bd - ae = mnp^3 + b^2 n^2 g$.

Now substituting those Values in the general Formulas $x = \frac{af - cd}{bf - ce}$, and $y = \frac{bd - ae}{bf - ce}$, we will have at length $x = \frac{p^5 - mn b^2 g}{m^2 n^2 + p^2 n^2}$, $y = \frac{mnp^3 + b^2 n^2 g}{m^2 n^2 + p^2 n^2}$.

LXVII.

Another example.

Let the Equations $\frac{3mpx}{p-q} = \frac{p^2 y}{p+q} - \frac{2nq}{p-q}$ and $mx + (p+q) \times y = \frac{nq}{p-q}$ be proposed to be solved, giving the first this Form $\frac{3mpx}{p-q} - \frac{p^2}{p+q} y = -\frac{2nq}{p-q}$, and comparing it with the general Equation $bx + cy = a$, $b = \frac{3mp}{p-q}$, $c = -\frac{p^2}{p+q}$, $a = -\frac{2nq}{p-q}$, comparing the second with the Equation $ex + fy = d$, we will have $e = m$, $f = p+q$, $d = \frac{nq}{p-q}$, and those Values being substituted in the Formula $x = \frac{af - cd}{bf - ce}$, gives $x = \frac{-\frac{2nq}{p-q} \times (p+q) + \frac{p^2}{p+q} \times \frac{qn}{p-q}}{\frac{3mp}{p-q} \times (p+q) + \frac{p^2}{p+q} \times m}$.

To reduce this Quantity, I first multiply the Numerator and Denominator of the Fraction $\frac{-2nq \times (p+q)}{p-q}$ by $p+q$, which

changes it into $\frac{-2nq \times (p+q)^2}{(p-q) \times (p+q)}$, by this Means the whole

Numerator of the Value of x becomes $\frac{-2nq \times (p+q)^2 + p^2 \times qn}{(p+q) \times (p-q)}$

or $\frac{[p^2 - 2 \times (p+q)^2] \times nq}{(p-q) \times (p+q)}$ or $-qn \times \frac{+p^2 + 4pq + 2q^2}{(p-q) \times (p+q)}$.

I operate afterwards on the Denominator of the Value of x , reducing its two Parts to the same Denominator.

Which gives $\frac{3mp \times (p+q)^2 + p^2 \times (p-q) \times m}{(p+q) \times (p-q)}$, or
 $\frac{mp \times [p^2 - pq + 3 \times (p+q)^2]}{(p+q) \times (p-q)}$, or $\frac{mp \times (4p^2 + 5pq + 3q^2)}{(p+q) \times (p-q)}$.

Those two Operations change the foregoing Value of x into - -

$$-qn \times \frac{p^2 + 4pq + 2q^2}{(p+q) \times (p-q)} - \frac{mp \times (4p^2 + 5pq + 3q^2)}{(p+q) \times (p-q)}$$

; but as the Numerator and Denominator of this Fraction are each divided by $(p+q) \times (p-q)$, I take away this Divisor, and the Value of x becomes - - -

$$-qn \times \frac{p^2 + 4pq + 2q^2}{mp \times (4p^2 + 5pq + 3q^2)}, \text{ or } -\frac{qn}{mp} \times \frac{pp + 4pq + 2qq}{4pp + 5pq + 3qq},$$

or finally $x = \frac{-nqp - 4pqqn - 2nq^3}{4mp^3 + 5mp^2q + 3mpq^2}$. Substituting in like Manner the same Values of a, b, c , &c. in the general Formula

$$y = \frac{bd - ae}{bf - ce}, \text{ we will have } y = \frac{\frac{3mp}{p-q} \times \frac{nq}{p-q} + \frac{2nq}{p-q} \times m}{\frac{3mp}{p-q} \times (p+q) + \frac{pp}{p+q} \times m}$$

whose Numerator I reduce to this Form $\frac{3mnpq + 2mnq \times p - q}{(p-q)^2}$

by multiplying the Numerator & Denominator of the Fraction $\frac{2mnq}{p-q}$ by $p - q$. I afterwards reduce this new Form and it becomes $\frac{mnq \times (5p - 2q)}{(p-q)^2}$.

As to the Denominator of the Value of y , it being the same as that of x , it will be reduced after the same Manner, and consequently we

$$\text{will have } y = \frac{mnq \times \frac{5p - 2q}{(p-q)^2}}{mp \times \frac{4pp + 5pq + 3qq}{(p+q) \times (p-q)}}, \text{ which by taking}$$

away the common Divisors $p - q$, and reducing, there results - -

$$y = \frac{qn \times \frac{5p - 2q}{p - q}}{p \times \frac{4pp + 5pq + 3qq}{p + q}}, \text{ which, by putting the Divisor}$$

$p + q$ at Top, and the Divisor $p - q$ at Bottom, according to the Rules of the Division of Fractions, will at length become

$$y = \frac{qn \times (p+q) \times (5p - 2q)}{p \times (p-q) \times (4p^2 + 5pq + 3q^2)},$$

$$\text{or } y = \frac{-2q^3n + 5p^2qn + 3pq^2n}{-2p^2q^2 + 4p^4 + p^3q - 3pq^3}.$$

LXVIII.

If we had freed the Equations proposed in this Example from Fractions, before we compared them with the general Formulas, they would have been more readily solved.

Another manner of solving the foregoing example.

Let the two Members of the Equation $\frac{3mpx}{p-q} = \frac{ppy}{p+q} - \frac{2qn}{p-q}$ or $\frac{3mpx}{p-q} - \frac{ppy}{p+q} = -\frac{2nq}{p-q}$ be multiplied by $pp - qq$, Product of the two Divisors $p - q$, $p + q$, and we will have the Equation $(3mpp + 3mpq) \times x + (ppq - p^3) \times y = -2npg - 2nqq$.

In like Manner, let the two Members of the Equation $mx + (p+q) \times y = \frac{nq}{p-q}$ be multiplied by $p - q$, and we will have $mp - mq \times x + (pp - qq) \times y = qn$.

Now comparing those two new Equations with the two general Formulas $b = 3mpp + 3mpq$, $c = ppq - p^3$, $a = -2npg - 2nqq$, $e = mp - mq$, $f = p^2 - q^2$, $d = qn$.

From whence is deduced $cd = p^2q^2n - p^3qn$, $af = -2np^3q - 2np^2q^2 + 2pq^3n + 2nq^4$, $ae = 2nmq^3 - 2nm p^2q$, $bd = 3mnp^2q + 3mnpq^2$; $ce = 2p^3qm - mp^4 - mp^2q^2$, $bf = 3mp^3q + 3mp^4 - 3mpq^3 - 3mp^2q^2$, hence $af - cd = 2nq^4 + 2npq^3 - 3np^2q^2 - np^3q$, $bf - ce = -3mpq^3 - 2mp^2q^2 + mp^3q + 4mp^4$, $bd - ae = -2nmq^3 + 3mnpq^2 + 5mnp^2q$,

which gives $x = \frac{-np^3q - 3np^2q^2 + 2pq^3n + 2nq^4}{mp^3q + 4mp^4 - 3mpq^3 - 2mp^2q^2}$.

or $x = \frac{qn p^3 + 3ppqqn - 2npq^3 - 2nq^4}{2mppqq - 4mp^4 - mp^3q + 3mpq^3}$,

and $y = \frac{2nmq^3 - 5mnp^2q - 3mnpqq}{2mppqq - 4mp^4 - mp^3q + 3mpq^3}$.

LXIX.

Comparison of the two foregoing solutions.

Comparing actually those two Values of x and of y with those found in the foregoing Article, first the Identity of the two Values of y is easily perceived. As to the Values of x to discover how the former can be the same with the latter, we are to observe, that the Equality which should subsist between those two Expressions, necessarily supposes that the Numerator $qn p^3 + 3np^2q^2 - 2npq^3 - 2nq^4$ of the latter contains the Numerator $-qn p^2 - 4pq^2n - 2q^3n$ of the former, after the same Manner that the Denominator $2mp^2q^2 - 4mp^4 - mp^3q + 3mpq^3$ of the latter contains the Denomina-

tor $4p^3mq + 5p^2qm + 3mpq^2$ of the former, and dividing the second Numerator by the first, we find in effect the same Quotient $p - q$, as results by dividing the second Denominator by the first, that is, the Expression $\frac{qn p^3 + 3p^2 q^2 n - 2npq^3 - 2nq^4}{2mp^2q^2 - 4mp^4 - mp^3q + 3mpq^3}$ is changed into $\frac{p - q \times (-nqp^2 - 4pq^2n - 2nq^3)}{p - q \times (4mp^3q + 5mp^2q + 3mpq^2)}$, or by taking away $p - q$, becomes $\frac{-nqp^2 - 4pq^2n - 2nq^3}{4mp^3q + 5mp^2q + 3mpq^3}$.

LXX.

It was easy to discover the Method employed to reduce the more complex Value of x to the most simple, when both one and the other of those two Expressions were known; but if the more complex Value was only known, and it was proposed to reduce it, to its lowest Terms, it would have been much more embarrassing, since we would not know by what Quantity to divide the Numerator and Denominator of the Fraction. Now as it would be an Imperfection in the Solution of a Problem that a Quantity should be reducible and not reduced, the Analysts have sought a Method for reducing Fractions to their least Terms, or which comes to the same, a Method for finding the greatest common Divisor of any two given Quantities.

To explain what they have imagined in this Respect, let us first suppose those two Quantities to be Numbers; let it be proposed, for Example, to find the greatest common Divisor of the Numbers 637 and 143, or to reduce the Fraction $\frac{637}{143}$, to its least Terms.

Dividing 637 by 143, the Quotient is 4 and there remains 65, that is, the Fraction $\frac{637}{143}$ is changed into $4 + \frac{65}{143}$, hence the next Step is to reduce the Fraction $\frac{65}{143}$, or to find the greatest common Divisor of the Numbers 143 and 65. For when this Number is found, it is manifest that it will be the greatest common Divisor of the Numbers 637 and 143, since we can not reduce the Fraction $\frac{637}{143}$ to its lowest Terms, without reducing at the same Time $4 + \frac{65}{143}$ or $\frac{637}{143}$ to its least Terms also.

The two Numbers 143 and 65, on which we are now to operate, being much simpler than the two first 637, 143, it is easy to perceive that the Difficulty is lessened, and that proceeding after the same Manner it will be still lessened. Instead of the Fraction $\frac{65}{143}$ to be reduced, I write $\frac{143}{65}$, not that those Fractions are the same; but because one cannot be reduced without the other being reduced also after the same Manner. To reduce $\frac{143}{65}$, I divide 143 by 65, and the Quotient is 2, the Remainder being 13. We have therefore no more to do according to the same Principle, than to find the greatest common Divisor of 13 and 65, because the greatest common Divisor of those two Numbers,

will be also that of 143 and 65, since the Fraction $\frac{143}{65}$ is changed into $2 + \frac{13}{65}$.

Now the greatest common Divisor of 13 and 65 is 13 itself, since it divides 65 exactly. Wherefore 13 is also the greatest common Divisor of 143 and 65, and consequently of the proposed Numbers 637, 143. In effect 637 is 49×13 and 143, 11×13 , from whence is deduced $\frac{637}{143} = \frac{49}{11}$ an irreducible Fraction.

LXXI.

General method for finding the greatest common divisor of two numbers.

It is easy to perceive that the Method employed in the foregoing Example, is applicable to any Numbers whatsoever. Let A and B express in general any two Numbers, and let a be the Quotient arising from the Division of the first by the second, and C the Remainder, the Question is reduced to find the greatest common Divisor of B and C ; then b being supposed to be the Quotient resulting from the Division of B by C , and D the Remainder, we have no more to do than to find the greatest common Divisor of C and D , that is, to divide C by D , and if there is a Remainder to divide D by it, and to proceed thus continually dividing until we find two Numbers, the least of which is contained exactly in the greatest, and this Number which is contained exactly in the other, will be the greatest common Divisor of the two first Numbers A and B .

This Rule in its full Extent, is founded, as, in the foregoing Example, on this Principle, that the Fraction $\frac{A}{B}$ being changed into $a + \frac{C}{B}$, cannot be reduced until $\frac{C}{B}$ is reduced, and that $\frac{C}{B}$ cannot be reduced but after the same Manner that $\frac{B}{C}$, and that $\frac{B}{C}$ being changed into $b + \frac{D}{C}$ cannot be reduced until $\frac{D}{C}$ is reduced, and so on.

LXXII.

We now proceed to shew what Alterations are to be made in this Method to render it applicable to specious or algebraic Quantities, to understand which, let us first take an Example.

Let it be proposed to find the greatest common Divisor of the Quantities $3a^3 - 3baa + bba - b^3$, and $4aa - 5ba + bb$. We are, according to the foregoing Method to divide the first of those two Quantities by the second; but as the Division cannot be performed, the first Term $3a^3$ of the Dividend not containing exactly the first Term of the Divisor, I multiply the first Quantity by 4, observing that as 4 is not a Divisor of the second Quantity $4aa - 5ba + bb$, the

greatest common Divisor of $12a^3 - 12baa + 4bba - 4b^3$, and $4aa - 5ba + bb$ must be also the greatest common Divisor of $3a^3 - 3baa + bba - b^3$ and $4aa - 5ba + bb$.

I divide then, according to the foregoing Rules, $12a^3 - 12baa + 4bba - 4b^3$ by $4aa - 5ba + bb$; the Quotient is $3a$, and there remains $3baa + bba - 4b^3$, by which, according to the same Rules we should divide $4aa - 5ba + bb$; but as the Division of those Quantities can not be performed without first preparing them, I observe 1^o. that b being common to all the Terms of the first Quantity, and not to those of the second, it cannot be a Part of the greatest common Divisor of those Quantities, wherefore I take it away from all the Terms of this first Quantity, and I assume in its Place $3aa + ba + 4bb$. I observe 2^o. that if the second Quantity $4aa - 5ba + bb$ be multiplied by 3, which is no Divisor of $3aa + ba - 4b^2$ the Division will succeed; I divide therefore $12aa - 15ab + 3b^2$ by $3aa + ba - 4b^2$, the Quotient is 4, and there remains $-19ab + 19b^2$.

We have therefore no more to do than to find the greatest common Divisor of $3a^2 + ba - 4b^2$, and $-19ab + 19b^2$. As this Operation requires that we should divide the first of those two Quantities by the second, and that it is necessary in order to make the Division of the two first Terms succeed, to multiply the first by $19b$, which is an exact Divisor of the second, I take away this Divisor from the second, whereby it is reduced to $-a + b$.

But the greatest common Divisor of $3aa + ba - 4bb$ and $-a + b$, is $-a + b$ itself, since the Division of those two Quantities is performed exactly. Wherefore $-a + b$ is the greatest common Divisor of $3aa + ba - 4bb$ and of $-19ab + 19b^2$, wherefore it is also the greatest common Divisor of $12aa - 15ab + 3bb$, and of $3aa + ba - 4b^2$, and consequently of $4aa - 5ba + b^2$, and $3baa + bba - 4b^3$, as also of $12a^3 - 12baa + 4bba - 4b^3$, and $4aa - 5ba + bb$, and finally it is the greatest common Divisor of the proposed Quantities $3a^3 - 3baa + bba - b^3$ and $4a^2 - 5ba + b^2$.

LXXIII.

It is easy to perceive that the greatest common Divisor of any other two Quantities may be found after the same Manner. The only Principle which we are obliged to add in this Enquiry to the Method of Art. LXXI. is that any two Quantities A and B will preserve their greatest common Divisor when one of those two Quantities, for Example, A , is multiplied or divided by a Quantity which has no common Divisor with B .

General method for finding the greatest common divisor of species or algebraic quantities.

The general Method for determining the greatest common Divisor, may be expressed thus. Let A and B be the two proposed Quantities, first order those two Quantities according to the same Letter, examine afterwards what Quantity m is most proper to multiply A by, in order that the Terms affected by the highest Power of the Letter according to which it is ordered may be divisible by the Terms of B affected by the highest Power of the same Letter; if this multiplier m has no common Divisor with B , multiply A by it, but if it has a common Divisor n , take away this common Divisor, both from m and B , and multiply A by $\frac{m}{n}$, which will form a new Quantity C , which assume in the Place of A . In like Manner assume in the Place of B the Quantity D , which results when B is divided by the Divisor n , common with m .

Then divide C by D , and the Division being performed, if it be exact, D will be the greatest common Divisor sought of A and B , but if there be a Remainder E , perform upon D and E the same Operation as on A and B , and so on until two Quantities are obtained, of which one divides the other without a Remainder, and that Quantity will be the greatest common Divisor required.

It is proper to observe, that if before you attempt this Operation, you can discover in one of the proposed Quantities A or B any Quantity which is an exact Divisor of it and not of the other, it will be convenient to take away this Divisor.

To render the Application of this Method familiar to Beginners, here follow more Examples.

LXXIV.

First example.

Let there be given the Quantities $q n p^3 + 3 n p^2 q^2 - 2 n p q^3 - 2 n q^4$, and $2 m p^2 q^2 - 4 m p^4 - m p^3 q + 3 m p q^3$, of which we found $p - q$ to be a common Divisor, only because we knew before Hand that the first of those two Quantities, divided by the second should give the same Quotient as the Quantity $-n q p^2 - 4 p q^2 n - 2 n q^3$, divided by $4 m p^3 + 5 m p^2 q + 3 m p q^2$.

To reduce now those two Quantities by the foregoing Method, I take away $q n$ which is common to all the Terms of the first of those two Quantities, and is not contained in those of the second; I take away likewise $p m$, which is common to all the Terms of the second without being contained in those of the first, and thereby the Operation is reduced to find the greatest common Divisor of the Quantities (A) $-4 p^3 - p^2 q + 2 p q^2 + 3 q^3$ and (B) $p^3 + 3 p p q - 2 p q^2 - 2 q^3$.

Dividing A by B the Quotient is -4 , and there remains (C) $11 p^2 q - 6 p q^2 - 5 q^3$, as B should be multiplied by $11 q$ in order to render its first Term divisible by the first Term of C , and that q is contained

in all the Terms of C , I multiply B simply by 11, and I divide C by q , hence the Question is reduced to find the greatest common Divisor of the two Quantities (D) $11p^3 + 33p^2q - 22pq^2 - 22q^3$, and (E) $11p^2 - 6pq - 5q^2$.

I divide the first by the second, the Quotient is p and there remains (F) $39p^2q - 17pq^2 - 22q^3$. As (E) must be multiplied by $39q$ to render its first Term divisible by that of this new Quantity F , and that q is common to all the Terms of F , I multiply E only by 39, and I divide its Product (G) $429p^2 - 234pq - 195q^2$ by (H) $39p^2 - 17pq - 22q^2$. The Quotient is 11, and the Remainder (I) $-47pq + 47q^2$.

To render H divisible by I , we must multiply all its Terms by $47q$; but this Quantity is a Divisor of H , wherefore I take it away from H , and there remains $q - p$ for Divisor to $39p^2 - 17pq - 22q^2$. And as it divides it exactly, I conclude that $q - p$ is the greatest common Divisor sought of the proposed Quantities.

LXXV.

Let there be given the two Quantities $ab + 2aa - 3bb - 4bc$ Second example. $-ac - cc$ and $9ac + 2aa - 5ab + 4cc + 8bc - 12bb$, ranging those two Quantities according to the Dimensions of a , we have $2aa + ba - ca - 3bb - 4bc - cc$ and $2aa + 9ca - 5ba - 12bb + 8bc + 4cc$, or (A) $2aa + (b - c) \times a - 3bb - 4bc - cc$ and (B) $2aa + (9c - 5b) \times a - 12bb + 8bc + 4cc$.

Dividing the first by the second, the Quotient is 1, and there remains (C) $(6b - 10c) \times a + 9bb + 12bc - 5cc$. To render B divisible by this Quantity, I observe that it must be multiplied by $3b - 5c$, but before I perform this Operation, I try the Division of C by $3b - 5c$, which succeeds, and gives (D) $2a + 3b + c$ for Quotient; the Question is therefore reduced to find the greatest common Divisor of B and D ; but B is divisible exactly by D ; wherefore D , or $2a + 3b + c$ is the greatest common Divisor sought of $ab + 2aa - 3bb - 4bc - ac - cc$ and $9ac + 2aa - 5ab + 4cc + 8bc - 12bb$. The first of those Quantities being the Product of $2a + 3b + c$ into $a - b - c$, the second the Product of $2a + 3b + c$ into $a - 4b + 4c$, and those two Quantities $a - b - c$, $a - 4b + 4c$ have no common Divisor.

LXXVI.

Let there be given the two Quantities (A) $(dd - cc) \times a^2 + c^4$ Third example. $-ddcc$, and (B) $4da^2 - (2cc + 4cd) \times a + 2c^3$ ranged according to the Dimensions of a . I first change B into (C) $2da^2 - (cc + 2cd) \times a + c^3$ by taking away from all its Terms the Divisor 2, which is not common with A . I afterwards multiply A by

$2d$, in order to make the Division succeed, the Quotient is $dd - cc$, and there remains $(D) (dd - cc) \times (cc + 2cd) \times a - (dd - cc) \times c^3 + 2dc^4 - 2d^3cc$. To make this Quantity serve as a Divisor to C , we must first multiply C by $(dd - cc) \times (cc + 2cd)$.

But before I perform this Multiplication, I examine whether $(dd - cc) \times (cc + 2cd)$ be not a Divisor or a Multiple of some Divisor of D . By investigating the greatest common Divisor of $(dd - cc) \times (cc + 2cd)$ and $-(dd - cc) \times c^3 + 2dc^4 - 2d^3cc$, that is, of $ddcc - c^4 + 2cd^3 - 2c^3d$ and $-ddc^3 + c^5 + 2dc^4 - 2d^3c^2$; and I find that the second of those Quantities is the Product of the first into $-c$, and consequently that the Quantity D is the Product of $(dd - cc) \times (cc + 2cd)$ into $a - c$; wherefore instead of multiplying C by $(dd - cc) \times (cc + 2cd)$. I divide D by this Quantity, and the Quotient is $(E) a - c$, and as $a - c$ divides C exactly, I conclude, that $a - c$ is the greatest common Divisor sought.

LXXVII.

Another manner of finding the greatest common divisor of the quantities in the foregoing example.

The greatest common Divisor of two Quantities may sometimes be obtained without having recourse to the general Method. For Example, the two foregoing Quantities $(dd - cc) \times aa + c^4 - ddc$ and $4daa - (2cc + 4cd) \times a + 2c^3$ being ordered according to d , and consequently being reduced to this Form $(aa - cc) \times dd + c^4 - aacc$ and $(4aa - 4ac) \times d + 2c^3 - 2c^2a$, it is easy to perceive that $aa - cc$ is a Divisor of the first, and $c - a$ a Divisor of the second. But $a^2 - c^2$ is divisible by $c - a$, wherefore $c - a$ is a Divisor of the two proposed Quantities; I divide therefore both one and the other by $c - a$, and the Quotients are $(cc - dd) \times (c + a)$; and $4ad + 2cc$; which by Inspection are found to have no common Divisor, consequently $c - a$ or $a - c$ is the greatest common Divisor of the proposed Quantities.

LXXVIII.

Other quantities whose greatest common divisor is found independent of the foregoing method.

Let it be proposed to find the greatest common Divisor of the two Quantities $6a^5 + 15a^4b - 4a^3c^2 - 10aabbcc$ and $9a^3b - 27aabc - 6abcc + 18bc^3$, I first take away aa from all the Terms of the first, and $3b$ from all those of the second, whereby they are reduced to $6a^3 + 15a^2b - 4acc - 10bcc$ and $3a^3 - 9aac - 2acc + 6c^3$; but as b is not contained in any of the Terms of the second Quantity, I conclude, that if it has a common Divisor with the first, it must have one with its two Parts $6a^3 - 4acc$ and $15a^2b - 10bcc$, and that those two Parts should also have the same common Divisor. But it appears by Inspection that $3aa - 2cc$ is the common Divisor of those two Parts, wherefore it is the greatest common Divisor of the proposed Quantities, if they have one, and dividing, in effect, those two

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Quantities by it, the Division succeeds, consequently it is their greatest common Divisor.

LXXIX.

From what precedes, it is easy to conclude, that when two Quantities are required in a Problem, there must be two Equations given. Likewise when there are three unknown Quantities there must be three Equations in order to determine them, and in general there must be as many Equations as Quantities required. As to the Manner of disengaging the unknown Quantities involved in those Equations, it is precisely the same as for Equations involving two unknown Quantities. For let three Equations be given, each involving three unknown Quantities x, y, z ; if a Value of x be deduced from each of those Equations, and those different Values be put equal to one another, it is obvious there will result two new Equations, involving only y and z , which will be resolved by the foregoing Method. If four Equations are given, involving four unknown Quantities, their Values may be found after the same Manner, and so on.

When there are three quantities required in a problem, there must be three equations given to find them.

How the values of the unknown quantities involved in those equations are found.

The Use of this Method will be more clearly understood by Help of the following Problem, containing the greatest Complication that Equations of the first Degree involving three unknown Quantities are susceptible of.

LXXX.

The Prices of three Magazines, each containing three Sorts of Grain, and the Number of Measures which each Magazine contains of those three different Sorts of Grain being given; to determine the Price of a Measure of each Sort of Grain.

Problem in which three unknown quantities are required.

Let a, b, c express the Numbers of Measures of each Sort of Grain contained in the first Magazine, and let m be the Price of this Magazine.

Let d, e, f express the Numbers of Measures of each Sort of the same Kinds of Grain, contained in the second Magazine, and let n be the Price of this Magazine.

Let g, h, k express the Numbers of Measures of each Sort of the same Kinds of Grain contained in the third Magazine, and let p be the Price of this Magazine.

And let x, y, z express the Prices of a Measure of each Sort of Grain.

It is manifest that the Price of the Quantity of the first Sort of Grain contained in the Magazine m will be expressed by ax , since a is the Number of Measures of this Sort of Grain, and x the Price of a Measure; in like Manner the Price of the Quantity of the second Sort of Grain contained in the same Magazine, will be expressed by by , and the Price of the Quantity of the third Sort of Grain contained in the same Magazine, will be expressed by cz . Wherefore since the Price

m of this Magazine is equal to those three Sums together, we will have $ax + by + cz = m$.

Expressing in like Manner the Conditions respecting the two other Magazines, we will have $dx + ey + fz = n$ and $gx + hy + kz = p$.

It is now Question to deduce from those Equations the Values of x, y, z ; with this View I first deduce the Value of x from the first Equation, which is $\frac{m - by - cz}{a}$, and putting this Value of x

equal to that deduced from the second, we will have the Equation $\frac{m - by - cz}{a} = \frac{n - ey - fz}{d}$, putting afterwards the same

Value $\frac{m - by - cz}{a}$ equal to that deduced from the third Equation, we will have $\frac{m - by - cz}{a} = \frac{p - hy - kz}{g}$.

From the first of those two Equations I deduce $an - dm = aey - dby + afz - dcz$, or $y = \frac{an - dm + dcz - afz}{ae - db}$, from the second I deduce $ap - gm = aby + akz - bgy - czg$, or $y = \frac{ap - gm + czg - akz}{ab - gb}$.

Putting those two Values of y equal, it is manifest that we will have an Equation involving no other unknown Quantity but z , and that solving this Equation the Value of z will be found. As the Calculations in this Operation would be considerable, I shall shew how they may be avoided by employing some Abbreviations which the first Analysts who were engaged in great Calculations easily imagined.

LXXXI.

Manner of
abridging
the calcula-
tions by par-
ticular deno-
minations.

Those Abbreviations consist in substituting new Letters in the Place of several Terms composed of known Quantities.

Instead of $an - dm$ I substitute A , for $dc - af$, B , for $ae - db$, C , for $ap - gm$, D , for $gc - ak$, E , for $ab - gb$, F .

By those new Denominations the foregoing Equations will become

$$y = \frac{A + Bz}{C} \text{ and } y = \frac{D + Ez}{F}, \text{ which gives } AF + BFz =$$

$$DC + CEz, \text{ from whence is deduced } z = \frac{DC - AF}{BF - CE} =$$

$$\frac{aep - abn + dbm - dbp + gbn - gem}{aek - abf + dbc - dbk + gb f - gec} = \dots\dots\dots$$

$$\frac{(ae - db)p + (gb - ab)n + (db - ge)m}{(ae - db)k + (gb - ab)f + (db - ge)c}, \text{ substituting after-}$$

terwards this Value of z in one of the two foregoing Values of y , in

the first, for Example, we will have $y = \frac{A + \frac{BDC - BAF}{BF - CE}}{C}$, or

$$y = \frac{BD - AE}{BF - CE} = \frac{akn - afp + dcp - dkm + gfm - gcn}{aek - abf + dbc - dbk + gbf - gec} =$$

$$\frac{(ak - gc)n + (dc - af)p + (gf - dk)m}{(ae - db)k + (gb - ab)f + (db - ge)c}.$$

This being done, I substitute those Values of y and z in one of the foregoing Values of x , in $\frac{m - cz - by}{a}$, for Example, and there

results $x = \frac{m}{a} - \frac{c}{a} \times \left(\frac{DC - AF}{BF - CE} \right) - \frac{b}{a} \times \frac{BD - AE}{BF - CE}$,

or $x = \frac{m \times (BF - CE) - c \times (DC - AF) - b \times (BD - AE)}{a \times (BF - CE)} =$

$$\frac{(ek - fb)m + (cb - bk)n + (bf - ce)p}{(ae - db)k + (gb - ab)f + (db - ge)c}.$$

LXXXII.

To apply this Method to an Example, let us suppose that the first Magazine contains 30 Measures of Rye, 20 of Barley, 10 of Wheat, and that it cost £11 10s.

That the second contains 15 Measures of Rye, 6 of Barley, and 12 of Wheat, and that it cost £6 18s.

That the third Magazine contains 10 Measures of Rye, 5 of Barley, and 4 of Wheat, and that it cost £3 15s. To determine the Prices of a Measure of Rye, that of Barley, and that of Wheat, we must put

$a = 30, b = 20, c = 10, m = 230, d = 15, e = 6, f = 12, n = 138, g = 10, h = 5, k = 4, p = 75$.

Substituting those Values in the Formulas, you will find $x = 4$, $y = 3$, and $z = 5$, consequently the Price of a Measure of Rye is 4 Shillings, that of a Measure of Barley 3 Shillings, and that of a Measure of Wheat 5 Shillings.

LXXXIII.

As the Equations of the foregoing Problem are the most general of the first Degree involving three unknown Quantities, since each involves three unknown Quantities combined with known Quantities, it follows that every Problem of the first Degree involving three unknown Quantities, is included in the foregoing, as soon as it is expressed analytically. To give an Example, let the following Problem be proposed.

There are three Ingots composed of different Metals melted down together, of the first of which a Pound (Averdupois) contains, of Silver 7 Ounces, of Brass 3 Ounces, and of Tin 6 Ounces; of the second a Pound con-

All problems of the first degree, in which three quantities are required when reduced to equations are contained in the foregoing problem.

tains, of Silver 12 Ounces, of Brass 3 Ounces, of Tin 1 Ounce; and a Pound of the third contains, of Silver 4 Ounces, of Brass 7 Ounces, and of Tin 5 Ounces: How much of each Ingot must be taken to make a fourth, which shall contain, of Silver 8 Ounces, of Brass $3\frac{1}{2}$ Ounces, and of Tin $4\frac{1}{4}$ Ounces.

Let x, y, z express the Number of Ounces to be taken of each of those Ingots.

It is manifest that $\frac{7}{8}x$ will express the Quantity of Silver in the Portion of the first Ingot, that $\frac{12}{8}y$ will express what is contained in the Portion of the second Ingot, and that $\frac{4}{8}z$ that contained in the Portion of the third.

As the Sum of those three Quantities should be 8 Ounces of Silver, there results the Equation $\frac{7}{8}x + \frac{12}{8}y + \frac{4}{8}z = 8$, or $7x + 12y + 4z = 128$.

In like Manner the Quantity of Brass taken out of each of the three Ingots, will be expressed by $\frac{3}{8}x, \frac{7}{8}y$, and $\frac{7}{8}z$ the Sum of which should be $3\frac{1}{2}$ Ounces, wherefore $\frac{3}{8}x + \frac{7}{8}y + \frac{7}{8}z = \frac{15}{4}$, or $3x + 7y + 7z = 60$.

The Quantity of Tin taken of each Ingot, will, in like Manner be expressed by $\frac{1}{8}x, \frac{5}{8}y, \frac{5}{8}z$, the Sum of which should be $4\frac{1}{4}$ Ounces, wherefore $\frac{1}{8}x + \frac{5}{8}y + \frac{5}{8}z = 4 + \frac{1}{4}$, or $6x + y + 5z = 68$.

We have no more to do now than to resolve those three Equations, which will be effected by putting in the foregoing Solution $a = 7, b = 12, c = 4, m = 128, d = 3, e = 3, f = 7, n = 60, g = 6, b = 1, k = 5, p = 68$.

Substituting those Values in the Formulas, you will find $x = 8, y = 5$ and $z = 3$, that is, we must take 8 Ounces of the first Ingot, 5 Ounces of the second, and 3 of the third to form the Ingot required.

LXXXIV.

In what cases problems are indeterminate.

It is easy to perceive that if there are more Quantities required than Equations given, the Question is not limited to determinate Quantities, but is capable of a Number of Solutions, which however are confined within certain Limits, as will appear by the following Example.

First example.

*There are three Ingots of Gold: The Mark * of the first contains 23 Carats of fine Gold, that of the second 21, and that of the third 18. How much of each Ingot must be taken to compose a fourth, weighing 9 Marks, each Mark containing 22 Carats of fine Gold.*

Let x, y, z express the Number of Marks to be taken of each of the Ingots.

It is manifest that $23x$ will express the Quantity of pure Gold in the Portion of the first Ingot, $21y$ that in the Portion of the second, and $18z$ that in the Portion of the third. Adding therefore those three

* The Mark contains 8 Ounces. The Carat is the $\frac{1}{24}$ of a Mark.

Quantities, their Sum should contain 9 Times 22 Carats, consequently $23x + 21y + 18z = 22 \times 9$. And as the fourth Ingot composed of those three Portions should weigh 9 Marks, $x + y + z = 9$.

All the Conditions of the Problem being expressed, it is manifest that the Question is not limited to determinate Quantities, because there are three unknown Quantities and only two Equations, but it is evident that if one of the three unknown Quantities be determined, the other two will be also determined, and the Suppositions which may be made and the Limits within which they are confined, will be discovered by solving those Equations; which will be effected by putting $a = 23$, $b = 21$, $c = 18$, $m = 22 \times 9$, $d = 1$, $e = 1$, $f = 1$, $n = 9$.

For then the Equation $an - md = aey - bdy + afz - cdz$ is reduced to $an - m = (a - b) \times y + (a - c) \times z$,

or $y = \frac{an - m - (a - c)z}{a - b}$, where I observe, 1°. that the Product $(a - c)z$ of the greatest Difference $a - c$ into z cannot exceed the Product $an - m$ of the Number of Marks into the least Difference, otherwise y would become negative. I observe, 2°. that z cannot be supposed $= \frac{an - m}{a - c}$, because y would in this Case become $= 0$, that is, the fourth Ingot would contain none of the second, which is contrary to the Conditions of the Problem, hence the Limits within which the Values of z are confined, are $z < 1 + \frac{1}{3}$ and $z > 0$. The Sign $>$ being employed by the Analysts to denote the Inequality of the two Quantities between which it is placed, the Point being always directed towards the least Quantity.

It is manifest that all the Suppositions for z included within those Limits will solve the Problem.

Let, for Example, $z = 1 + \frac{1}{3}$ Marks, then $y = \frac{9 - 5 \times \frac{2}{3}}{2} = \frac{9 - 8}{2} = \frac{1}{2}$; which denotes that if we take $1 + \frac{1}{3}$ Marks of the third Ingot, we must take $\frac{1}{2}$ a Mark of the second, and consequently $6 \frac{1}{2}$ of the first. In effect $1 + \frac{1}{3}$ Marks of the third Ingot contains $28 \frac{1}{3}$ Carats of fine Gold, $\frac{1}{2}$ a Mark of the second contains $10 \frac{1}{2}$ Carats, and $6 \frac{1}{2}$ Marks of the first contains $138 \frac{1}{2}$ Carats, and consequently the three Portions together contain 198 Carats of pure Gold, the same Quantity that is contained in 9 Marks, each Mark consisting of 22 Carats of fine Gold.

If we suppose $z = 1$, then $y = 2$, which shews that if we take 1 Mark of the third Ingot we must take 2 of the second, and consequently 6 of the first. In effect 1 Mark of the third furnishes 18 Carats, 2 Marks of the second will give 42, and 6 Marks of the first makes 138,

which together amount to 198 Carats, which is precisely the same Quantity, as is contained in 9 Marks, each Mark consisting of 22 Carats of fine Gold.

LXXXV.

To accustom Beginners to the Method of limiting the Answers to all Sorts of Questions of this Kind, here follows another Example.

Second example.

Suppose a Piece of Metal to be composed of Tin, Gold and Copper, melted down together, its weight to be 375 Ounces, its Bulk 80 cubick Inches, let the Weight of a cubick Inch of Tin be $4\frac{3}{8}$ Ounces, that of Gold $11\frac{5}{8}$ Ounces, and that of Copper $5\frac{1}{8}$ Ounces, it is required to determine how much of each of those Metals are contained in the proposed Mass.

Let x, y, z express the Quantities of each of those Metals.

As the Quantity of Tin added to that of Gold and Copper should compose the whole Bulk, $x + y + z = 80$. And it being manifest that $4\frac{3}{8}x$ expresses the Weight of the Quantity of Tin in the Mixture, $11\frac{5}{8}y$ that of the Gold, and $5\frac{1}{8}z$ that of the Copper; and that those three Weights together should be equal to the whole Weight of the Mixture $4\frac{3}{8}x + 11\frac{5}{8}y + 5\frac{1}{8}z = 375$ or taking away the Fractions $35x + 93y + 41z = 3000$.

Those two Equations expressing the Conditions of the Problem, it is manifest that it is not limited to determinate Quantities, but if one of the unknown Quantities be determined, the two others will be determined also. In order therefore to solve those two Equations, and thereby discover what Suppositions may be made, and within what Limits they are confined, I put $a = 1, b = 1, c = 1, m = 80, d = 35, e = 93, f = 41, n = 3000$, substituting those Values in the Equation $an - dm = (ae - bd)y + (af - cd)xz$, there results $58y + 6z = 200$.

From whence it appears that $z < 33\frac{1}{3}$, that is, that there cannot be $33\frac{1}{3}$ cubick Inches of Copper in the Mixture, for the Equation $58y + 6z = 200$, becoming in this Case $58y + 6 \times 33\frac{1}{3} = 200$ will be changed into $58y + 200 = 200$, from whence is deduced $58y = 200 - 200 = 0$, that is, there would be no Gold in the Mixture, which is contrary to the Supposition, but all the Suppositions for the Value of z less than $33\frac{1}{3}$ will answer the Conditions of the Problem.

Let us suppose $z = 4$, consequently $6z = 24$, and $58y = 176$, or $y = 3\frac{1}{19}$, which denotes that if the Mixture be supposed to contain 4 cubick Inches of Copper, it will contain $3\frac{1}{19}$ cubick Inches of Gold, consequently $72\frac{28}{19}$ cubick Inches of Tin, since those three Quantities should compose 80 cubick Inches, and $4\frac{3}{8} \times 72\frac{28}{19} + 11\frac{5}{8} \times 3\frac{1}{19} + 5\frac{1}{8} \times 4 = 375$ Ounces.

If we suppose $z = 3$, then $6z = 18$, and the Equation $58y + 6z = 200$ will be changed into $58y + 18 = 200$, wherefore $y = \frac{182}{58} = 3\frac{4}{29}$, which denotes that if the Mixture is supposed to contain 3 cubick Inches of Copper, it will contain $3\frac{4}{29}$ cubick Inches of Gold, and consequently $73\frac{25}{29}$ cubick Inches of Tin. Those three Quantities together making 80 cubick Inches, and $4\frac{2}{3} \times 73\frac{25}{29} + 11\frac{1}{3} \times 3\frac{4}{29} + 5\frac{1}{3} \times 3 = 375$ Ounces.

LXXXVI.

It is obvious that when the Number of Equations and unknown Quantities is considerable, the Calculation for finding their Values according to the foregoing Method would be very laborious, we shall therefore proceed to explain what Means the Analysts have found to remedy this Inconveniency.

Inconveniency to which the foregoing method for exterminating unknown quantities is liable.

On examining the Formulas expressing the Values of the unknown Quantities in the general Solutions of the Problems of Art. LVI and LXXX, it will appear, that the common Denominator of those Values is formed of all the Products that can be made of a Number of Coefficients equal to the Number of Equations, and that are taken each from a different Equation, and are prefixed to a different unknown Quantity; those which involve the Products of two such Coefficients having contrary Signs.

Observations which have served to improve this method.

For Example, in two Equations involving two unknown Quantities $ax + by = c$ and $dx + ey = f$, all the Products that can be made of two Coefficients taken each from a different Equation, and are prefixed to a different unknown Quantity, are ae, bd , to which if contrary Signs be given, there will result $ae - bd$, which is the common Denominator of the Values of x and y deduced from those Equations Art. LVI. In like Manner in three Equations including three unknown Quantities $ax + by + cz = m$, $dx + ey + fz = n$, and $gx + hy + iz = p$, all the Products that can be made of three Coefficients taken each from a different Equation, and are prefixed to a different unknown Quantity, are $ae, k, abf, dbc, dbk, gbf, gec$, and if contrary Signs be given to those which involve the Products of two such Coefficients, there will result $ae - k - abf + dbc - dbk + gbf - gec$, which is the common Denominator of the Values of x, y and z , deduced from those Equations Art. LXXX.

It also appears that the common Denominator is changed into the Numerator of any of those Values, by substituting in this Denominator in the Place of the Coefficients of the unknown Quantity in the given Equations those which affect no unknown Quantity.

For Example, if we substitute in the common Denominator $ae - db$, for the Coefficients e and b of y , in the Equations $ax + by = c$, $dx + ey = f$, those f, c which affect no unknown Quantity, there will

result $af - dc$, which is the Numerator of the Value of y , in like Manner substituting in this Denominator for the Coefficients a and d of x , those c and f , which affect no unknown Quantity, there will result $ce - bf$, which is the Numerator of the Value of x .

If we substitute in the common Denominator $ae k - abf + dbc - dbk + gb f - gec$ for the Coefficients k, f, c of z in the Equations $ax + by + cz = m$, $dx + ey + fz = n$, $gx + hy + kz = p$, those p, n, m which affect no unknown Quantity, there will result $ae p - ab n + db m - db p + gb n - gec m$, which is the Numerator of the Value of z .

In like Manner substituting in this Denominator for the Coefficients b, e, b of y , those p, n, m , there will result $akn - afp + dcp - dkm + gfm - gc n$, which is the Numerator of the Value of y .

Finally substituting in this Denominator for the Coefficients g, d, a of x , those p, n, m , there will result $ekm - fbm + cbn - bkn + bfp - cep$, which is the Numerator of the Value of x .

LXXXVII.

From these Observations the Analysts have deduced this general Rule, for exterminating unknown Quantities in Equations of the first Degree.

General rule for finding the values of any number of quantities required when as many simple equations are given.

Let a, b, c, d , &c. be the Coefficients of those unknown Quantities and A that which affects no unknown Quantity, in the first Equation.

Let a', b', c', d' , &c. be the Coefficients of the same unknown Quantities, and A' that which affects no unknown Quantity, in the second Equation.

Let a'', b'', c'', d'' , &c. be the Coefficients of the same unknown Quantities, and A'' that which affects no unknown Quantity in the third, and so on.

Form the two Permutations ab and ba and write down $ab - ba$; with those two Permutations and the Letter c , form all the possible Permutations, observing to change the Sign as often as c will change its Place in ab , as likewise in ba ; and there will result

$$abc - acb + cab - bac + bca - cba.$$

With those six Permutations and the Letter d , form all the Permutations possible, observing to change the Sign as often as d will change its Place in the same Term, and there will result

$$abcd - abdc + adbc - dab c - acbd + acdb - adcb + dacb + cabd - cadb + cdab - dcab - bacd + badc - bdac + dbac + bcad - bcda + bdca - dbca - cbad + cbda - cdba + dcba$$

and proceed in this Manner until all the Coefficients of the first Equation are exhausted.

Afterwards preserve the Letters which occupy the first Place, give those which occupy the second Place the same Mark they have in the second Equation, and those which occupy the third Place the same Mark they have in the third Equation, and so on. And this Result will be the common Denominator of the Values of the unknown Quantities.

The common Denominator being thus formed, the Value of x will be obtained by giving to this Denominator the Numerator which is found by changing in all its Terms a into A , and the Value of y is the Fraction which has the same Denominator, and for Numerator the Quantity which results by changing b into A in all the Terms of the Denominator, and in like Manner the Values of the other unknown Quantities will be obtained.

Hence if there be two Equations and two unknown Quantities $ax + by = A$, $a'x + b'y = A'$, the common Denominator will be $ab' - ba'$ or $a'b' - a'b$.

If there be three Equations and three unknown Quantities $ax + by + cz = A$, $a'x + b'y + c'z = A'$, $a''x + b''y + c''z = A''$, the common Denominator will be $ab'c'' - ac'b'' + ca'b'' - ba'c'' + b'ca'' - cb'a''$, or $a'b'c'' - a'b''c' + a'b''c' - a'b'c'' + a'b'c' - a'b'c'$, or $(ab' - a'b)c'' + (a'b'' - a'b'')c' + (a'b'' - a'b')c$.

If there be four Equations and four unknown Quantities $ax + by + cz + dt = A$, $a'x + b'y + c'z + d't = A'$, $a''x + b''y + c''z + d''t = A''$, $a'''x + b'''y + c'''z + d'''t = A'''$, the common Denominator will be found after ranging the Letters in alphabetic Order to be

$$\begin{aligned} & a b' c'' d''' - a b' c''' d'' + a b'' c''' d' - a b'' c''' d - a b'' c' d''' \\ & + a b''' c' d'' - a b''' c'' d' + a' b''' c'' d + a' b''' c' d''' - a' b''' c' d'' \\ & + a'' b''' c' d' - a'' b''' c' d - a' b' c'' d''' + a' b' c''' d'' - a'' b' c''' d' \\ & + a'' b' c''' d + a'' b' c' d''' - a''' b' c' d'' + a''' b' c'' d' - a''' b' c' d' \\ & - a'' b' c' d''' - a''' b' c' d'' - a''' b'' c' d' + a''' b'' c' d \end{aligned}$$

$$\begin{aligned} & \text{or } [(a' b' - a' b) c'' + (a'' b - a' b'') c' + (a' b'' - a'' b') c] d''' \\ & + [(a' b - a' b') c''' + (a' b''' - a''' b) c' + (a''' b' - a' b''') c] d'' \\ & + [(a''' b - a' b''') c'' + (a' b'' - a'' b) c''' + (a'' b''' - a''' b'') c] d' \\ & + [(a' b''' - a''' b') c'' + (a''' b'' - a'' b''') c' + (a'' b' - a' b'') c''] d \end{aligned}$$

LXXXVIII.

Now it is easy to observe, 1^o that the first Term of any one of those Denominators, is formed of the foregoing Denominator multiplied by the next Letter in the Order of the Alphabet, which it does not include, this Letter being affected by the Mark which immediately follows the highest of those in this same Denominator.

Observations which have served to render this rule more simple

2^o. That the second Term is formed of the first, by changing the highest Mark in it into that which is immediately below it, and that which is immediately below the highest into the highest, as also by changing the Signs.

3^o. The third Term is formed of the first, by changing in it the highest Mark into that of two Numbers below it, and that of two Numbers below the highest into the highest, as also by changing the Signs.

4°. The fourth is formed of the first by changing the highest Mark in it, into that which is three Marks below it, and reciprocally, as also by changing the Signs.

For Example, the second Denominator has for first Term $(a.b' - a'b)c''$ which is the first Denominator multiplied by c'' , which is the Letter which follows immediately the Letters a and b , and which has the Mark $''$, which follows immediately the Mark $'$, the highest of those, which enter in the Expression $a.b' - a'b$.

The second Term of this second Denominator is $(a''b - a'b'')c'$, which is no other than $(a.b' - a'b)c''$, in which the Signs have been changed, and the highest Mark $''$ into $'$ which is immediately below it, and $'$ which is immediately below the highest into the highest $''$, and so on.

Hence the common Denominator of the Values of the unknown Quantities in any Number of Equations is easily determined, for Example, if there be five Equations and five unknown Quantities, the common Denominator will be found by this Method to be

$$\begin{aligned}
 & \left\{ \begin{aligned} & [(a.b' - a'b)c'' + (a''b - a'b'')c' + (a'b'' - a''b'')c] d''' \\ & + [(a'b - a'b')c''' + (a'b''' - a''b'')c' + (a''b'' - a'b''')c] d'' \\ & + [(a'''b - a'b''')c'' + (a'b'' - a''b'')c''' + (a''b''' - a'''b'')c] d' \\ & + [(a'b''' - a'''b'')c'' + (a''b'' - a''b''')c' + (a'b'' - a'b''')c] d \end{aligned} \right\} e''' \\
 & \left\{ \begin{aligned} & + [(a'b - a'b')c'' + (a'b'' - a''b'')c' + (a''b' - a'b'')c] d''' \\ & + [(a'b' - a'b'')c''' + (a''b - a'b''')c' + (a'b'' - a''b'')c] d'' \\ & + [(a'b''' - a'''b'')c'' + (a''b - a'b''')c''' + (a''b'' - a''b''')c] d' \\ & + [(a'''b' - a'b''')c'' + (a''b'' - a''b''')c' + (a'b'' - a'b'')c] d \end{aligned} \right\} e'' \\
 & \left\{ \begin{aligned} & + [(a'b - a'b')c''' + (a'b'' - a''b'')c' + (a''b' - a'b'')c] d''' \\ & + [(a'b' - a'b'')c''' + (a''b - a'b''')c' + (a'b'' - a''b'')c] d''' \\ & + [(a'b'' - a''b'')c''' + (a''b - a'b''')c''' + (a''b'' - a''b''')c] d' \\ & + [(a''b' - a'b'')c''' + (a''b'' - a''b''')c' + (a'b'' - a'b'')c] d \end{aligned} \right\} e' \\
 & \left\{ \begin{aligned} & + [(a'''b - a'b''')c'' + (a'b'' - a''b'')c''' + (a''b'' - a''b'')c] d' \\ & + [(a'b'' - a''b'')c''' + (a''b - a'b''')c''' + (a''b'' - a''b'')c] d'' \\ & + [(a'b'' - a''b'')c''' + (a''b - a'b''')c''' + (a''b'' - a''b'')c] d''' \\ & + [(a''b' - a'b'')c''' + (a''b'' - a''b'')c''' + (a''b'' - a''b'')c] d \end{aligned} \right\} e \\
 & \left\{ \begin{aligned} & + [(a'b'' - a''b'')c'' + (a''b'' - a''b'')c' + (a''b' - a'b'')c] d''' \\ & + [(a''b' - a'b'')c''' + (a''b'' - a''b'')c' + (a'b'' - a''b'')c] d'' \\ & + [(a''b'' - a''b'')c''' + (a''b'' - a''b'')c' + (a''b'' - a''b'')c] d' \\ & + [(a''b' - a'b'')c''' + (a''b'' - a''b'')c' + (a'b'' - a''b'')c] d \end{aligned} \right\} e
 \end{aligned}$$

LXXXIX.

When there are as many simple Equations given as Quantities required, generally speaking, the Problem is limited, but in some particular Cases it may be unlimited or indeterminate, and in others impossible, this happens when the common Denominator is found equal to nothing; that is, if there be two Equations given, when $ab' - a'b = 0$, if there be three, when $(ab' - a'b)c'' + (a''b - ab'')c' + (a'b'' - a''b')c = 0$, &c. then if the Quantities A, A', A'', A''' , &c. are such that the Numerators are also equal to nothing, the Problem is indeterminate; because the Fractions $\frac{0}{0}$ which should express the Values of the unknown Quantities are indeterminate; but if the Quantities A, A', A'', A''' , &c. are such that the common Denominator being equal to nothing, the Numerators or any one of them is not equal to nothing, the Problem is impossible, or at least the Quantities required are all or some of them greater than any assignable Quantities. For Example, let there be given the two Equations $2 = 3x - 2y$ and $5 = 6x - 4y$, there will result $x = \frac{2}{3}$ and $y = \frac{5}{3}$. Now as 0 is less than any assignable Quantity, it follows that x and y are greater than any assignable Quantities.

If the unknown Quantities be disengaged according to the ordinary Method, there would result this absurd Equation $\frac{2}{3} = \frac{5}{3}$, for the first Equation gives $x = \frac{2}{3}y + \frac{2}{3}$, and the second $x = \frac{4}{6}y + \frac{5}{6}$. Wherefore $\frac{2}{3}y + \frac{2}{3} = \frac{4}{6}y + \frac{5}{6}$, or $\frac{2}{3} = \frac{5}{6}$, which is absurd, if x and y are assignable Quantities, but if they are greater than any assignable Quantities, it may be said without absurdity, that $x = \frac{2}{3}y + \frac{2}{3}$, at the same Time that $x = \frac{4}{6}y + \frac{5}{6}$; because the finite Magnitudes $\frac{2}{3}$ and $\frac{5}{6}$ vanishing in Respect of the infinite Magnitudes x and $\frac{2}{3}y$, the two Equations $x = \frac{2}{3}y + \frac{2}{3}$ and $x = \frac{4}{6}y + \frac{5}{6}$ are reduced to $x = \frac{2}{3}y$, which involves no Contradiction.

XC.

The greatest Difficulty which commonly occurs in the Solution of a Problem consists in finding the Equations arising from the Conditions, because it often happens that the Relations necessary for forming Equations are not expressed in the Conditions of the Problem. In this Case the Analysts examine, 1^o: whether some Quantity known or unknown, whose Relation with the other Quantities can be expressed by an Equation, may not be introduced into the Problem; and if no such Quantity can be found, or if this Quantity is not sufficient, they examine, 2^o: whether among the known and unknown Quantities there be any which may be expressed by new Letters, whereby a new Equation may result. To understand which, let the following Problem be proposed.

Problems which furnish as many equations as quantities required are not always limited, but in some cases are unlimited and in others impossible.

Rules for bringing problems to equations.

Problem.

There are three Meadows a, b, c , of the same Quality, each of a given Extent, in which the Grass grows uniformly; a Number of Oxen d will eat up the Pasture a in a Number of Days e , and a Number of Oxen f will eat up the Pasture b in a Number of Days g . It is required to find the Number of Oxen x which will eat up the Pasture c in a Number of Days h .

It is plain that the Conditions of the Problem do not express the Relations requisite for forming Equations, but as it is Question of the Quantity of Grass in each Meadow when the Oxen entered & what grew during their Stay, I divide the Oxen of each Meadow into two Herds, I suppose the first to eat up the Grass grew in each Meadow when they entered, and the second to eat up the Grass which grows, hence I suppose $d = y + z$, $f = t + u$, $x = s + r$.

I first consider the Oxen which eat up the Grass already grew, and observing that the Number of Oxen to eat up a Meadow should be greater in Proportion as the Meadow is greater and the Time is less, I make the two Proportions $y : t = \frac{a}{e} : \frac{b}{g}$ and $y : s = \frac{a}{e} : \frac{c}{h}$, from whence I deduce $t = \frac{e b y}{a g}$ and $s = \frac{e c y}{a h}$.

I next consider the Oxen which eat up the Grass which grows whilst the others eat up the Grass already grew, and observing that their Number should be greater in Proportion as the Meadows are greater, without any Regard being had to the Time. I make the two Proportions $z : u = a : b$ and $z : r = a : c$, from whence I deduce the Values of z and r , Viz. $z = \frac{a u}{b}$ and $r = \frac{c z}{a}$.

Now having seven Equations and seven unknown Quantities, $d = y + z$, $f = t + u$, $x = s + r$, $t = \frac{e b y}{a g}$, $s = \frac{e c y}{a h}$, $z = \frac{a u}{b}$, $r = \frac{c z}{a}$. Substituting in the third Equation for s and r their Values, there results $x = \frac{e c y}{a h} + \frac{c u}{b}$, and deducing from the Equations $d = y + \frac{a u}{b}$, and $f = \frac{e b y}{a g} + u$, $y = \frac{b d g - a f g}{g b - b e}$ and $u = \frac{a f g - e b d}{a g - a e}$, and substituting those Values of y and u in the Equation $x = \frac{e c y}{a h} + \frac{c u}{b}$, I find $x = \frac{a c f g b - a c e f g - b c d e b + b c d e g}{a b g b - a b e b}$.

To apply this general Solution to an Example, let the first Meadow contain $3 \frac{1}{2}$ Acres, the second 10 Acres, and the third 24 Acres; and let 12 Oxen eat up the Pasture of the first in 4 Weeks, 21 Oxen eat up the Pasture of the second in 9 Weeks; and let it be required to find how many Oxen will eat up the third in 18 Weeks. By substituting those Values in the general Solution, the Number will be found to be 36.

C H A P. II.

Of the Resolution of Equations of the second Degree.

HAVING fully explained what concerns the Solution of Problems of the first Degree, Order requires that we should pass to those of the second Degree, which we propose to treat of in this Chapter. As to the Manner of expressing their Conditions, it is the same as for Problems of the first Degree; it is only to solve the Equations to which the Problems are brought, that different Methods are employed according to the Degrees of those Equations. Of this we have an Instance in the following Problem, which in its full Extent includes Problems of every Degree, and is not more difficult to be expressed analytically in the most complicated Case as in the most simple.

I.

A Merchant having placed a Sum a in Trade, finding himself to be a loser, is willing to withdraw at the End of the first Year, but having missed the Opportunity and not being able to obtain it until the End of the second, third, or in general until the End of the n th Year, he finds that the Sum is diminished by the Quantity b , more than it was at the End of the first Year. It is required to determine how much per Cent. his Loss amounted to yearly.

Problem which in its full extent includes problems of every degree.

Let x be the Number sought, that is, what each £100 lost at the End of the first Year. Making the Proportion $100 : 100 - x = a : a \times \frac{100 - x}{100}$, the fourth Term $a \times \frac{100 - x}{100}$, or $a \times \left(1 - \frac{x}{100}\right)$ will express what the Sum a is reduced to at the End of the first Year.

If this Proportion be continued by saying $100 : 100 - x = a \times \frac{100 - x}{100} : a \times \frac{(100 - x)^2}{10000}$, the fourth Term $a \times \frac{(100 - x)^2}{10000}$, or $a \times \left(1 - \frac{x}{100}\right)^2$ will express what the Sum a is reduced to at the End of the second Year, and what the same Sum a is reduced to at the End of the third Year will be expressed by $a \times \left(1 - \frac{x}{100}\right)^3$, and in general what it is reduced to at the End of the n th Year will be expressed by $a \times \left(1 - \frac{x}{100}\right)^n$, that is, by a multiplied by the Quantity $1 - \frac{x}{100}$ raised to the Power n .

II.

If now it was proposed to find the Equation to be solved, supposing the Merchant to have withdrawn at the End of the second Year, it is

Equation of
the forego-
ing problem
for the se-
cond degree

manifest that the Quantity $a \times \left(1 - \frac{x}{100}\right)^2$ should be put equal to the Quantity $a \times \left(1 - \frac{x}{100}\right)$ diminished by the Quantity b , which will give $a \times \left(1 - \frac{x}{100}\right)^2 = a \times \left(1 - \frac{x}{100}\right) - b$, or multiplying $1 - \frac{x}{100}$ by itself, as indicated by the Exponent 2,
 $a \times \left(1 - \frac{2x}{100} + \frac{x^2}{10000}\right) = a \times \left(1 - \frac{x}{100}\right) - b$, which is reduced to $x^2 - 100x = -10000 \frac{b}{a}$, Equation of the second Degree, for to solve which the foregoing Methods are insufficient.

III.

For the
third degree

If the Merchant is supposed to withdraw at the End of the third Year, the Equation to be solved will be $a \times \left(1 - \frac{x}{100}\right)^3 = a \times \left(1 - \frac{x}{100}\right) - b$, which by multiplying $1 - \frac{x}{100}$ twice by itself, as the Exponent 3 indicates, becomes

$a \times \left(1 - \frac{3x}{100} + \frac{3x^2}{10000} - \frac{x^3}{1000000}\right) = a \times \left(1 - \frac{x}{100}\right) - b$,
 or $x^3 - 300x^2 + 20000x = 1000000 \frac{b}{a}$, an Equation more difficult to be solved than the foregoing.

IV.

As to the other Cases it is easy to perceive how the Equations they furnish may be formed, and that the Equation will be always of a Degree expressed by the Number n . If this Equation in general without specifying the Number n be required, it may be obtained by employing the general Expression $a \times \left(1 - \frac{x}{100}\right)^n$ of the Quantity to which a is reduced to after the n th Year, and the Equation will be

$a \times \left(1 - \frac{x}{100}\right)^n = a \times \left(1 - \frac{x}{100}\right) - b$, or
 $\left(1 - \frac{x}{100}\right)^n = 1 - \frac{x}{100} - \frac{b}{a}$.

V.

We shall for the present confine ourselves to the Investigation of the Solution of that Case of the Problem in which its Equation is of the second Degree, that is, when it is $x^2 - 100x = -10000 \frac{b}{a}$, or ra-

ther we shall investigate a Method for solving in general all Equations of the second Degree. Those who are desirous of solving the higher Cases of the same Problem will easily effect it after they have seen the general Methods to be explained hereafter corresponding to the different Degrees of those Equations.

What most naturally occurs in searching for a Method for solving in general Equations of the second Degree, is to examine the Relation which subsists between those Equations and the Equations of the first Degree. Now it is evident that every Equation of the first Degree will become one of the second, if the two Members be squared, for Example, $x + a = b$ becomes, when squared, $x^2 + 2ax + a^2 = b^2$, it remains therefore to know whether by a contrary Operation every Equation of the second Degree may not be reduced to one of the first.

Investigation of the method for solving equations of the second degree

Let us take, for Example, the Equation $x^2 - px = -q$, and let us try if $x^2 - px = -q$ is not the Square of some Quantity, the first Part of which is x and the second a known Quantity, in order to find by this Means the Equation of the first Degree, which being squared, would become $x^2 - px = -q$. Now it is easy to perceive that $x^2 - px$ is not a Square, but at the same Time it is manifest that it may be made one by an Addition of some Quantity, and we are at Liberty to make this Addition, provided the same Quantity be added to the other Side of the Equation.

To find this Quantity which added to $x^2 - px$ will render it a compleat Square, we have no more to do than to compare it with the Square $x^2 + 2ax + a^2$, the Term px corresponding with $2ax$, p will correspond to $2a$, and consequently a to $\frac{1}{2}p$. Now as a is what compleats $x^2 + 2ax$ into a Square, the Square of $-\frac{1}{2}p$, that is, $\frac{1}{4}p^2$ will compleat $x^2 - px$ into a Square, that is, $x^2 - px + \frac{1}{4}p^2$ will be a Square, it is one in effect, Viz. that of $x - \frac{1}{2}p$ when $x > \frac{1}{2}p$, and that of $\frac{1}{2}p - x$ when $x < \frac{1}{2}p$. Having therefore added $\frac{1}{4}p^2$ to the first Member of the Equation, the same must be added to the second, and there results $x^2 - px + \frac{1}{4}p^2 = \frac{1}{4}p^2 - q$. Now either the Quantity $x - \frac{1}{2}p$ or the Quantity $\frac{1}{2}p - x$ multiplied by itself, gives $x^2 - px + \frac{1}{4}p^2$, wherefore $x - \frac{1}{2}p$ or $\frac{1}{2}p - x$ will be equal to the Number which multiplied by itself will give $\frac{1}{4}p^2 - q$, which is expressed in general thus, $\sqrt{\frac{1}{4}p^2 - q}$, the Sign $\sqrt{\quad}$ being employed by the Analysts to indicate that the square Root of the Quantity before which it is placed is to be extracted.

The sign $\sqrt{\quad}$ indicates the square root.

Employing therefore this Denomination $x - \frac{1}{2}p = \sqrt{\frac{1}{4}p^2 - q}$ and $\frac{1}{2}p - x = \sqrt{\frac{1}{4}p^2 - q}$, from whence is deduced $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$. The Sign $\pm$ being employed by the Analysts to include in one and the same Expression the two Values of x .

An equation of the second degree has two roots.

VI.

Let us now apply this general Solution to the Equation $x^2 - 100x = -10000 \frac{b}{a}$, to which we were led in the foregoing Problem. Comparing this Equation with $x^2 - px = -q$, we will have $p = 100$, $q = 10000 \frac{b}{a}$, and substituting those Values in the Place of p and q in the general Formula $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$, we will have $x = 50 \pm \sqrt{(2500 - 10000 \frac{b}{a})}$.

VII.

Reduction
of the value
of x by re-
solving the
root of the
product into
those of its
factors.

We may reduce to a more simple Form the radical Part $\sqrt{(2500 - 10000 \frac{b}{a})}$ of this Value of x , by observing, that the square Root of the Product of two or more Quantities is the Product of the square Roots of those Quantities; for resolving $2500 - 10000 \frac{b}{a}$ into its Factors $2500, 1 - \frac{4b}{a}$, and extracting the Roots of those two Quantities we will have 50 and $\sqrt{(1 - \frac{4b}{a})}$ whose Product $50 \sqrt{(1 - \frac{4b}{a})}$ will be $\sqrt{(2500 - 10000 \frac{b}{a})}$ that is, that the Value of x will be $50 \pm 50 \sqrt{(1 - \frac{4b}{a})}$.

As to the Demonstration of this Principle, that the Root of any Product is obtained by multiplying the Roots of its Factors, it easily may be found, by observing that the Square of a Product, as ab , is the Product $aa \times bb$, of the Squares aa & bb of its Factors a and b .

VIII.

Example of
this prob-
lem.

To make Use of this Value of x , nothing more is required than to know the Ratio that b bears to a . Let b , for Example, be the $\frac{6}{25}$ of a , that is, let us suppose the Merchant to have found the Sum at the End of the second Year, diminished by a Quantity equal to the $\frac{6}{25}$ of the Whole more than it was at the End of the first Year, according to this Supposition $\frac{4b}{a} = \frac{24}{25}$ and $1 - \frac{4b}{a} = \frac{1}{25}$, hence the Root $\sqrt{(1 - \frac{4b}{a})} = \frac{1}{5}$ and consequently will give $x = 50 \pm 50 \times \frac{1}{5} = 60$ or 40 .

In effect those two Values of x resolve the Equation $x^2 - 100x = -2400$ into which the Equation $x^2 - 100x = -10000 \frac{b}{a}$ is transformed by the Supposition of $\frac{b}{a} = \frac{6}{25}$, for whether x be supposed $= 60$ or 40 , $x^2 - 100x$ becomes $= 2400$.

The Necessity of the two Solutions 60 and 40 will appear after a more satisfactory Manner, as follows. Let us suppose first $x = 60$, that is, that the Sum of £100000, for Example, is diminished 60 per Cent. yearly, it is manifest that at the End of the first Year it would be reduced to £40000, and at the End of the second Year it will be reduced to £16000; but £16000 are less than £40000 by £24000, which are the $\frac{6}{25}$ of £100000, as the Problem requires.

Let us now suppose the same Sum of £100000 to be diminished 40 per Cent. yearly, at the End of the first Year it will be reduced to £60000, and at the End of the second to £36000, but £36000 are less than the Sum £60000 by the Quantity £24000, or the $\frac{6}{25}$ of £100000.

IX.

If b was supposed to be the $\frac{4}{25}$ of a , then $x = 50 \pm 50 \sqrt{1 - \frac{16}{25}}$ Another example. $= 50 \pm 50 \sqrt{\frac{9}{25}} = 50 \pm 30$, that is, 80 or 20, both of which will be found to solve the Problem.

X.

If $\frac{b}{a}$ be put $= \frac{1}{3}$, there will result $x = 50 \pm 50 \sqrt{1 - \frac{4}{9}}$, or $50 \pm 50 \sqrt{-\frac{1}{9}}$, but as the Squares of all Quantities are affirmative, it follows that the Quantity $\sqrt{-\frac{1}{9}}$ cannot be assigned, or which comes to the same that the Problem is impossible in this Case. Third example in which the root of a negative quantity being required is impossible

Hence we may conclude that there is no possible Value which substituted for x in the Equation $x^2 - 100x = -\frac{10000}{3}$ will render the two Members equal, or what comes to the same Thing, that the Sum a cannot be lessened each Year according to any given Proportion, so that from the second to the third Year the Diminution may amount to a Quantity equal to a Third of the whole Sum. The Analysts however regard as a Kind of a Solution or Root of the Equation $x^2 - 100x = -\frac{10000}{3}$, the Value $50 \pm 50 \sqrt{-\frac{1}{9}}$ which results from thence, but they call it an imaginary Root, and this imaginary Root on Account of the Sign $\pm$ is always considered as a double Solution. Those roots are called imaginary.

From the general Value of $x, \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$, it appears that as often as the Quantity designed by q is negative and greater than $\frac{1}{4}p^2$, the two Roots of the Equation $x^2 - px = -q$ will be both imaginary. Equations of the second degree whose roots are imaginary.

XI.

Though the Roots of an Equation or the Values of the unknown Quantity are real and positive, however they do not always, as in the foregoing Example both solve the Problem. Let the following Question be proposed. The real and positive values of the unknown quantity do not always solve the problem

The Sum of £190 was divided between three Persons; whose Shares were in geometrical Proportion, and the greatest of them exceeded the least by £50. What were the several Shares?

If x be put for the least of them, then the greatest will be $x + 50$, First example. the Sum of which two subtracted from (190) the Whole, leaves $140 - 2x$

for the mean Share. Therefore $x : 140 - 2x :: 140 - 2x : x + 50$, and consequently $(140 - 2x)^2 = x(x + 50)$, that is, $19600 - 560x + 4x^2 = x^2 + 50x$, whence $3x^2 - 610x = -19600$, and $x^2 - \frac{610x}{3} = -\frac{19600}{3}$, hence by completing the Square $x^2 - \frac{610x}{3} + \frac{93025}{9} = \frac{19600}{3} + \frac{93025}{9} = \frac{34225}{9}$, and by extracting the Root $x = \frac{305}{3} \pm \frac{185}{3}$, or $x = 40$ and $x = 163 \frac{1}{3}$, two real and positive Roots, however the first Root only solves the Problem, the other two Shares being 60 and 90 Pounds.

As to the Root $163 \frac{1}{3}$ it solves the following Question.

A Sum of Money was divided between three Persons, whose Shares were in geometrical Proportion, the greatest exceeding the least by £50, and their Sum exceeding the mean Share by £190. What were the several Shares?

For the analytical Expression of this Question is $(2x - 140)^2 = x^2 + 50x$, from whence results the same Equation as in the foregoing Question, and in this Case $x = 163 \frac{1}{3}$. Because the Equation $19600 - 560x + 4x^2 = x^2 + 50x$, includes not only $(140 - 2x)^2 = x^2 + 50x$, but also $(2x - 140)^2 = x^2 + 50x$, which are the analitick Translations of two very different Questions, the first of which gives $x = 40$, the latter $163 \frac{1}{3}$, wherefore the Roots of an Equation, though real and positive, do not both solve the Question in the Sense it is proposed, but considered in two different Lights, which Difference cannot be expressed analitically. For a further Illustration, here follows another Example.

XII.

Two Travellers set out at the same Time, the one from Dublin, the other from Londonderry, the former for Londonderry, the latter for Dublin; when they met and had computed their Journey, it was found, that the former had travelled thirty Miles more than the latter, and that at their Rate of Travelling, the former expected to reach Londonderry in four Days, and the latter to reach Dublin in nine Days; the Distance between Dublin and Londonderry is required.

I put x for the Number of Miles between Dublin and Londonderry; and since the Travellers both together had travelled x Miles when they met; it follows that the former had travelled $\frac{x+30}{2}$ Miles, and the latter $\frac{x-30}{2}$ Miles, according to the first Condition of the Problem, therefore the remaining Part of the formers Journey to Londonderry is $\frac{x-30}{4}$ Miles, which he expects to perform in four Days, and the

remaining Part of the latters Journey is $\frac{x+30}{2}$ Miles, which he expects to perform in nine Days. I now enquire into the Number of Days each hath travelled already, and I say $\frac{x-30}{2} : \frac{x+30}{2} = 4 :$

$\frac{4 \times (x+30)}{x-30}$ Number of Days travelled by the Former, and $\frac{x+30}{2} : \frac{x-30}{2} = 9 : \frac{9 \times (x-30)}{x+30}$ Number of Days travel-

led by the Latter, but as they both set out at the same Time and are now met, they must have travelled the same Number of Days, therefore

$\frac{4 \times (x+30)}{x-30} = \frac{9 \times (x-30)}{x+30}$, which after the necessary Re-

ductions, becomes $x^2 - 156x = -900$, whence by compleating the Square $x^2 - 156x + 6084 = 5184$, consequently $x = 150$ or $x = 6$, therefore the Distance between *Dublin* and *Londonderry* must either be 150 or 6 Miles; but 6 Miles it cannot be, because when the first Traveller came up to the Latter, he had travelled 30 Miles more than him, and had not yet reached *Londonderry*; therefore the Distance between *Dublin* and *Londonderry* is 150 Miles; for then the first Traveller must have made $75 + 15$ or 90 Miles, and the second $75 - 15$ or 60 Miles, from the Time of their setting out; therefore the Former has 60 Miles and the Latter 90 Miles to travel; but if the Former could travel 60 Miles in 4 Days, he must at the same Rate have travelled 90 Miles in 6 Days, and if the Latter could travel 90 Miles in 9 Days, he must have travelled 60 Miles in 6 Days, as the Problem requires.

As to the second Value, $x = 6$, it solves the following Problem.

Two travellers set out at the same Time, the one from Dublin, the other from Swords, the Former with a Design to pass through Swords, and the Latter with a Design to travel the same Way. The Former had overtaken the Latter, and having computed their Journey, it was found that they had both together travelled 30 Miles, that the Former had passed through Swords 4 Days before, and that the Latter at his Rate of Travelling, was a nine Days Journey distant from Dublin. The Distance of Dublin from Swords is required.

For if x be put for the Number of Miles from *Dublin* to *Swords*, then it is plain that the first Traveller must have travelled more Miles than the Latter by x ; but they both together travelled 30 Miles when they met, wherefore the first must have travelled $\frac{30+x}{2}$, and the Latter must have travelled $\frac{30-x}{2}$ Miles, therefore the Distance of the first from *Swords*, after he had overtaken the second, was $\frac{30-x}{2}$ Miles, which

ELEMENTS OF

he had travelled in 4 Days, and the Distance of the second from *Dublin* was $\frac{30+x}{3}$ Miles, which he could travel in 9 Days; therefore to find

how many Days each had travelled already, I say $\frac{30-x}{2} : \frac{30+x}{3}$

$= 4 : \frac{4 \times (30+x)}{30-x}$ Number of Days travelled by the Former, and

$\frac{30+x}{2} : \frac{30-x}{2} = 9 : \frac{9 \times (30-x)}{30+x}$ Number of Days travel-

led by the Latter, and as they both set out at the same Time, they must have both travelled the same Number of Days, wherefore

$\frac{4 \times (30+x)}{30-x} = \frac{9 \times (30-x)}{30+x}$, which after the necessary Reduc-

tions, becomes $x^2 - 156x = -900$, the same Equation as found above, whose Roots are $x = 150$ and $x = 6$. Wherefore the Distance between *Dublin* and *Swords* must either be 6 or 150 Miles, but 150 Miles it cannot be, because after the first Traveller had passed from *Dublin* beyond *Swords*, and at last had overtaken the second, they had both travelled but 30 Miles, therefore the Distance from *Dublin* to *Swords* must be 6 Miles, and this Number will answer the Conditions of the Problem, for then the first Traveller when he had overtaken the Latter, had travelled $15 + 3$ or 18 Miles, and the Latter $15 - 3$ or 12 Miles. Therefore the first Traveller had got 12 Miles beyond *Swords* in 4 Days Time, and the Latter was 18 Miles distant from *Dublin*, which he could travel in 9 Days, but at the Rate of 12 Miles in 4 Days, the Former must have performed his 18 Miles Journey in 6 Days, and at the Rate of 18 Miles in 9 Days, the Latter must have performed his 12 Miles Journey also in 6 Days, therefore from the Time of their first setting out to the Time of their Meeting, they had both travelled the same Number of Days as the Problem requires.

XIII.

General
rule for re-
solving equa-
tions of the
second de-
gree.

It is easy to perceive that any Equation of the second Degree may be resolved by repeating the Process pursued in resolving the foregoing Equations, which may be expressed thus. 1°. Transpose all the Terms that involve the unknown Quantity to one Side, and the known Terms to the other Side of the Equation. 2°. If the Square of the unknown Quantity is affected with a Coefficient, you are to divide all the Terms by that Coefficient, and if the Sign of that Power be negative you are to change all the Signs of the Terms. 3°. Add to both Sides the Square of half the Coefficient prefixed to the unknown Quantity itself, and the Side of the Equation that involves the unknown Quantity will then be a compleat Square. 4°. Extract the Square Root from both Sides of the Equation, which you will find on one Side always to be the unknown

Quantity, with half the foresaid Coefficient subjoined to it, so that by transposing this half you may obtain the Value of the unknown Quantity expressed in known Terms, for Example, suppose $x^2 - 6x = 27$, adding 9 the Square of the half of 6 to both Sides, there results $x^2 - 6x + 9 = 9 + 27 = 36$, and extracting the Square Root you will find $x - 3 = 6$ and $3 - x = 6$, according as x is $>$ or $<$ 6, wherefore $x = +3 \pm 6$ or $x = 9$ and $x = -3$, which both resolve the Equation $x^2 - 6x = 27$, in like Manner if $x^2 + 8x = 9$, adding 16 Square of the half of 8 to both Sides, there will result $x^2 + 8x + 16 = 16 + 9 = 25$, and extracting the Square Root you will find $x + 4 = \pm 5$, that is, $x = -4 \pm 5$, or $x = -9$ and $x = 1$.

XIV.

To accustom Beginners to the Difficulties which occur in the Solution of Problems of the second Degree, here follows another Problem.

Let it be proposed to find in the Line which joins two Lights the Point where a Body would be equally illuminated by them, according to this Principle of Physicks, that the Effect of a Light is four Times greater when it is twice nearer, nine Times greater when it is three Times nearer, or increases as the Square of the Distance decreases.

Another problem of the second degree.

Let a express the Distance between the two given Lights, and let the Ratio of m to n express that of the Effect of the least Light at a certain Distance, to the Effect of the greatest Light at the same Distance.

Let x express the Distance of the least of the two Lights from a Point taken at will in the Line which joins the two Lights, it is manifest that $a - x$ will express the Distance of the other Light from the same Point, that the Squares of those two Distances will be x^2 and $x^2 - 2ax + a^2$, and consequently the Quantities which decrease as the Squares of those Quantities increase will be to one another as $\frac{1}{x^2}$ and $\frac{1}{x^2 - 2ax + a^2}$.

From whence it follows that if the Lights were of equal Force, the Effects they would each produce in this same Point would be to each other as $\frac{1}{x^2}$ to $\frac{1}{x^2 - 2ax + a^2}$; but the absolute Quantities of those Lights being to each other in the Ratio of m to n , their Effects therefore will be to each other as $\frac{m}{x^2}$ to $\frac{n}{x^2 - 2ax + a^2}$.

Now that the Point taken at will may become the Point required, I put those two Quantities equal to each other, which gives the Equation $m a^2 - 2 a m x + m x^2 = n x^2$.

To solve this Equation, I transpose the Terms $m x^2$ and $2 a m x$ into the other Member, and there results $(n - m) x x + 2 a m x = m a a$.

or $x x + \frac{2 a m}{n - m} x = \frac{a a m}{n - m}$. I afterwards add to the two Members of this Equation the Square of half the Coefficient of the second Term, and there results $x^2 + \frac{2 a m x}{n - m} + \frac{a^2 m^2}{(n - m)^2} = \frac{a a m}{n - m} + \frac{a a m m}{(n - m)^2}$, whose second Member becomes $\frac{a^2 m n}{(n - m)^2}$, by reducing the two Terms $\frac{a a m}{n - m} + \frac{a a m m}{(n - m)^2}$ to the same Denominator.

Then extracting the Square Root of the two Members, there results $x + \frac{a m}{n - m} = \pm \sqrt{\frac{a a m n}{(n - m)^2}}$, or $x = -\frac{a m}{n - m} \pm \frac{a}{n - m} \sqrt{m n}$.

By extracting the Root of the Part $\frac{a a}{(n - m)^2}$ which is a perfect Square, and leaving under the radical Sign its Multiplier $m n$, which is no Square, at least for all the Values of m and n . Wherefore the two Values of x which solve the foregoing Equation, and consequently the Problem which leads to this Equation are expressed by the Formula $x = -\frac{a m}{n - m} \pm \frac{a}{n - m} \sqrt{m n}$, or $x = \frac{a}{n - m} \times -m \pm \sqrt{m n}$.

XV.

Of the two foregoing values, one is necessarily positive and the other negative.

From this Expression it appears that one of the Values is necessarily negative and the other positive, for 1°. if we take the radical Quantity $\sqrt{m n}$ with the Sign $-$, there is no doubt but the whole Quantity will be negative, 2°. if we take $\sqrt{m n}$ with the Sign $+$, $-m + \sqrt{m n}$ which we will have then will be positive, because n being greater m , $\sqrt{m n}$ must be greater than m .

XVI.

Use of the negative value.

If we now enquire into the Use of the negative Value of the Distance x of the least Light to the assumed Point, we will find by recalling what had been demonstrated (Chap. I. Art. LXIII.) with Respect to those Values in Equations of the first Degree, that its Direction is opposite to that of the first, that is, that the Point which it gives for solving this Problem, instead of being placed between the two Lights, will be placed in the Line on the other Side of the weakest Light.

There can be no Difficulty in admitting this Position of the negative Value of x , when it is observed, that this same Value was found negative only, because the Problem was solved in the Supposition that the Point sought was placed between the two Lights; for if it had been supposed (as it might be) to be placed in the Line which joins them, produced on the Side of the weakest Light, there would have resulted another Computation relative to this Position, and x which in this Case

is taken in the Line produced on the Side of the weakest Light would be positive.

XVII.

In order to make this appear, we will resume the Problem, the Point sought being supposed in the Line produced on the Side of the weakest Light. The Distance of this Point from the least Light being expressed by x , its Distance from the greatest Light will be expressed by $a + x$, the Squares of those Distances will be x^2 and $a^2 + 2ax + x^2$, and the two Quantities of Light $\frac{m}{x^2}$ and $\frac{n}{a^2 + 2ax + x^2}$, which being equal by the Conditions of the Problem, will give

$\frac{m}{x^2} = \frac{n}{a^2 + 2ax + x^2}$, or $ma^2 + 2amx + mx^2 = nx^2$,
or $(n - m)x^2 - 2amx = ma^2$, or $x^2 - \frac{2amx}{n - m} = \frac{m}{n - m}a^2$,
which being resolved will give $x = \frac{a \times (m \pm \sqrt{mn})}{n - m}$, the first Value of which $\frac{a \times (m + \sqrt{mn})}{n - m}$ will be positive, and solves the Problem in the Sense it is proposed.

As to the second Value $\frac{a \times (m - \sqrt{mn})}{n - m}$ of the Distance x , being negative, it should be taken in an opposite Direction to the first, that is, the Point which it gives is not situated in the Line, which joins the two Lights produced, but in the Line itself.

Hence in this Solution the Values of x differ from those in the former with Respect to the Signs, and those two Solutions confirm what has been already proved in the foregoing Chapter, Art. LXIII. that the unknown Quantities which become negative, should be always considered as being of an opposite Kind from what they have been supposed in expressing the Conditions of the Problem.

XVIII.

In order to remove whatever Difficulties may occur to the Reader with Respect to this Problem, we shall apply it to an Example. Let us suppose $n = 4m$, that is, that the greatest Light has four Times the Force of the other. Substituting this Value of n in the general Formula of Art. XIV. $x = \frac{a}{n - m} \times (m \pm \sqrt{mn})$, it will be transformed into $x = \frac{a}{3} \times \pm (2 - 1)$, that is, $+\frac{1}{3}a$, or $-a$, which furnish two Points, both of which solve the Problem, one situated between

Example of
the foregoing
problem

the two Lights twice nearer the weakest than the strongest, and the other in the line produced, at a Distance from the weakest equal to the Distance between the two Lights.

XIX.

The positive & negative values of the unknown quantity do not always both solve the problem

The positive and negative Values of the unknown Quantity do not always, as in the foregoing Example both solve the Problem. Let the following Question be proposed.

A Draper buys some Ells of Cloth for 70 Crowns, and finds that if he had 4 Ells more, he had bought every Ell two Crowns cheaper. How many Ells did he buy?

Let x denote the Number of Ells bought, then dividing the whole Price by the Number of Ells, we have $\frac{70}{x}$ for the Price of one Ell, and

$\frac{70}{x+4}$ for the Price of one Ell if he had got 4 Ells more, consequently

ly $\frac{70}{x} - 2 = \frac{70}{x+4}$ whence $x^2 + 4x = 140$, consequently $x = -2 \pm 12$, or $x = 10$ and $x = -14$, the first Value of x solves the Problem, as to $x = -14$, it does not solve the proposed Question, it solves the following.

A Draper buys some Ells of Cloth for 70 Crowns, and finds that if he had 4 Ells less, he would have bought every Ell 2 Crowns dearer. How many Ells did he buy?

In Effect the analitick Expression of this Question is $\frac{70}{x} + 2 = \frac{70}{x-4}$, or $x^2 - 4x = 140$, from whence is deduced $x = 14$ and $x = -10$, which are precisely the Roots of the foregoing Equation with contrary Signs. From whence it appears that the negative Roots solve the Question, not as it is proposed, but with some Alterations which consist in subtracting what should be added, or in adding what should be subtracted. The Sign $-$ which precedes the Root $x = -14$, for Example, indicating that in the foregoing Question the 4 Ells are to be subtracted, and the 2 Crowns to be added to the Price.

XX.

If the Roots of an Equation or Values of the unknown Quantity are both negative, then the Question has been wrong stated, as will appear by the following Example.

A and B take in Trade £2112 per Annum each, but A whose Profits are 2 per Cent. greater than those of B, clears £100 per Annum more than B. What are the Profits of each per Cent. and what do they clear per Annum.

Let x denote what A gains per Cent. Now since A for every £100 he layed out in purchasing Goods, recovered in the Sales $100 + x$ Pounds, and 2112 being the whole Sum recovered, $100 + x : x = 2112 :$

When a question is wrong stated the values of the unknown quantity become both negative.

$\frac{2112x}{100+x}$ the whole Profit of *A*, and in like Manner $98+x : x-2$

$= 2112 : \frac{2112x-4224}{98+x}$ the whole Profit of *B*. Therefore

$\frac{2112x}{100+x} - \frac{2112x-4224}{98+x} = 100$ by the Question, hence

$x^2 + 198x = -5576$, whence by compleating the Square and extracting the Root $x = -99 \pm 65$, both which Roots are negative, which denotes that the Question should be expressed thus:

A and *B* take in Trade £2112 per Annum each, but *A* whose Losses are 2 per Cent. less than those of *B*, loses 100 per Annum less than *B*. What are the Losses each sustained per Cent. and what did their Losses amount to yearly?

In effect, since *A* for every £100 he layed out in purchasing Goods, recovered but $100-x$ Pounds, and £2112 being the whole Sum recovered, $100-x : x = 2112 : \frac{2112x}{100-x}$ the whole Loss of *A*, in like

Manner $98-x : x+2 = 2112 : \frac{2112x(x+2)}{98-x}$ the whole Loss

of *B*, wherefore $\frac{2112x+4224}{98-x} - \frac{2112x}{100-x} = 100$ by the Question. Hence $x^2 - 198x = -5576$, and compleating the Square and extracting the square Root $x = 99 \pm 65$, which are the same Roots as found before but affirmative.

From which it appears that *A* lost £34 per Cent. and his whole Loss amounted to £1088 per Annum, and that *B* lost £36 per Cent. and his whole Loss amounts to £1188 per Annum.

XXI.

The Principles which we have explained are sufficient for solving all Equations of the second Degree, but to render the Application of them easy to Beginners, we shall proceed to exercise them in the Resolution of several Equations, there will result this Advantage, that besides their becoming better acquainted with the Method, they will learn at the same Time the new Operations of specious Arithmetick, which, without doubt, are owing to the Researches which the first Analysts have made on the Equations of the second Degree.

Let $bx^2 = 2c^2x + 2c^2a$, transposing the Terms affected with x on one Side, and dividing all the Terms by the Coefficient of xx , we will have $x^2 - \frac{2c^2x}{b} = \frac{2c^2a}{b}$, to the two Members of which, adding $\frac{c^4}{b^2}$, and afterwards extracting the Square Root, we will have

$$x = \frac{c^2}{b} \pm \sqrt{\left(\frac{2c^2ab + c^4}{b^2}\right)} = \frac{c^2 \pm c\sqrt{2ab + c^2}}{b}.$$

Other examples of the resolution of equations of the second degree.

Let $f^2 + g^2 - 2gx + x^2 = \frac{m^2 x^2}{n^2}$ which I first reduce to
 $\left(\frac{mm - nn}{nn}\right) \times x^2 + 2gx = f^2 + g^2$, and afterwards to
 $x^2 + \frac{2gn^2}{m^2 - n^2} x = \frac{f^2 n^2 + g^2 n^2}{m^2 - n^2}$, or $x^2 + \frac{2gn^2 x}{m^2 - n^2}$
 $+ \frac{g^2 n^4}{(m^2 - n^2)^2} = \frac{f^2 n^2 + g^2 n^2}{m^2 - n^2} + \frac{g^2 n^4}{(m^2 - n^2)^2} =$
 $\frac{ffnnmm + ggnnmm - ff n^4}{(mm - nn)^2}$, from whence is deduced
 $x + \frac{gnn}{mm - nn} = \pm \frac{n \sqrt{(ffmm + ggmm - ffnn)}}{mm - nn}$, or
 $x = \frac{n}{mm - nn} \times [-gn \pm \sqrt{(ffmm + ggmm - ffnn)}]$.

Let $abc - af^2 + 2afz = az^2 - bz^2$ be given, whose Terms
 being disposed in order, will become $z^2 - \frac{2af}{a-b} z = \frac{abc - af^2}{a-b}$,
 and compleating the Square $z^2 - \frac{2af}{a-b} z + \frac{a^2 f^2}{(a-b)^2} =$
 $\frac{aabc - abbc + abff}{(a-b)^2}$, hence $z = \frac{af \pm \sqrt{(aabc - abbc + abff)}}{a-b}$.

Let the Equation $4a^2 - 2x^2 + 2ax = 18ab - 18b^2$ be pro-
 posed, disposing its Terms in order and reducing it, there results.
 $x^2 - ax = 2a^2 - 9ab + 9b^2$, and compleating the Square, be-
 comes $x^2 - ax + \frac{1}{4}a^2 = \frac{9}{4}a^2 - 9ab + 9b^2$, which gives
 $x = \frac{1}{2}a \pm \sqrt{(\frac{9}{4}a^2 - 9ab + 9b^2)}$.

The young Analyst would have easily reduced this Quantity if he
 had perceived, and he could not but have perceived by what precedes,
 that the Square of a Quantity composed of two Terms, is equal to the
 Sum of the Squares of each of those two Terms, and to double of the
 Product of those two Terms.

For finding in the Quantity $\frac{9}{4}a^2 - 9ab + 9b^2$, the Terms $\frac{9}{4}a^2$
 and $9b^2$, which are the Squares of $\frac{3}{2}a$ and $3b$, and the Term $9ab$
 which is double the Product of $\frac{3}{2}a$ into $3b$. It is easy to conclude that
 this Quantity $\frac{9}{4}a^2 - 9ab + 9b^2$ is the Square of $\frac{3}{2}a - 3b$, there-
 fore instead of the Expression $\sqrt{(\frac{9}{4}a^2 - 9ab + 9b^2)}$, we may write
 simply $\frac{3}{2}a - 3b$: Wherefore the Value of x is $\frac{1}{2}a \pm \frac{3}{2}a \mp 3b$,
 that is, either $2a - 3b$, or $-a + 3b$. In effect, both those Values
 solve the given Equation.

XXII.

Process of
 the extrac-
 tion of the
 square root

Among the different Equations of the second Degree, which are re-
 quired to be resolved, Cases similar to the foregoing may occur; it is
 therefore necessary to have a general Method for discovering such Quan-

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ties as are Squares, and for finding their Roots; this Method is easily deduced from the Principles employed in the foregoing Example. The Process of this Method applied to an Example, is as follows.

Let the square Root of the Quantity $30ab + 9b^2 + 25a^2$ be required.

$$\begin{array}{r} 25a^2 + 30ba + 9b^2 \\ - 25a^2 \\ \hline 30ba + 9b^2 \\ - 30ba - 9b^2 \\ \hline 0 \end{array} \quad \left| \begin{array}{l} 10a + 3b \\ 5a + 3b \end{array} \right.$$

I first range the Terms of this Quantity according to the Dimensions of the Letter a , for Example, as above.

I afterwards extract the Root of the first Term $25a^2$ which is $5a$, which will be the first Term of the Root, and I write it down beside the proposed Quantity $25a^2 + 30ba + 9b^2$, separating them by a Line in order to avoid Confusion. I then write under the proposed Quantity the Square $25a^2$ of $5a$, prefixing to it the Sign $-$. I draw a Line and reduce, and there remains $30ab + 9b^2$, which I write under the Line, which being done, I double $5a$ which gives $10a$, and divide the first Term $30ab$ of the Quantity $30ab + 9b^2$ by $10a$, and I write the Quotient $3b$, which is the second Term of the Root sought, beside $5a$, and I place it at the same Time beside $10a$, and I multiply the superior Quantity by this new Term $3b$ of the Root, observing as in Division to change the Signs, in writing the Product under the Quantity $30ab + 9b^2$, then reducing and finding that all the Terms destroy each other, I conclude that $5a + 3b$ is the Root required.

XXIII.

To render the Method of extracting the square Root familiar to Beginners, here follow more Examples.

Let it be proposed to extract the square Root of the Quantity $4a^2 - 4ab + 4ac + b^2 - 2cb + c^2$, whose Terms are ranged according to the Dimensions of the Letter a .

Another example of the extraction of the square root.

$$\begin{array}{r} 4a^2 - 4ab + 4ac + b^2 - 2cb + c^2 \\ - 4a^2 \\ \hline - 4ba + 4ca + b^2 - 2cb + c^2 \\ + 4ba \quad \quad - b^2 \\ \hline 4ca \quad \quad - 2cb + c^2 \\ - 4ca \quad \quad + 2cb - c^2 \\ \hline 0 \end{array} \quad \left| \begin{array}{l} 4a - 2b + c \\ 2a - b + c \\ 4a - b \end{array} \right.$$

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Example 3.

$$\begin{array}{r}
 9x^4 - 24a^2x^2 + 16a^4 \\
 + 12b^2x^2 - 16a^2b^2 \\
 - 9x^4 \qquad \qquad \qquad + 4b^4 \\
 \hline
 - 24a^2x^2 + 16a^4 \\
 + 12b^2x^2 - 16a^2b^2 \\
 \qquad \qquad \qquad + 4b^4 \\
 + 24a^2x^2 - 16a^4 \\
 - 12b^2x^2 + 16a^2b^2 \\
 \qquad \qquad \qquad - 4b^4 \\
 \hline
 0
 \end{array}
 \quad \left| \begin{array}{l}
 6x^2 - 4a^2 \\
 \qquad + 2b^2 \\
 3x^2 - 4a^2 \\
 \qquad + 2b^2
 \end{array} \right.$$

Example 4.

$$\begin{array}{r}
 y^4 + 4y^3 \quad \bullet \quad - 8y + 4 \\
 - y^4 \\
 \hline
 + 4y^3 \quad \bullet \quad - 8y + 4 \\
 - 4y^3 - 4y^2 \\
 \hline
 - 4y^2 - 8y + 4 \\
 + 4y^2 + 8y - 4 \\
 \hline
 0
 \end{array}
 \quad \left| \begin{array}{l}
 2y^2 + 4y - 2 \\
 y^2 + 2y - 2 \\
 2y^2 + 2y
 \end{array} \right.$$

XXIV.

Method of
extracting
the square
root of num-
bers.

The square Root of any Number may be found out after the same Manner. If it be a Number under 100, its nearest square Root is found by the following Table.

Squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.

Roots 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Where it is easy to observe, that the Square of a simple Number cannot consist of more than two Places of Figures, since the Square of 10 the least Number consisting of two Places is 100, the least Number consisting of three Places. That the Square of a Number consisting of two Places, cannot consist of more than four Places; for the Square of 100, the least Number consisting of three Places, is 10000, the least Number consisting of five Places; and in general, that the Square of any Number cannot consist of more than the Double of its Places.

From whence it follows that a Number consisting of less than three Places, can have but one Figure in its Root; that a Number consisting of more than three, but less than five Places, can have but two Figures in its Root; that a Number consisting of more than five, but less than seven Places, can have but three Figures in its Root, and so on, taking for Limits the odd Numbers 1, 3, 5, 7, 9, 11, &c. whose common

Difference is 2. Consequently the Number of Places which the square Root of any Number consists of, is discoverable by Inspection. How the Numbers corresponding to those Places are found, we shall now explain.

XXV.

Let it be proposed to extract the square Root of 99856, as it consists of five Places, I conclude that the Root will consist of three. To determine the Numbers corresponding to those Places, I express them by x , y and z , consequently $x^2 + 2xy + y^2 + (2x + 2y)z + z^2 = 99856$. Hence the Difficulty is reduced to find the Terms of this Quantity in the given Number 99856. First example.

The first Number x being Hundreds, its Square x^2 will be Tens of Thousands, or Tens of Thousands with Hundreds of Thousands; consequently this Term should be contained in 9, the Number corresponding to the fifth Place of the Proposed, and its square Root 3 (Hundreds) is the first Figure of the Root, which I place in the Quotient, and subtract the Square of 3 (Hundreds) from the proposed Number.

The Number y being Tens and multiplied by $2x$, the Double of the first, the Product should be Thousands or Thousands with Tens of Thousands, consequently the Term $2xy$ should be contained in 9, the Number corresponding to the fourth Place of the Proposed. I divide therefore 9 by 6, the Quotient is 1 (Ten) which being the second Part of the Root, I place it after the first Figure 3, and as y^2 the Square of the second Part of the Root is Hundreds, I conclude that the Terms $2xy + y^2$ should be contained in 98, the Numbers corresponding to the third and fourth Places of the Proposed. I therefore multiply 6 Hundreds and 1 Ten or 61 by 1, and subtract the Product from 98, and the Remainder is 37 (Hundreds).

It remains to find in the given Number the Terms $2xz + 2yz$, the Product of double the first Part of the Root (Hundreds) into the last (Units) which will produce either Hundreds or Tens of Hundreds, and the Product of the Double of the second Part of the Root (1 Ten) into the last (Units), from whence can arise only Tens or Tens with Hundreds, they will therefore be contained in 375, omitting the 6 Units which does not affect those two Products. I divide therefore 375 by 62, the Double of the first and second Figures of the Root, and the Quotient is 6, which being the third Figure of the Root, I place it after the second (1), and as z^2 the Square of this third Figure is Units or Units with Tens, I conclude that 3756 should contain the Terms $(2x + 2y)z + z^2$, the Product of Double of the first and second Figures of the Root into the third, together with the Square of the third; I multiply therefore 626 by 6, and subtract the Product from 3756, and as there nothing remains, I conclude that 316 is the Root required.

$$\begin{array}{r}
 9 \cdot 98 \cdot 56 \quad (316 \\
 \underline{9} \\
 61 \overline{) 98} \\
 \underline{\times 1} \quad 61 \\
 \hline
 626 \overline{) 3756} \\
 \underline{\times 6} \quad 3756 \\
 \hline
 0
 \end{array}$$

XXVI.

Process of
the extracti-
on of the
square root
of numbers.

It is easy to perceive that the square Root of any Number may be discovered by the same Method of Reasoning, the Process of which may be expressed thus. Put a Point in the Place of the Tens, and omitting one, point every other Figure towards the left Hand, and by these Points the Number will be distinguished into as many Periods as there are Figures in the Root. Then find the square Root of the first Period, and it will give the first Figure of the Root, subtract its Square from that Period, and annex the second Period of the given Number to the Remainder, then divide this new Number (neglecting its last Figure) by the Double of the first Figure of the Root, and quote the Number of Times, then annexing the Quotient to that Double, multiply the Number thence arising by the said Quotient, and subtract the Product from the whole Dividend, as before. Annex the third Part of the given Number to the Remainder, which is to be managed exactly as the last, and proceed thus until all the Periods are brought down. If at last there be no Remainder, then will the Quotient express the true Root.

XXVII.

Second ex-
ample.

Let the square Root of 27394756 be required. I first point it into Periods of two Figures each, and as there are four Periods, I conclude there will be four Places in the Root, then I find the nearest square Root of 27 to be 5, which therefore is the first Figure of the Root. I subtract 25, the Square of 5 from 27, and to the Remainder 2, I annex the second Period 39, and I divide (neglecting the last Figure 9) by the Double of 5 or by 10, and I place the Quotient after 5, and then multiply 102 by 2, and subtract the Product 204 from 239; then to the Remainder 35, I annex the third Period 47, and dividing 3547 (neglecting the last Figure 7) by the Double of 52, that is, by 104, I place the Quotient after 2, and multiplying 1043 by the Quotient 3, I find the Product to be 3129, which I subtract from the Dividend 3547, and to the Remainder 418 I annex the last Period 56, and dividing 41856 (neglecting the last Figure 6) by the Double of 523, that is, by 1046 I place the Quotient after 3, and multiplying 10464 by the Quotient 4, I find the Product to be 41856, which subtracted from the Dividend and leaving no Remainder, the exact Root must be 5234.

$$\begin{array}{r}
 27 \cdot 39 \cdot 47 \cdot 56 \quad (5234 \\
 \underline{25} \\
 102 \overline{) 239} \\
 \times 2 \quad 204 \\
 \hline
 1043 \overline{) 3547} \\
 \times 3 \quad 3129 \\
 \hline
 10464 \overline{) 41856} \\
 \times 4 \quad 41856 \\
 \hline
 \end{array}$$

XXVIII.

Let the square Root of 389489 be required. After having first Pointed it, I then find the nearest square Root of 38 to be 6 which therefore is the first Figure of the Root; I subtract 36, the Square of 6, from 38, and to the Remainder I annex the second Period 94, and I divide (neglecting the last Figure 4) by the double of 6 or by 12 and I place the Quotient after 6, and then multiply 122 by 2, and subtract the Product 244 from 294. then to the Remainder (50) I annex the third Period 89, and dividing 5089 (neglecting the last Figure 9) by the Double of 62, or by 124, I place the Quotient after 2, and Multiplying 1244 by the Quotient 4, I find the Product to be 5976, which subtracted from the Dividend leaves the Remainder 113, and as there are no more Periods to bring down there are to be no more Figures in the Root; because a Number consisting of six Places cannot have more than three Figures in its Root; but as 624 Multiplied by Itself gives for Product 389376, which is less than the proposed Number 389489, and that 625 multiplied by Itself gives for Product 390625, which is greater than the proposed Number, I conclude that 624 is the nearest Root in Integers.

$$\begin{array}{r}
 38.94.89.(624. \\
 \underline{36} \\
 122 \overline{)294} \\
 \times 2 \quad \underline{244} \\
 1244 \overline{)5089} \\
 \times 4 \quad \underline{4976} \\
 113
 \end{array}$$

Third example.

XXIX.

As this Reasoning may be applied to any other Example whatsoever, it appears in general, that the square Root a of any Number A , found by the foregoing Method, when there is a Remainder left, is less than $a + 1$ but greater than a ; consequently if it be determinable it will be $a + \frac{m}{n}$, $\frac{m}{n}$ expressing any proper Fraction reduced to its least Terms.

The root of a number which is not a perfect power of the same degree as the root required is not determinable.

Now since $\frac{m}{n}$ is a Fraction reduced to its least Terms, then $\frac{an + m}{n}$ will be also a Fraction reduced to its least Terms; for if n and $an + m$ have a common Divisor, then the Denominator of the Fraction $\frac{an + m}{n}$ when reduced would be less than n , consequently the Fraction $\frac{m}{n}$ has not been reduced to its least Terms, which is against the Supposition. if therefore the square Root of a Number which is not an exact Square is determinable. A Fraction reduced to its least Terms when squared must give an Integer, or which comes to the

same Thing, two Numbers a and b that have no common Divisor and two Others c and d that have no common Divisor with one another or with the two First, in their Product $\frac{ac}{bd}$, ac and bd would have a common Divisor.

But it is Easy to perceive 1^o that ac and b can have no common Divisor or $\frac{ac}{b}$ cannot be reduced, for if possible let $\frac{ac}{b} = \frac{ec}{gb}$ since $b = gb$, b is a Divisor of b , but b has no common Divisor with a or c wherefore neither has b for if $b = rs$ and, for Example, $a = ts$, then $b = grs$, that is a , and b would have a common Divisor which is against the Supposition, Again since $ac = eb$, $\frac{b}{a} = \frac{c}{e}$ but $\frac{b}{a}$ is a Fraction reduced to its least Terms, wherefore b is a Divisor of c , and consequently b and c have a common Divisor which is likewise against the Supposition. In like manner it will appear 2^o that ac and d can have no common Divisor, and in general that any two Numbers which have no common Divisor with a third Number, their Product will have no common Divisor with this third Number, wherefore since b and d have no common Divisor with ac their Product bd can have no common Divisor with ac . hence we may conclude that if $\frac{a}{b}$ is a Fraction reduced to its least Terms $\frac{a^2}{b^2}$, $\frac{a^3}{b^3}$ and in general $\frac{a^n}{b^n}$ will be a Fraction reduced to its least Terms, wherefore a Fraction either pure or mixed raised to any Power will give always a Fraction, wherefore if the square Root, cube Root &c. of an integer Number is not an Integer, it cannot be determined: as also the Root of a Fraction or of an Integer joined to a Fraction which when reduced to a Fraction each of its Terms is not a perfect Power of the same Degree as the Root required, for each Term of the Fraction may be considered as an Integer a part, whose Root cannot be determined.

xxx.

Method of
approximat-
ing to the
square root
of numbers.

Though the exact Root of an Integer which is not a perfect Square cannot be found, however we may Approximate to it to any degree of Exactness, for Example, to extract the square Root of 389489, instead of 389489 we may write $\frac{3894890000}{10000}$ which is equal to it, observing that the Denominator 10000 be a square Number, that is, should contain an even number of Cyphers then extracting the square Root of the Numerator which may be found true to an Unit, and dividing this Root by 100 which is the square Root of the Denominator, the square Root of

$\frac{3894890000}{10000}$, or of 389489, will be obtained true to $\frac{1}{100}$, if we had wrote $\frac{389489000000}{1000000}$ it is easy to perceive that the Root would be found true to $\frac{1}{1000}$, and so on. Hence in extracting the square Root after the Number proposed is gone through, if there is a Remainder the Operation being continued by adding periods of Cyphers to that Remainder, the true Root will be obtained in Decimals to any degree of Exactness.

xxxI.

To Approximate, for Example, to the Root of 389489, having found its nearest Root in Integers, I annex to the Remainder 113 a period of Cyphers, I divide (neglecting the last Figure) by the double of 624 or by 1248, and place the Quotient (0) after 4 and then multiply 12480 by 0 and Subtract (0) from 11300 then to the Remainder (11300) I annex another Period of Cyphers, and dividing 1130000 (neglecting the last Figure) by the double of 6240 or by 12480 I place the Quotient after 0, and multiplying 124809 by the Quotient 9 I subtract the Product 1123281 from 1130000, and to the Remainder 6719 I annex another Period which I manage as the last, and proceeding thus it is Manifest that the Root may be Approximated to any degree of Exactness. If we neglect the Remainder 4780975, the Root will be $\frac{6240905}{10000}$ which is true to $\frac{1}{10000}$, for if instead of $\frac{6240905}{10000}$

we take $\frac{6240906}{10000}$ the Root would be too great, since this Quantity squared gives $\frac{38948907700836}{100000000}$ or $389489 + \frac{7700836}{100000000}$ which is greater than 389489.

xxxII.

After the same way the square Root of decimal Numbers, may be extracted, but before the Operation the Number of decimal Places are to be made even, and after the Operation as many places are to be pointed off in

First Example.

$$\begin{array}{r}
 38.94.89(624,0905 \\
 36 \\
 \hline
 122)294 \\
 \times 2)244 \\
 \hline
 1244)5089 \\
 \times 4)4976 \\
 \hline
 12480)11300 \\
 \times 0)0 \\
 \hline
 124809)1130000 \\
 \times 9)1123281 \\
 \hline
 1248180)671900 \\
 \times 0)0 \\
 \hline
 12481805)67190000 \\
 \times 5)62409025 \\
 \hline
 4780975
 \end{array}$$

Second example.

the Root as there are Periods in the Fractional part of the proposed Number. Thus if the square Root of 3297,6 be required True to four decimal Places, instead of 3297,6 I write $\frac{329760000000}{100000000}$ which is equal to it, whose Root is 57,4247 in like manner to extract the square Root of 0,99856 true to six decimal places, instead of 0,99856 I write $\frac{998560000000}{1000000000000}$ which is equal to it, whose Root is 0,999279 as will appear by the following Diagrams.

$ \begin{array}{r} 32.97;60(57,4247 \\ \underline{25} \\ 107)797 \\ \times 7)749 \\ \hline 1144)4860 \\ \times 4)4576 \\ \hline 11482)28400 \\ \times 2)22964 \\ \hline 114844)543600 \\ \times 4)459376 \\ \hline 1148487)8422400 \\ \times 7)8039409 \\ \hline 3829918c. \end{array} $	$ \begin{array}{r} 0,99.85.60(0,999279 \\ \underline{81} \\ 189)1885 \\ \times 9)1701 \\ \hline 1989)18460 \\ \times 9)17901 \\ \hline 19982)55900 \\ \times 2)39964 \\ \hline 199847)1593600 \\ \times 7)1398929 \\ \hline 1998549)19467100 \\ \times 9)17986941 \\ \hline 1480159 \end{array} $
---	--

xxxiii.

Besides the foregoing Method for Approximating to the square Root of Numbers, the Analysts have imagined others which we shall proceed to explain.

Another method of approximating to the square root of numbers.

Let it be proposed to Approximate to the square Root of 50, the nearest less Root of 50 in Integers is 7, let $7 + z$ be the true Root, wherefore $49 + 14z + z^2 = 50$, or $-1 + 14z + z^2 = 0$: let this Quantity be multiplied by $1 + Az + Bz^2$, the Product will be

$-1 + (14 - A)z + (1 + 14A - B)z^2 + (A + 14B)z^3 + Bz^4 = 0$ in which the Terms $(1 + 14A - B)z^2$ and $(A + 14B)z^3$ may be made to vanish by determining A and B by means of the Equations $1 + 14A - B = 0$ and $A + 14B = 0$, from whence results

$A = -\frac{14}{197}$ and $B = -\frac{A}{14} = \frac{1}{197}$, substituting those Values in

the foregoing Equation, we will have $-1 + (14 + \frac{14}{197})z + \frac{z^4}{197} = 0$ which by rejecting the exceeding small Quantity $\frac{z^4}{197}$ becomes

$-1 + (14 + \frac{14}{197})z = 0$; hence $-197 + 2772z = 0$, and

$z = \frac{197}{2772} = .0710678$, which Value is true to the last decimal Place, and if more terms of the Series $1 + Az + Bz^2 + Cz^3 \&c.$ had been taken, the Root would be Exact in Proportion.

In general to Approximate to the square Root of any Number A let r be the nearest less Root in Integers, and $r + z$ the true Root, then will $r^2 + 2rz + z^2 = A$, $2rz + z^2 = A - r^2$, or $-\frac{m}{2r} + z + \frac{z^2}{2r} = 0$

(putting $A - r^2 = m$) then multiplying by $1 + Az$ there will result

$$-\frac{m}{2r} + \left(1 - \frac{mA}{2r}\right)z + \left(\frac{1}{2r} + A\right)z^2 + \frac{A}{2r}z^3 = 0,$$

here by making $\frac{1}{2r} + A = 0$, A is found $= -\frac{1}{2r}$, and our

Equation becomes $-\frac{m}{2r} + \left(1 + \frac{m}{4r^2}\right)z - \frac{1}{4r^2}z^3 = 0$, which

by neglecting the last Term as being very small, will be reduced to

$$-\frac{m}{2r} + \left(1 + \frac{m}{4r}\right)z = 0, \text{ whence } z = \frac{r m}{2r^2 + \frac{1}{2}m}.$$

To obtain a more approximate Value of z , we have only to multiply

$-\frac{m}{2r} + z + \frac{z^2}{2r} = 0$ by $1 + Az + Bz^2$ whence there arises

$$-\frac{m}{2r} + \left(1 - \frac{mA}{2r}\right)z + \left(\frac{1}{2r} + A - \frac{mB}{2r}\right)z^2 + \left(\frac{A}{2r} + B\right)z^3 + \frac{B}{2r}z^4 = 0$$

in which the third and fourth Terms will vanish by determining A and B

by means of the Equations, $\frac{1}{2r} + A - \frac{mB}{2r} = 0$ & $\frac{A}{2r} + B = 0$,

from whence A is found $= -\frac{2r}{4r^2 + m}$ and $B = -\frac{A}{2r} = \frac{1}{4r^2 + m}$;

wherefore $-\frac{m}{2r} + \left(\frac{2r^2 + m}{2r^2 + \frac{1}{2}m}\right)z + \frac{1}{2r(4r^2 + m)}z^4 = 0$;

which by rejecting the last Term as exceeding small, will become

$$-\frac{m}{2r} + \left(\frac{2r^2 + m}{2r^2 + \frac{1}{2}m}\right)z = 0 \text{ and } z = \frac{m}{2r} \times \frac{2r^2 + \frac{1}{2}m}{2r^2 + m}.$$

To obtain still a more Approximate Value of z , it suffices to multiply

$-\frac{m}{2r} + z + \frac{z^2}{2r}$ by $1 + Az + Bz^2 + Cz^3$, & there results

$$-\frac{m}{2r} + \left(1 - \frac{mA}{2r}\right)z + \left(\frac{1}{2r} + A - \frac{mB}{2r}\right)z^2 + \left(\frac{A}{2r} + B - \frac{mC}{2r}\right)z^3 + \left(\frac{B}{2r} + C\right)z^4 + \frac{C}{2r}z^5.$$

in which Equation the third, fourth, and fifth Terms may be made to vanish by determining A , B , and C by means of the Equations

$$\frac{1}{2r} + A - \frac{mB}{2r} = 0, \quad \frac{A}{2r} + B - \frac{mC}{2r} = 0, \quad \frac{B}{2r} + C = 0$$

whence $B = \frac{2rA + 1}{m}$, $C = \frac{2rB + A}{m}$ and $C = -\frac{B}{2r}$,

exterminating C , there arises $\frac{2rB + A}{m} + \frac{B}{2r} = 0$, which by substituting

the value of B , becomes $\frac{4r^2A + 2r}{m^2} + \frac{2rA + 1}{2rm} + \frac{A}{m} = 0$, where-

fore $A = -\frac{4r^2 + m}{8r^3 + 4rm}$, and $C = -\frac{1}{8r^3 + 4rm}$, and of course

$$-\frac{m}{2r} + \left(\frac{16r^4 + 12r^2m + m^2}{16r^4 + 8r^2m} \right) z - \frac{1}{16r^4 + 8r^2m} z^5 = 0,$$

from which, rejecting the last Term as exceeding small, you will find

$$z = \frac{8r^3m + 4rm^2}{16r^4 + 12r^2m + m^2} = \frac{rm(2r^2 + m)}{r^2(4r + 3m) + \frac{1}{4}m^2}.$$

xxxiv.

To shew the Use and great Exactness of these Approximations by an Example, let it be proposed to extract the square Root of 441.

First example.

Supposing r to be assumed = 20, then $m = 41$, and $z = \frac{1640}{1641}$ &

consequently $r + z = 21 - \frac{1}{1641}$ by the first Approximation.

By the Second $r + z = 20 + \frac{67281}{67280} = 21 + \frac{1}{2758481}$ more nearly.

And by the Third $r + z = 20 + \frac{2758480}{2758481} = 21 - \frac{1}{2758481}$,

still more nearly.

Let it be proposed to extract the square Root of 108, here $r = 10$, $m = 8$, and $z = \frac{80}{204} = .392$. Hence the Root required is 10.392 nearly, which value is true to the last decimal Place.

Second Example.

By the Second $r + z = 10 + \frac{1632}{4160} = 10.39230$ more nearly.

And by the Third $r + z = 10 + \frac{16640}{42416} = 10.3923048$, true to the last decimal Place.

It is to be observed that a vulgar Fraction cannot always rigourously and exactly be reduced to a Decimal. let $\frac{p}{q}$, for Example, be pro-

posed to be reduced to a Decimal $\frac{r}{10^n}$, then $r = \frac{p \times 10^n}{q}$ but

$10^n = 2^n \times 5^n$, now $\frac{p \times 2^n \times 5^n}{q}$ cannot give an Integer r , unless q be some Power of 2, or of 5, or of 2×5 , less than n , for $\frac{p}{q}$ is supposed to be a Fraction reduced to its least Terms, in every other Case $\frac{p \times 10^n}{q}$ can never give an Integer r , but it is manifest the greater n is, the nearer $\frac{r}{10^n}$ will approach to $\frac{p}{q}$, the Error being always less than $\frac{1}{10^n}$, since dividing $p \times 10^n$ by q the Quotient which results, and which is too little will be too great if increased by an Unit, wherefore $\frac{r}{10^n} < \frac{p}{q}$ and $\frac{r+1}{10^n} > \frac{p}{q}$.

A vulgar fraction cannot always be reduced to a decimal.

XXXV.

To shew the use of the foregoing Rules, in the Solution of Problems, here follow some Examples.

A Gentleman left an Estate of r Pounds per Annum to his Son, who being a Minor, his Guardian allowed him a Sum a the first Year, and puts out at Interest the Remainder $r - a$. He allowed him the second Year a Sum b , and the over-plus $r - b$ was also put out at Interest, and the Guardianship being ended a few Days after, the Revenue of the young Gentleman was found to be Increased during the two Year's Guardianship by a Sum d . What rate of Interest was allow'd.

Let $\frac{100}{x}$ express the yearly Interest of 100*l*. I say $100 : \frac{100}{x} = r - a :$
 $\frac{r - a}{x}$ Sum added to the Revenue of the young Gentleman at the End of the first Year, wherefore the Sum saved at the End of the first Year will be $r - a + \frac{r - a}{x}$, the Interest of which Sum at the End of the Second Year will be $\frac{r - a}{x} + \frac{r - a}{x^2}$, which together with $\frac{r - b}{x}$ the Interest of $r - b$ put out at Interest at the beginning of the second Year, will express the Increase, which the Revenue of the young Gentleman has received during the two year's Guardianship. Consequently $\frac{2r - a - b}{x} + \frac{r - a}{x^2} = d$, putting $2r - a - b = m$, $r - a = c$ (to abridge the Computation) there results the Equation $\frac{m}{x} + \frac{c}{x^2} = d$, or $x^2 - \frac{mx}{d} = \frac{c}{d}$, and compleating the Square and extracting the

$$\text{Square Root } x = \frac{m}{2d} \pm \sqrt{\left(-\frac{c}{d} + \frac{m^2}{4d^2}\right)} = \frac{m \pm \sqrt{4cd + m^2}}{2d}$$

XXXVI.

Application
of the fore-
going Pro-
blem to an
example.

To apply this general Solution to an Example, let the Estate be 4800 *l.* per Annum, and suppose the young Gentleman to have expended the first Year 2400 *l.* and the second Year 3120 *l.* and let the increase of his Revenue at the End of the second Year be found to be 356 *l.* 13 *s.* 4 *d.*

We will have $r = 4800$, $a = 2400$, $b = 3120$, $d = \frac{1070}{3}$
 $2r - a - b = m = 4080$, Substituting those Values in the Expression
 $x = \frac{m \pm \sqrt{4cd + m^2}}{2d}$ we will have $x = \frac{3[4080 \pm \sqrt{(20070400)}]}{2140}$
 $= \frac{3(4080 \pm 4480)}{2140}$, wherefore $x = 12$ and Consequently the rate

of Interest allowed was 8 *l.* 6 *s.* 8 *d.* per Cent. in Effect the Interest of 2400 *l.* for one Year at 8 *l.* 6 *s.* 8 *d.* per Cent. amounts to 200 *l.* consequently the Sum saved at the End of the first Year will be 2600 *l.* the interest of which for one Year at 8 *l.* 6 *s.* 8 *d.* per Cent. will be 216 *l.* 13 *s.* 4 *d.* and the Interest of 1680 *l.* at 8 *l.* 6 *s.* 8 *d.* for one Year is 140 *l.* and consequently the increase of the Revenue of the young Gentleman during the two Years Guardianship will be 356 *l.* 13 *s.* 4 *d.* agreeable to the Conditions of the Problem.

If the Revenue of the young Gentleman was supposed to be increased 432 *l.* during the two Years Guardianship, then x will be found = 10 and consequently the rate of Interest allowed will be 10 per Cent. in effect the Interest of 2400 *l.* for one Year at 10 per Cent. will be 240 *l.* the Sum saved therefore, at the End of the first Year will be 2640 *l.* the Interest of which for one Year at 10 per Cent. will be 264 *l.* and the Interest of 1680 *l.* for one Year at 10 per Cent. will be 168 *l.* wherefore the increase of the Revenue of the young Gentleman during the two Years Guardianship will be 432 *l.* as the Conditions of the Problem require.

XXXVII.

Another
Problem.

Two Notes one of 120 *l.* payable in 6 Months and the other of 150 *l.* payable in 9 Months, were discounted for 81 *l.* 10 *s.* what rate of Interest were they discounted at.

Let x denote the Interest of one Pound for 12 Months, then the amount of 1 *l.* in 6 Months being $1 + \frac{x}{2}$ and in 9 Months $1 + \frac{3x}{4}$ the present Value of the Bill due at the end of the 6 Months will therefore be $\frac{120}{1 + \frac{1}{2}x}$, and that of the Bill due at the end of 9 Months

$$\frac{150}{1 + \frac{3}{4}x}.$$

whence we have $\frac{120}{1 + \frac{1}{2}x} + \frac{150}{1 + \frac{3}{4}x} = 120 + 150 - 8, 5$
 $= 261, 5$ by the Question, which, by Reduction, becomes $120 + 90x$
 $+ 150 + 75x = 261, 5 \times (1 + \frac{1}{2}x + \frac{3}{8}x^2)$ or, $270 + 165x$
 $= \frac{261, 5 \times (8 + 10x + 3x^2)}{8}$, therefore $\frac{(270 + 165x) \times 8}{261, 5}$

$= 3x^2 + 10x + 8$, that is, $\frac{2160}{261, 5} + \frac{1320}{261, 5}x = 3x^2 + 10x + 8$,

from which we have $x^2 + \frac{2590}{3 \times 523}x = \frac{136}{3 \times 523}$, whence by completing the Square, &c.

x will be found $= \sqrt{\left[\frac{136}{1569} + \left(\frac{1295}{1569} \right)^2 \right]} - \frac{1295}{1569} =$
 $\frac{\sqrt{(136 \times 1569 + 1295 \times 1295)} - 1295}{1569} = 0, 05093$, which

multiplied by 100 gives $\pounds 5, 093$, or $\pounds 5. 1s. 10d. \frac{1}{2}$. nearly, for the Rate per Cent. at which the Notes were discounted.

XXXVIII.

In the different Examples which we have given of the Resolution of Equations of the second Degree, the young Analyst could scarce meet with any Difficulty except when it was proposed to reduce radical Quantities by taking from under the Sign, the square Quantities which were Factors of the radical Quantity: In Effect this Operation is the nicest that occurs in the Resolution of Equations of the second Degree; to assist therefore the young Analyst to perform it with Facility, here follow some Examples.

Examples of the reduction of radical quantities.

$$\sqrt{48 a a b c} = 4 a \sqrt{3 b c}, \sqrt{\frac{(a^3 b + 4 a a b b + 4 a b^3)}{c c}} = \frac{(a + 2 b \sqrt{a b})}{c},$$

$$\sqrt{\left(\frac{a a b b m m}{p p z z} + \frac{4 a a m^3}{p z z} \right)} = \frac{a m}{p z} \sqrt{(b b + 4 m p)},$$

$$6 \sqrt{\frac{75}{98} a b b} = \frac{30 b}{7} \sqrt{\frac{3 a}{2}}.$$

XXXIX.

No sooner had Equations of the second Degree made known the irrational or incommensurable Quantities (it is thus Quantities which have no exact Root are called) but the Analysts found it necessary to perform on those Quantities the same Operations as on rational or commensurable Quantities, that is, they had to add, subtract, multiply, and divide Quantities either entirely incommensurable, or partly incommensurable and partly commensurable.

Quantities which have not an exact root are called incommensurable or irrational quantities.

As to the Addition and Subtraction of radical Quantities, the only Difficulty consists in reducing them to their least Expressions.

The addition and subtraction of

ELEMENTS OF

irrational
quantities
presupposes
only their
reduction to
their least
terms.

For Example, Let it be proposed to add $\sqrt{48abb}$ and $b\sqrt{75a}$ together, I reduce the first of these Quantities to $4b\sqrt{3a}$, and the second to $5b\sqrt{3a}$, whose Sum is $9b\sqrt{3a}$.

In like Manner $\sqrt{48c^3} = \sqrt{\frac{16}{3}}c^3 = \frac{4}{\sqrt{3}}c\sqrt{3c}$,

$$\sqrt{\frac{ab^3}{cc}} + \frac{1}{2c}\sqrt{(a^3b - 4aabb + 4ab^3)} = \frac{a}{2c}\sqrt{ab}.$$

$$a\sqrt{\left(\frac{a^3b}{3aa + 6ac + 3cc}\right)} - \frac{bc\sqrt{ab}}{a+c} = \left(\frac{aa}{\sqrt{3}} - bc\right) \times \frac{\sqrt{ab}}{a+c}.$$

XL.

Multiplication
of
Surd.

With Respect to Multiplication, if the Quantities to be multiplied are both incommensurable, there is no more to be done than to multiply the Quantities which are under the radical Sign, and set the common radical Sign over the Products; which afterwards is to be reduced to its least Expression.

Let it be proposed for Example, to multiply $\sqrt{ab}$ by $\sqrt{ac}$, I write down $\sqrt{aabc}$ or $a\sqrt{bc}$. In like Manner

$$\sqrt{3cd} \times \sqrt{4fgc} = \sqrt{12fgccd} = 2c\sqrt{3fgd}.$$

When the radical Quantities to be multiplied are equal, the Multiplication is performed by taking away the radical Sign. The Product, for Example, of $\sqrt{a^3cd}$ into $\sqrt{a^3cd}$ is a^3cd .

If the Quantity by which the Surd is to be multiplied be rational, it suffices to write it before the Sign between Parentheses when it consists of several Terms; and before it can be brought under the Sign it must be first squared.

For Example, the Product of $a + b$ into $\sqrt{\frac{ffg}{aa-bb}}$ is

$$(a+b)\sqrt{\frac{ffg}{aa-bb}}, \text{ or } \sqrt{\left[\frac{ffg \times (aa + 2ab + bb)}{aa-bb}\right]}$$

$$\text{or } \sqrt{\left[\frac{ffg \times (a+b)}{a-b}\right]}, \text{ or } \frac{f\sqrt{(ag+bg)}}{\sqrt{(a-b)}},$$

$$2a\sqrt{\frac{aa+bb}{c}} \times -3a\sqrt{\frac{(aa+bb)}{d}} =$$

$$-\frac{6aa \times (aa+bb)}{\sqrt{cd}} = -\frac{6a^4 + 6aabb}{\sqrt{cd}}.$$

$$\sqrt{(a+b)} \times \sqrt{(a-b)} = \sqrt{(aa-bb)}.$$

If the Quantities to be multiplied consist of several Parts, either all irrational or partly irrational and partly rational, the Operation is performed by observing the same Rules as in the Multiplication of compound Quantities, for Example, $(3a\sqrt{bc} - 2b\sqrt{ac}) \times 2c\sqrt{ab} = 6abc\sqrt{ac} - 4abc\sqrt{bc}$. $[\sqrt{ab} + \sqrt{(aa-xx)}] \times [\sqrt{ab} + \sqrt{(aa-xx)}] = aa + ab - xx + 2\sqrt{(a^3b - abxx)}$. $[a + \sqrt{(aa-bb)}] \times [a + \sqrt{(aa-bb)}] = 2aa - bb + 2a\sqrt{(aa-bb)}$, $[a + \sqrt{(aa-xx)}] \times [a - \sqrt{(aa-xx)}]$

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$$=xx, \frac{2ax}{c} - \sqrt{\frac{a^3}{c}} \times 3a + \sqrt{\frac{abb}{c}} = \frac{6aax}{c}$$

$$- 3a \sqrt{\frac{a^3}{c}} + \frac{2ax}{c} \sqrt{\frac{aab}{c}} - \frac{aab}{c}.$$

XLI.

Irrational Quantities are divided one by the other, by dividing the Quantities under the Sign and setting the Sign over the Quotient.

Division of incommensurable quantities.

If an irrational Quantity is to be divided by a rational one, the irrational Quantity is to be placed above a small Line and the rational Quantity under it, and before it can be brought under the Sign it must be first squared.

If the Surds have any rational Coefficients, their Quotient is to be prefixed to the radical Quotient. All which will appear plain by the following Examples.

$$\frac{\sqrt{ab}}{\sqrt{a}} = \sqrt{b}, \quad \frac{ac\sqrt{bc}}{a\sqrt{b}} = c\sqrt{c}, \quad \frac{12ac\sqrt{6bc}}{4c\sqrt{2b}} = 3a\sqrt{3c},$$

$$\frac{\sqrt{(aa - xx)}}{\sqrt{(a + x)}} = \sqrt{(a - x)}, \quad \frac{\sqrt{aabb - bbxx}}{a - x} = b\sqrt{\frac{a + x}{a - x}}.$$

If the Quantities to be divided consist of several Parts rational and irrational, the Operation will be performed as the Division of compound Quantities.

For Example, $a^4 + b^4$ being divided by $a^2 + ab\sqrt{2} + b^2$ the Quotient will be found to be $a^2 - ab\sqrt{2} + b^2$, $a^3b - abbc - aab\sqrt{bc} + bbc\sqrt{bc}$ divided by $a - \sqrt{bc}$ the Quotient is $aab - bbc$. In like Manner, $a^3 - abc + aa\sqrt{bc} - bc\sqrt{bc}$ divided by $a - \sqrt{bc}$ the Quotient will be $aa + bc + 2a\sqrt{bc}$.

XLII.

The foregoing Principles are sufficient for resolving Equations of the second Degree when they include only one unknown Quantity; but as there frequently occur Problems where several unknown Quantities are concerned, we shall proceed to explain how they are to be managed; with this View we shall propose the following Problems.

The Joint-Stock of two Partners A and B was £165. (a) A's Money was in Trade 12 Months, and B's 8 Months, when they shared Stock and Gain, A received £67. (b) and B £126. (c) what was each Man's Stock?

First Problem of the second degree where two unknown quantities are concerned.

Suppose A advanced x Pounds, then B advanced $a - x$ Pounds, and their respective Shares of the Gain may be considered as the Interest of their respective Capitals during the Time they were in Trade, the Rate of Interest being the same for both: Let z therefore denote the Interest of £1. for 12 Months, then the Amount of £1. for 12 Months being $1 + z$, and in 8 Months $1 + \frac{2}{3}z$, the Dividend of A will be $x(1 + z)$,

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the Dividend of $B(a - x) \times (1 + \frac{2}{3}z)$, hence $x(1 + z) + (a - x) \times (1 + \frac{2}{3}z) = b + c$, from whence is deduced $z = \frac{3m - 3a}{x + 2a}$ (putting $b + c = m$ to abridge the Computation) substituting this Value of z in $x + zx$, the Dividend of A , there will result $x + \frac{3mx - 3ax}{x + 2a} = d$, or $x^2 + 3mx - ax - dx = 2ad$, now putting $3m - a - d = n$ compleating the Square and extracting the Root there will result $x = -\frac{1}{2}n \pm \sqrt{2ad + \frac{1}{4}n^2} = 55$.

XLIII.

Second Problem.

A Grocer sold 80lb. of Mace, and 100lb. of Cloves for £65. he sold 60lb. more of Cloves for £20. than he did of Mace for £10. what Price did he sell each at?

Suppose he sold the Mace at x and the Cloves at y Shill. a lb. then $x : 1 \text{ lb.} = £10. \text{ or } 200s. : \frac{200}{x}$ the Number of Pounds of Mace sold for £10. and $y : 1 \text{ lb.} = £20. \text{ or } 400s. : \frac{400}{y}$ Number of Pounds of Cloves sold for £20. now £65. or 1300 = $80x + 100y$ and $\frac{200}{x} + 60 = \frac{400}{y}$ by the Question. From the first Equation I deduce $\frac{130 - 8x}{10} = y$, and from the second Equation $\frac{20x}{10 + 3x} = y$, wherefore $\frac{130 - 8x}{10} = \frac{20x}{10 + 3x}$, or $1300 - 80x + 390x - 24x^2 = 200x$, that is, $24x^2 - 110x = 1300$, or $x^2 - \frac{110}{24}x = \frac{1300}{24}$, and compleating the Square and extracting the Root, there results $x - \frac{55}{24} = \frac{185}{24}$ or $x = 10$, consequently $y = 5$.

XLIV.

Third Problem.

The Sum and the Sum of the Squares of three Numbers in geometrical Proportion being given to find the Numbers.

Let the Sum of the three Numbers sought be denoted by x, y, z , from the Nature of Proportion $x : y = y : z$, that is, $yy = xz$; moreover, since their Sum is given, denoting this Sum by a we will have $x + y + z = a$, and expressing the Sum of their Squares by bb we will have by the last Condition of the Problem $xx + yy + zz = bb$.

To make use of those three Equations, I first exterminate z by Means of its Value $a - x - y$ deduced from the Equation $x + y + z = a$; substituting therefore this Value of z in the two other Equations they will be transformed into $2yy + 2xx + 2xy + aa - 2ax - 2ay$

$= bb$, and $yy = ax - xx - xy$. To exterminate out of those two Equations one of the two unknown Quantities they include, it suffices to find the Value of this unknown Quantity in each of those Equations, and to equate them; let x be the unknown Quantity to be exterminated, the first Equation will give

$$x = -\frac{1}{2}y + \frac{1}{2}a \pm \sqrt{\left(\frac{bb}{2} - \frac{aa}{4} - \frac{1}{2}yy + \frac{1}{2}ay\right)},$$

and the second $x = -\frac{1}{2}y + \frac{1}{2}a \pm \sqrt{\left(\frac{1}{4}a^2 - \frac{1}{2}ay - \frac{1}{2}y^2\right)}$.

Equating those two Values of x , there will result the Equation

$$-\frac{1}{2}y + \frac{1}{2}a \pm \sqrt{\left(\frac{bb}{2} - \frac{aa}{4} - \frac{1}{2}yy + \frac{1}{2}ay\right)} = -\frac{1}{2}y + \frac{1}{2}a \pm \sqrt{\left(\frac{1}{4}aa - \frac{1}{2}ay - \frac{1}{2}yy\right)}$$

to solve which I observe first, that the Terms $-\frac{1}{2}y + \frac{1}{2}a$ being common to both Members of the Equation, the Equation may be reduced to

$$\pm \sqrt{\left(\frac{bb}{2} - \frac{aa}{4} - \frac{1}{2}yy + \frac{1}{2}ay\right)} = \pm \sqrt{\left(\frac{1}{4}aa - \frac{1}{2}ay - \frac{1}{2}yy\right)}$$

the two Members of which being squared, and reduced, there results

$$ay = \frac{1}{2}aa - \frac{1}{2}bb, \text{ from whence is deduced } y = \frac{aa - bb}{2a},$$

this Value of y substituted in one of the foregoing Values of x , will give

$$x = \frac{1}{4}a + \frac{bb}{4a} \pm \sqrt{\left(\frac{1}{4}aa - \frac{1}{4}aa + \frac{1}{4}bb - \frac{1}{16}aa + \frac{1}{8}bb - \frac{3b^4}{16aa}\right)},$$

or $x = \frac{bb + aa \pm \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$ and substituting

the Values of x and y in the Equation $z = a - x - y$ we will have

$$z = \frac{aa + bb \pm \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}.$$

XLV.

As in this Solution we arrived at a very simple Value of y , after having had very complex Radicals, it should incline us to think that this Value might be obtained after a more concise Manner: And it was not difficult to imagine the following Method.

The foregoing equations solved by another Method.

Let the two Equations $x^2 - ax + yx = \frac{bb}{2} - \frac{aa}{2} - y^2 + ay$

and $x^2 + xy - ax = yy$ be resumed; subtracting one from the other there results $0 = \frac{bb}{2} - \frac{aa}{2} + ay$, from whence is deduced

$$y = \frac{aa - bb}{2a}, \text{ which being substituted in either one or the}$$

other of those two Equations, gives

$$x^2 + \frac{aax - bbx}{2a} - ax = \frac{-a^4 + 2aabb - b^4}{4aa}, \text{ or}$$

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$x^2 - \frac{(bb + aa)x}{2a} = \frac{-a^4 + 2aabb - b^4}{4aa}$ from whence
is deduced the same Value of x as above.

XLVI.

To apply this general Solution to an Example, let the Sum of the three Numbers in continual Proportion be 20, and the Sum of their Squares 140; in this Case $a = 20$, and $bb = 140$, consequently $y = 6\frac{1}{2}$, and $x = 6\frac{1}{2} \pm \sqrt{3\frac{5}{16}}$; Viz. $6\frac{1}{2} + \sqrt{3\frac{5}{16}}$ is the greatest of the three Numbers sought, and $6\frac{1}{2} - \sqrt{3\frac{5}{16}}$ the least. For as the Values of x and z in the foregoing Solution are exactly alike, x denotes ambiguously either of the extreme Numbers, and thence there will come out two Values either of which may be x , the other being z .

XLVII.

When the values of two unknown quantities in a problem are brought out perfectly alike, how the ambiguity thence arising may be avoided.

When there happens to be such an Affinity or Similitude of the Relation of the two Terms to the other Terms of the Question, that in making use of either of them, their Values would be brought out exactly alike; (which will be easily seen) the Analysts to avoid Ambiguity make use of neither of them, but in their Room chuse some third which shall bear a like Relation to both, as suppose the half Sum or half Difference, or any other Quantity related to both indifferently and without alike.

Thus in the precedent Problem, having found $y = \frac{aa - bb}{2a}$, to avoid the Ambiguity arising from the Substitution of this Value in one of the foregoing Values of x , I put $\frac{x - z}{2} = u$, and as $\frac{x + z}{2} = \frac{a - y}{2} = \frac{aa + bb}{4a}$, consequently $x = \frac{aa + bb}{4a} + u$ and $z = \frac{aa + bb}{4a} - u$; but $xz = yy = \frac{a^4 + 2aabb + b^4}{16aa} - uu = \frac{a^4 - 2aabb + b^4}{4aa}$ wherefore $uu = \frac{10bbaa - 3a^4 - 3b^4}{16aa}$, and $u = \frac{\sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$, consequently $x = \frac{bb + aa + \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$ and $z = \frac{bb + aa - \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}$.

XLVIII.

In this Problem, there occurred Quantities which destroyed each other, whereby the Computation was rendered extremely simple, but as this seldom happens in Equations of the second Degree including several unknown Quantities, we shall proceed to explain the Methods employed by the Analysts in the more complicated Cases. Let, for Example, the two Equations $x^2 + ax - 2xy = aa + 2yy$ and $xx - 2ax + xy = 2aa - yy$ be proposed, the first of those Equations gives $x = -\frac{1}{2}a + y \pm \sqrt{(\frac{1}{4}aa - ay + 3yy)}$ and the other gives $x = a - \frac{1}{2}y \pm \sqrt{(3aa - ay - \frac{1}{4}yy)}$, equating those two Values, and reducing, there results

$$-\frac{3a}{2} + \frac{1}{2}y \pm \sqrt{(\frac{1}{4}aa - ay + 3yy)} = \pm \sqrt{(3aa - ay - \frac{1}{4}yy)}.$$

To make the Radicals disappear in this Equation, I first square the two Members and there results $\frac{1}{4}yy - \frac{3}{2}ay + \frac{9}{4}aa \pm (3y - 3a)\sqrt{(\frac{1}{4}aa - ay + 3yy)} + \frac{1}{4}aa - ay + 3yy = 3aa - ay - \frac{1}{4}yy$, which still contains a radical Quantity. To make it disappear I write down this Equation thus,

$$-\frac{1}{2}aa + \frac{3}{2}ay - 6yy = \pm (3y - 3a)\sqrt{(\frac{1}{4}aa - ay + 3yy)}$$

reducing it and leaving the radical Quantity $\pm (3y - 3a)\sqrt{(\frac{1}{4}aa - ay + 3yy)}$ alone on one Side of the Equation; squaring afterwards the two Members $\frac{1}{4}a^4 - \frac{9}{2}a^3y + \frac{105}{4}a^2yy - 54ay^3 + 36y^4 = (9yy - 18ay + 9aa) \times (\frac{1}{4}aa - ay + 3yy)$ which when reduced becomes $9y^4 + 9ay^3 - 30aayy + 27a^3y = 11a^4$

Equation resulting from the two foregoing ones, which is to be resolved to obtain the Solution of the Problem which furnished those Equations: From whence it appears that in Equations of the second Degree involving several unknown Quantities it is not as in Equations of the first, in which the resulting Equation when one of the unknown Quantities is exterminated never rises to a Degree higher than that of the Equations themselves.

XLIX.

The Equation which results when an unknown Quantity is exterminated out of Equations of the second Degree, may be obtained without being at the Trouble of first solving them, and afterwards making their Radicals disappear.

That this may appear let the two foregoing Equations be resumed, and from each be deduced the Value of xx , the first will give

$$x^2 = aa + 2yy - ax + 2xy, \text{ the second } x^2 = 2aa - yy + 2ax - xy.$$

Equating those two Values, we will have $aa + 2yy + 2xy - ax = 2aa - yy - xy + 2ax$, from whence is deduced

$$x = \frac{3yy - aa}{3a - 3y}, \text{ which being substituted in one or other of the}$$

Example of equations of the second degree including two unknown quantities more complicated than the foregoing.

The resulting equation when one of the unknown quantities is exterminated.

Another method by which the unknown quantity is exterminated in the foregoing example.

ELEMENTS OF

$$x^2 - \frac{(bb + aa)x}{2a} = \frac{-a^4 + 2aabb - b^4}{4a^2} \text{ from whence}$$

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Thus in the precedent Problem, having found $y = \frac{aa - bb}{2a}$, to avoid the Ambiguity arising from the Substitution of this Value in one of the foregoing Values of x , I put $\frac{x - z}{2} = u$, and as

$$\frac{x + z}{2} = \frac{a - y}{2} = \frac{aa + bb}{4a}, \text{ consequently}$$

$$x = \frac{aa + bb}{4a} + u \text{ and } z = \frac{aa + bb}{4a} - u; \text{ but}$$

$$xz = yy = \frac{a^4 + 2aabb + b^4}{16aa} - uu = \frac{a^4 - 2aabb + b^4}{4aa}$$

wherefore $uu = \frac{10bbaa - 3a^4 - 3b^4}{16aa}$, and

$$u = \frac{\sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a}, \text{ consequently}$$

$$x = \frac{bb + aa + \sqrt{(10bbaa - 3a^4 - 3b^4)}}{4a} \text{ and}$$

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$$-\frac{3a}{2} + \frac{3}{2}y \pm \sqrt{(\frac{1}{4}aa - ay + 3yy)} = \pm \sqrt{(3aa - ay - \frac{1}{4}yy)}.$$

To make the Radicals disappear in this Equation, I first square the two Members and there results $\frac{9}{4}yy - \frac{9}{2}ay + \frac{9}{4}aa \pm (3y - 3a)\sqrt{(\frac{1}{4}aa - ay + 3yy)} + \frac{1}{4}aa - ay + 3yy = 3aa - ay - \frac{1}{4}yy$, which still contains a radical Quantity. To make it disappear I write down this Equation thus,

$$-\frac{1}{2}aa + \frac{9}{2}ay - 6yy = \pm (3y - 3a)\sqrt{(\frac{1}{4}aa - ay + 3yy)}$$

reducing it and leaving the radical Quantity $\pm (3y - 3a)\sqrt{(\frac{1}{4}aa - ay + 3yy)}$ alone on one Side of the Equation; squaring afterwards the two Members $\frac{1}{4}a^4 - \frac{9}{2}a^3y + \frac{105}{4}a^2yy - 54ay^3 + 36y^4 = (9yy - 18ay + 9aa) \times (\frac{1}{4}aa - ay + 3yy)$ which when reduced becomes $9y^4 + 9ay^3 - 30a^2yy + 27a^3y = 11a^4$

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That this may appear let the two foregoing Equations be resumed, and from each be deduced the Value of xx , the first will give

$$x^2 = aa + 2yy - ax + 2xy, \text{ the second } x^2 = 2aa - yy + 2ax - xy.$$

Equating those two Values, we will have $aa + 2yy + 2xy - ax = 2aa - yy - xy + 2ax$, from whence is deduced

$$x = \frac{3yy - aa}{3a - 3y}, \text{ which being substituted in one or other of the}$$

Example of equations of the second degree including two unknown quantities more complicated than the foregoing.

The resulting equation when one of the unknown quantities is exterminated.

Another method by which the unknown quantity is exterminated in the foregoing example.

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two given Equations, in $xx - 2ax + xy = 2aa - yy$, for Example, gives $\frac{9y^4 - 6aayy + a^4}{(3a - 3y)^2} + \frac{(y - 2a) \times (3yy - aa)}{3a - 3y}$
 $= 2aa - yy$ which when reduced, becomes
 $9y^4 + 9ay^3 - 30aayy + 27a^3y = 11a^4$ the same above.

L.

The foregoing method applied to another example.

Let the two Equations $x^2 + axy + bx = cy^2 + dy + e$ and $x^2 + fxy + gx = by^2 + iy + k$ be given, which are the most complicated of the second Degree involving two unknown Quantities.

Deducing a Value of xx from each of these Equations and equating them, we will have $(a - f) \times xy + (b - g) \times x = (c - b) \times y^2 + (d - i) \times y + e - k$, which, by putting, to abridge the Computation, $a - f = l$; $b - g = m$; $c - b = n$; $d - i = p$; $e - k = q$; will be transformed into $lxy + mx = ny^2 + py + q$. From whence is deduced $x = \frac{ny^2 + py + q}{ly + m}$. Which substituted in

one or other of the two given Equations, in the first, for Example, there will result $\frac{(ny^2 + py + q)^2}{(ly + m)^2} + \frac{(ay + b) \times (ny^2 + py + q)}{ly + m}$
 $= cy^2 + dy + e$, or $(ny^2 + py + q)^2 + (ay + b) \times (ny^2 + py + q) \times (ly + m) = (cy^2 + dy + e) \times (ly + m)^2$.

If we now perform the Operations which are indicated, reduce and order the Equation, there will result

$$y^4 + \frac{(bln + amn + alp + 2np - 2mlc - l^2d)}{aln + nn - l^2c} y^3$$

$$+ \frac{bmn + qal + pbl + pam + p^2 + 2uq - m^2c - 2mld - el^2}{aln + nn - l^2c} y^2$$

$$+ \frac{blq + amq + bmp + 2pq - m^2d - 2mel}{aln + nn - l^2c} y$$

$$= \frac{m^2e - q^2 - bmq}{aln + nn - l^2c}, \text{ Equation of the fourth Degree which}$$

results from two of the most general Equations of the second.

LI.

Whatever be the dimensions of y in two equations if those of x do not rise higher than two it may be exterminated by the foregoing method.

If we had two Equations such as $xxxy + axy = abb$ and $xxxy + ccyx = a^4$, they cannot be considered as Equations of the second Degree, because the unknown Product of x^2 into y is of three Dimensions, and that of x^2 into y^2 is of four Dimensions; however x may be exterminated out of those two Equations by the foregoing Method. That this may appear let A express all the Quantities composed of y and known Quantities which affect x^2 to whatever Degree they rise, (or in the Language of the Analysts any Function of y) in one of the given Equations; let B express the Function of y which affects x in the same Equation, and C the Quantity which is not affected by any Power of

x , that is, let the first Equation be $Ax^2 + Bx = C$, and in like Manner let the second be $A'x^2 + B'x = C'$.

From the first I deduce $x^2 = \frac{C - Bx}{A}$, and from the second $x^2 = \frac{C' - B'x}{A'}$, equating those two Values there results $C A' - B A' x = A C' - A B' x$, from whence is deduced $x = \frac{A C' - C A'}{A B' - B A'}$, which being substituted in the Equation $Ax^2 + Bx = C$, gives $A \times (A C' - C A')^2 + B \times (A B' - B A') \times (A C' - C A') = C \times (A B' - B A')^2$, in which substituting for A, B, C, A', B', C' , the Functions of y their Values, the Equation sought will be obtained.

LII.

If the unknown Quantity which is to be exterminated out of two given Equations, rises to higher Dimensions than two, it is easy to perceive that by an Operation similar to the foregoing they may be transformed into two others of lower Dimensions, and by repeating this Operation the unknown Quantity will at length be taken away.

If we had, for Example, $y^3 = xyy + 3x$ and $yy = xx - xy - 3$; to take away y I multiply the latter Equation by y , and you have $y^3 = xxy - xyy - 3y$, of as many Dimensions as the former, now by making the Values of y^3 equal to one another, I have $xxy + 3x = xxy - xyy - 3y$, where y is depressed to two Dimensions, from whence I deduce $yy = \frac{xx - 3y - 3x}{2x}$ by this therefore, and the most simple one of the Equations first proposed $yy = xx - xy - 3$, I find $y = \frac{2x^3 - 3x}{3xx - 3}$ which Value substituted in $yy = xx - xy - 3$ as being the most simple, there results $\frac{4x^6 - 12x^4 + 9xx}{9x^4 - 18xx + 9} = xx - \frac{2x^4 + 3xx}{3xx - 3} - 3$, whence $4x^6 - 12x^4 + 9xx = 9x^6 - 18x^4 + 9x^2 - 6x^6 + 9x^4 + 6x^4 - 9x^2 - 27x^4 + 54x^2 - 27$, which when reduced becomes $x^6 + 18x^4 - 45x^2 + 27 = 0$.

LIII.

If there were three Equations given and as many unknown Quantities of different Dimensions, mixed promiscuously in each Equation, they may likewise by the foregoing Method be reduced into one, affected with only one unknown Quantity; for neglecting at first one of the three unknown Quantities, two of the three Equations will be sufficient to find an Equation involving the unknown Quantity which was not attended to,

How the foregoing method is to be applied when the quantity to be exterminated rises to higher dimensions than two.

If there were more than two unknown quantities and equations given they may be reduced to

one by the foregoing method.

and either of the two others, and the same Operation being performed on one of the Equations employed in the first and on the third Equation, there will result another Equation involving the same unknown Quantities, hence the Question is reduced to that where two Equations involving each two unknown Quantities are concerned, whence a new Equation will be deduced, affected with only one unknown Quantity.

If there were four Equations given and as many unknown Quantities mixed promiscuously in each Equation, the Question will be reduced in like Manner to three Equations and three unknown Quantities, and afterwards to two Equations and two unknown Quantities, and at length to one Equation and one unknown Quantity; and the same Method is to be pursued if there were a greater Number of Equations and unknown Quantities.

LIV.

Inconveniency to which the foregoing method is liable.

When the Quantity to be exterminated is of several Dimensions, there would be required a very laborious Calculus to exterminate it out of the Equations by the foregoing Method, we shall therefore proceed to explain how the Analysts have remedied this Inconveniency.

$$\text{Let } \begin{cases} Ax^m + Bx^{m-1} + Cx^{m-2} + Dx^{m-3} + Ex^{m-4} \dots V = 0(L) \\ A'x^{m'} + B'x^{m'-1} + C'x^{m'-2} + D'x^{m'-3} + E'x^{m'-4} \dots V' = 0(L') \end{cases}$$

be two Equations in which $A, B, C, D, \&c.$ denote any Function of y , to exterminate x out of those two Equations, it is sufficient to find a Function of x by which the first Equation being multiplied, and another Function of x by which the second Equation being multiplied in the Sum of the Products, the Terms including the Powers of x will be destroyed.

How it has been removed.

$$\text{let then } \begin{cases} Mx^n + Nx^{n-1} + Px^{n-2} + Qx^{n-3} + Rx^{n-4} \dots T \\ M'x^{n'} + N'x^{n'-1} + P'x^{n'-2} + Q'x^{n'-3} + R'x^{n'-4} \dots T' \end{cases}$$

be those two Functions. After the Multiplication and Addition of the Products there will result,

$$\begin{aligned} & AMx^{m+n} + BMx^{m+n-1} + CMx^{m+n-2} + DMx^{m+n-3} \dots VT = 0 \\ & A'M'x^{m'+n'} + A'N'x^{m'+n'-1} + A'P'x^{m'+n'-2} + A'Q'x^{m'+n'-3} \dots V'T' = 0 \\ & + B'M'x^{m'+n'-1} + A'Px^{m+n-2} + BPx^{m+n-3} \\ & + A'N'x^{m'+n'-1} + CM'x^{m'+n'-2} + A'Qx^{m+n-3} \\ & + B'N'x^{m'+n'-2} + D'M'x^{m'+n'-3} \\ & + A'P'x^{m'+n'-2} + C'N'x^{m'+n'-3} \\ & + B'P'x^{m'+n'-3} \\ & + A'Q'x^{m'+n'-3} \end{aligned}$$

Where I observe, 1^o. That as the Condition that the Terms including the Powers of x should be destroyed, furnish the Equations by Means of which the unknown Coefficients $M, N, \&c.$ are to be determined, $m+n$ must be equal to $m'+n'$, whence results the following Equations for determining those Coefficients.

$$\begin{aligned}
 AM + A'M' &= 0 \\
 AN + A'N' + BM + B'M' &= 0 \\
 AP + A'P' + BN + B'N' + CM + C'M' &= 0 \\
 AQ + A'Q' + BP + B'P' + CN + C'N' + DM + D'M' &= 0 \\
 AR + A'R' + BQ + B'Q' + CP + C'P' + DN + D'N' + EM + E'M' &= 0
 \end{aligned}$$

Consequently, $VT + V'T' = 0$

I observe, 2°. That as the Number of Terms including the Powers of x is $m + n$, the Number of Equations they furnish when supposed equal to Nothing, will be also $m + n$, and as each Term of these Equations include an unknown Quantity, Viz. the Coefficient of a Term of the assumed Multinomials, it follows (LXXXIX. Chap. I.) that the Number of undetermined Coefficients $n + 1 + n' + 1$ will be $= m + n + 1$, or $m' + n' + 1$; consequently $n = m' - 1$ and $n' = m - 1$. Hence when two Equations each involving two unknown quantities are given, to exterminate one of them, it is sufficient to multiply the first Equation by a Multinomial as $Mx^{m'-1} + Nx^{m'-2} + \dots$, m' denoting the Degree of the second Equation; and the second by a Multinomial as $M'x^{m-1} + N'x^{m-2} + \dots$, m denoting the Degree of the first Equation: To add those two Products, and to put the Coefficients of each Power of x in this Sum equal to Nothing, the last Term of the Sum will be the Equation in y .

LV.

For Example, let it be proposed to exterminate x out of the Equations $Ax^2 + Bx + C = 0$ and $A'x^2 + B'x + C' = 0$, multiplying $Ax^2 + Bx + C$ by $Mx + N$, and $A'x^2 + B'x + C' = 0$ by $M'x + N'$, and adding the Products we will have $(AM + A'M')x^3 + (BM + AN + B'M' + A'N')x^2 + (CM + BN + C'M' + B'N')x + NC + C'N' = 0$, the Condition that the Terms in this Equation including the Powers of x should be destroyed will give $AM + A'M' = 0$, $BM + AN + B'M' + A'N' = 0$, $CM + BN + C'M' + B'N' = 0$, consequently $NC + C'N' = 0$, and as there is one Coefficient undetermined. I determine M by supposing it $= A'$, which gives $M' = -A$, $AN + A'N' = AB' - BA'$, $BN + B'N' = AC - CA'$, from which I deduce $N = \frac{AB'B' - BAA' - AA'C + CA'A'}{AB' - BA'}$, and

$$N' = \frac{AAC' - ACA' - ABB' + BBA'}{AB' - BA'} \text{ and those Values of}$$

N and N' being substituted in the Equation $CN + C'N' = 0$ there results $ACB'B' - BCA'B' - ACA'C' + CCA'A' + AAC'C' - ACA'C' - ABB'C' + BBA'C' = 0$, that is

RULE I.

$$(AC' - BB' - 2CA')AC' + (BC' - CB')BA' + (AB'B' + CA'A')C = 0.$$

If it was proposed to exterminate x out of the Equations Example, $xx + 5x - 3yy = 0$ and $3xx - 2xy + 4 = 0$, I respectively sub-

Investigation of Sir Isaac Newton's rule for exterminating an unknown quantity of two dimensions in each equation by the foregoing Method.

stitute in the preceding Expression for $A, B, C; A', B' C'$; these Quantities, 1, 5, $-3yy$; 3, $-2y$, 4; and duly observing the Signs $+$ and $-$, there arises $(4 + 10y + 18yy) \times 4 + (20 - y^3) \times 15 + (4yy - 27yy) \times -3yy = 0$, or $16 + 40y + 72yy + 300 - 90y^3 + 69y^4 = 0$.

LVI.

Investigation of Sir Isaac Newton's Rule for exterminating an unknown quantity of three dimensions in one, and of two dimensions in the other equation by the foregoing Method.

Let it be proposed to exterminate x out of the Equations $Ax^3 + Bx^2 + Cx + D = 0$, and $A'x^2 + B'x + C' = 0$, multiplying $Ax^3 + Bx^2 + Cx + D$ by $Mx + N$ and $A'x^2 + B'x + C'$ by $M'x^2 + N'x + P'$ and adding their Products there will result $(AM + A'M')x^4 + (BM + AN + B'M + A'N')x^3 + (CM + BN + CM' + B'N' + A'P')x^2 + (DM + CN + C'N' + B'P')x + DN + C'P' = 0$. The Condition that the Terms of this Equation including the Powers of x should be destroyed, will give $AM + A'M' = 0$, $BM + AN + B'M + A'N' = 0$, $CM + BN + CM' + B'N' + A'P' = 0$, $DM + CN + C'N' + B'P' = 0$, consequently $DN + C'P' = 0$, and as one of the Coefficients is undetermined I determine M by supposing it $= A'$, which gives $M' = -A$, $AN + A'N' = AB' - BA'$, $BN + B'N' + A'P' = AC' - CA'$, $CN + C'N' + B'P' = -DA'$, from those three last Equations I deduce the Values of N and P' , which may readily be obtained by Means of the Formulas of Article LXXXI. Chap. I. by putting $x = N$, $y = P'$, $z = N'$, $m = AB' - BA'$, $n = AC' - CA'$, $p = -DA'$, $a = A$, $b = 0$, $c = A'$; $d = B$, $f = B'$, $e = A'$; $g = C$, $b = B'$, $k = C'$,

then $x = \frac{ekm - fbm + cbn - cep}{aek - afb + cdb - ceg}$ gives

$$N = \frac{AA'B'C' - BA'A'C' - AB'^3 + BA'B'^2 + AA'B'C' - CA'A'B' + DA'^3}{AA'C' - AB'B' + BA'A'B' - CA'A'}$$

and $y = \frac{akn - gcn + dcp - aff + gfm - dkm}{ack - abf + dbc - ceg}$ becomes

$$N' = \frac{A^2C^2 - 2ACA'C + C^2A'^2 - BDA'^2 + ADA'B' + ACB'^2 - BCA'B' - ABB'C' + BB'A'C'}{AA'C' - AB'B' + BA'A'B' - CA'A'}$$

substituting those Values of N and N' in the Equation $DN + C'P' = 0$, there results $ADA'B'C' - BDA'A'C' - ADB'^3 + BDA'B'B' + ADA'B'C' - CDA'A'B' + DD A'^3 + AAC^3 - ACA'CC' - ACA'CC' + CCA'A'C' - BDA'A'C' + ADA'B'C' + ACB'B'C' - BCA'B'C' - ABB'C'C' + BB'A'C'C' = 0$.

Or $AAC^3 - ABB'C'^2 - 2ACA'CC' + B^2A'CC' - BCA'B'C' - 2BDA'A'C' + ACB'B'C' + CCA'A'C' - ADB'^3 - CDA'^2B' + 3ADA'B'C' + BDA'B'^2 + DD A'^3 = 0$, that is,

RULE II.

$(AC' - BB' - 2CA')AC'C' + (BC' - CB' - 2DA')BA'C' + (CC' - DB')(AB'B' + CA'A') + (3AB'C' + BB'B' + DA'A')DA' = 0$.

Example.

If it was proposed to exterminate x out of the Equations $y^3 - xyy - 3x = 0$ and $yy + xy - xx + 3 = 0$, I substitute in the precedent Expression

for A, B, C, D ; A', B', C' , and x , those Quantities 1, $-x$, 0, $-3x$; 1, x , $-xx + 3$, and y respectively; and there comes out $(3-xx+xx) \times (9-6xx+x^4) + (-3x+x^3+6x)(-3x+x^3)+3xx \times xx + (9x-3x^3-x^3-3x) \times -3x = 0$, then blotting out the superfluous Quantities and multiplying, you have $27 - 18xx + 3x^4 - 9xx + x^6 + 3x^4 - 18x^2 + 12x^4 = 0$, and ordering $x^6 + 18x^4 - 45xx + 27 = 0$.

LVII.

Let it be proposed to exterminate x out of the Equations $Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$, and $A'x^2 + B'x + C' = 0$, I multiply the first by $Mx + N$ and the second by $M'x^3 + N'x^2 + O'x + Q'$ and adding their Products there results, $(AM + A'M')x^5 + (BM + AN + B'M' + A'N')x^4 + (CM + BN + CM' + B'N' + A'O')x^3 + (DM + CN + C'N' + B'O' + A'Q')x^2 + (EM + DN + B'Q' + C'O')x + EN + C'Q' = 0$, and the Condition that all the Terms in this Equation including the Powers of x should be destroyed, gives the Equations $AM + A'M' = 0$, $BM + AN + B'M' + A'N' = 0$, $CM + BN + CM' + B'N' + A'O' = 0$, $DM + CN + C'N' + B'O' + A'Q' = 0$, $EM + DN + B'Q' + C'O' = 0$, consequently $EN + C'Q' = 0$, and as one of the Coefficients is undetermined I determine M by supposing it equal to A' ; whence

$$M' = -A, AN + A'N' = AB' - A'B, BN + B'N' + A'O' = AC - CA', CN + A'Q' + B'O' + C'N' = -DA', DN + B'Q' + C'O' = -EA'.$$

I deduce from those four last Equations the Values of N and Q' by Help of the Formulas of Art. LXXXVII, by putting $x = N$, $y = N'$, $z = O'$, $t = Q'$, $a = A$, $b = A'$, $c = 0$, $d = 0$, $a' = AB' - BA'$, $a'' = B$, $b' = B'$, $c' = A'$, $d' = 0$, $a'' = AC' - A'C$, $a''' = C$, $b'' = C'$, $c'' = B'$, $d'' = A'$, $a''' = -A'D$, $a'''' = D$, $b''' = 0$, $c''' = C'$, $b'' = C$, $c'' = B'$, $d'' = A'$, $A'' = -A'D$, $c''' = D$, $b''' = 0$, $c''' = C'$, $d''' = B'$, $a''' = -A'E$.

In this Case the Numerator of the Value of x is reduced to

$$[(a'b' - a'b)c'' + (a''b - a'b'')c']d''' + [(a'b - a'b')c''' - a'''b'c']d''$$

Consequently the Numerator of the Value of N will be $AB'^4 - BAA'B^3 - AA'B^{1/2}C' + CA'^2B^{1/2} - DA'^3B' - AA'B^{1/2}C' + BA'^2B'C' + AA'^2C'^2 - CA'^3C' - AA'B^{1/2}C' + EA'^4 + BA'^2B'C'$.

And the Numerator of the Value of t being reduced to

$$[(a'b' - a'b)c'' + (a''b - a'b'')c']a''' + [(a'b - a'b')c''' - a'''b'c']a'' + [a'''b'c']a' + (a'b'' - a''b)c'''a' + [-a'''b'c' + a'''b''c' + (a''b' - a'b'')c''']a.$$

Consequently, the Numerator of the Value of Q' will be

$$-AEAA'B^2 + BEAA^2B' - CEAA^3 + AEAA'^2C' - DBAA'^2C' + ADA'A'B'C' + D^2A'^3 + ADA'A'B'C' - CDA'^2B' + A^2C'^3 - 2ACA'A'C'^2 + C^2A'^2C' - ADB^3 + BDA'A'B^2 + ADA'A'B'C' - BDA'^2C' + ACB'^2C' - BCA'A'B'C' - ABB'B'C^2 + B^2A'C^2$$

Investigation of Sir Isaac Newton's Rule for exterminating an unknown quantity of four dimensions in one, and of two dimensions in the other equation by the foregoing Method.

wherefore $EN + CQ' = 0$ will be transformed into

$$\begin{aligned} & AEB^4 - BEAB^3 - AEA'B^2C + CE A^2B^2 - DE A^3B' - AEA'B^2C' \\ & + BEA^2B'C + AEA^2C^2 - CE A^3C' - AEA'B^2C' + BEA^2B'C' + E^2A^4 \\ & - AEA'B^2C + BEA^2B'C - CE A^3C' + AEA^2C^2 - BD A^2C^2 \\ & + ADA'B'C^2 + D^2A^3C' + ADA'B'C^2 - CDA^2B'C' \\ & + A^2C^4 - 2ACA'C^3 + C^2A^2C^2 - ADB^3C' + BDA'B^2C' \\ & + ADA'B'C^2 - BDA^2C^2 + ACB^2C^2 - BCA'B'C^2 - ABB'C^3 \\ & + B^2A'C^3 = 0. \text{ Or into,} \end{aligned}$$

$$\begin{aligned} & A^2C^4 - 2ACA'C^3 - ABB'C^3 + B^2A'C^3 - BCA'B'C^2 - 2BDA^2C^2 \\ & + ACB^2C^2 - ADB^3C' + AEB^4 - 2AEA'B^2C + C^2A^2C^2 \\ & - CDA^2B'C + CE A^2B^2 - 2CE A^3C' + 3ADA'B'C^2 \\ & + BDA'B^2C + D^2A^3C' + 2AEA^2C^2 + 3BEA^2B'C - DEA^3B' \\ & + E^2A^4 - BEA'B^3 - 2AEA'B^2C' = 0. \end{aligned}$$

RULE III.

That is $(AC - BB' - 2CA')AC^3 + (BC - CB' - 2DA')BA'C^2$
 $+ (AB^2 + CA^2)(CC^2 - DB'C + EB'B' - 2EA'C')$
 $+ (3AB'C + BB^2 + DA^2)DA'C + (2AC^2 + 3BB'C - DAB'$
 $+ EA^2)EA^2 + (-BB' - 2AC')EA'B^2 = 0.$

If it was proposed to exterminate x out of the Equations

$$x^4 + yyx^2 - 140xx + y^4 = 0, \text{ and } xx + yx - 20x + yy = 0;$$

Example.

Substitute for $A, B, C, D, E; A', B', C'$, respectively, $1, 0, y, y$
 $- 140, 0, y^4; 1, y - 20$, and yy ; and there will come out
 $(-yy + 280) \times y^5 + (2yy - 40y + 260) \times (260y^4 - 40y^5)$
 $+ 3y^4 \times y^4 - 2yy \times (y^6 - 40y^5 + 400y^4) = 0$; and by Multiplica-
 tion $1600y^6 - 20800y^5 - 67600y^4 = 0$, and by Reduction
 $4yy - 52y + 169 = 0.$

LVIII.

Let it be proposed to exterminate x out of the Equations

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terminating
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quantity of
three dimen-
sions in each
equation by
the forego-
ing Method.

$Ax^3 + Bx^2 + Cx + D = 0, A'x^3 + B'x^2 + C'x + D' = 0$, mul-
 tiplying $Ax^3 + Bx^2 + Cx + D$ by $Mx^2 + Nx + O$, and $A'x^3 + B'x^2$
 $+ C'x + D'$ by $M'x^2 + N'x + O'$, and adding their Products there
 will result $(AM + A'M')x^5 + (BM + AN + B'M' + A'N')x^4$
 $+ (CM + BN + AO + C'M' + B'N' + A'O')x^3 + (DM + CN$
 $+ BO + D'M' + C'N' + B'O')x^2 + (DN + CO + D'N' + C'O')x$
 $+ OD + O'D' = 0.$

The Condition that the Powers of x should be destroyed in this Equa-
 tion will give $AM + A'M' = 0, BM + AN + B'M' + A'N' = 0$
 $CM + BN + AO + C'M' + B'N' + A'O' = 0, DM + CN + BO$
 $+ D'M' + C'N' + B'O' = 0, DN + D'N' + CO + C'O' = 0,$
 consequently $DO + D'O' = 0$, and as there is one of the Coefficients in-
 determined, I determine M by supposing it $= A'$; whence
 $M' = -A, AN + A'N' = AB' - BA', BN + B'N' + AO + A'O'$
 $= AC - CA', CN + C'N' + BO + B'O' = AD' - DA,$
 $DN + D'N' + CO + C'O' = 0.$

I deduce from the four last Equations the Values of O and O' by the Help of the Formulas of Art. LXXXVII. by putting

$$x = N, y = N', z = O, t = O', a = A, b = A', c = o, d = o, a' = AB' - BA'; a'' = B, b'' = B', c'' = A, d'' = A, a''' = AC - CA'; a'''' = C, b'''' = C', c'''' = B, d'''' = B', a''''' = AD - DA'; a'''''' = D, b'''''' = D', c'''''' = C, d'''''' = C', a''''''' = o.$$

In this Case the Numerator of the Value of z will be reduced to

$$[(ab' - a'b)a'' + (a''b - a'b'')a' + (a'b'' - a''b')a]d'' + [(ab''' - a''b'')a' + (a''b' - a'b''')a]d' + [(a''b - a'b'')a'' + (a''b'' - a''b''')a]d$$

consequently the Numerator of the Value of O will be,

$$A^2B'CD' - ABACD' - ADA'B'C' + BDA^2C' + ACA^2C^2 - A^2C^3 - C^2A^2C' + ACA^2C'^2 + ABB'C^2 - ACB^2C - B^2AC'^2 + BCAB'C' + A^2B'CD' - ADA'B'C' - ACAB'D' + CDA^2B' + ADB^3 - ABB^2D' - BDA'B^2 + B^2A'B'D' + ADA^2D' - A^2A'D^2 - D^2A^3 + ADA^2D' + ACA^2B'D' - ADA'B'C' - BGA^2D' + BDA^2C'.$$

And the Numerator of the Value of t being reduced to

$$[(a'b - a'b')c''' + (a'b''' - a'''b)c']a'' + [(a'''b - a'b''')c'] + (a'b'' - a''b')c'' + [(a'b''' - a'''b')c'] + (a''b'' - a''b''')c']a.$$

Consequently the Numerator of the Value of O' will be

$$ABCA'D' - A^2CB'D' - BCD A^2 + ACDAB' + A^3D^2 - A^2DA'D' - A^2DA'D' + AD^2A^2 + ABD A^2C' - A^2BCD' - BDC A^2 + ABCA'D' + A^2CC'^2 - AC^2AC' - AC^2AC' + C^3A^2 + AB^2B'D' - ABD B^2 - B^3AD' + B^2DA'B' + A^2DB'C' - A^2CB'D' - ABD A^2C' + ABCA'D' + AC^2B^2 - ABCB'C' - BC^2A'B' + B^2CA'C'.$$

Wherefore $DO \mp D'O' = o$, will be transformed into

$$A^2DB'CD' - ABD A^2C'D' - AD^2A'B'C' + BD^2A^2C' + ACD A^2C^2 - A^2DC^3 - C^2DA^2C' + ACD A^2C'^2 + ABD B^2C^2 - ACD B^2C' - B^2DA^2C'^2 + BCD A^2B'C' + A^2DB'C'D' - AD^2A'B'C' - ACDAB'D' + CD^2A^2B' + AD^2B^3 - ABD B^2D' - BD^2A'B^2 + B^2DA'B'D' + AD^2A^2D' - A^2DA'D^2 - D^3A^3 + AD^2A^2D' + ACDAB'D' - AD^2A'B'C' - BCD A^2D' + BD^2A^2C' + ABCA'D^2 - A^2CB'D^2 - BCD A^2D' + ACDAB'D' + A^3D^3 - A^2DA'D^2 - A^2DA'D^2 + AD^2A^2D' + ABD A^2C'D' - A^2BCD^2 - BCD A^2D' + ABCA'D^2 + A^2CC^2D' - AC^2AC'D' - AC^2A'C'D' + C^3A^2D' + AB^2B'D^2 - ABD B^2D' - B^3AD^2 + B^2DA'B'D' + A^2DB'C'D' - A^2CB'D^2 - ABD A^2C'D' + ABCA'D^2 + AC^2B^2D' - ABCB'C'D' - BC^2A'B'D' + B^2CA'C'D' = o, or into$$

$$-A^2DC^3 + ABD B^2C^2 + 2ACDA^2C^2 + A^2CC^2D' - ABCB'CD' - 2AC^2A'C'D' - ABD A^2C'D' - B^2DA^2C^2 + BCD A^2B'C'$$

This Rule will serve to exterminate an unknown Quantity of a Number of Dimensions in each Equation not exceeding four; for if the unknown Quantity is only of three Dimensions in the Equation L' , we have no more to do than to put $A' = 0$, that is, to omit all the Terms in the resulting Equation among whose Factors A' is included, and then the whole Equation will be divisible by A . And if the unknown Quantity is only of two Dimensions in the Equation L' , we are to omit also all the Terms among whose Factors B' is included; and the resulting Equation will be again divisible by A , &c. But if the unknown Quantity rises only to three Dimensions in the Equation L , we are to omit all the Terms including E , and the whole Equation will be divisible by E' . And if the unknown Quantity rises only to two Dimensions in the Equation L , we are also to omit all the Terms which include D , and to divide a second Time the Equation by E' , and so on.

How an unknown quantity of a number of dimensions not exceeding four in each equation may be exterminated by the preceding Rule.

For Example, to exterminate x out of those two Equations $x^3 - 2ax^2 + 4axy - y^3 = 0$, and $ax^2 + y^2x - ay^2 = 0$, by the foregoing Rule; I put $E' = -ay^2$, $D' = y^2$, $C' = a$, $B' = 0$, $A' = 0$; $A = 1$, $B = -2a$, $C = 4ay$, $D = -y^3$, and $E = 0$.

Example.

Whence the Terms in the Formula, among whose Factors A' , B' , or E are included, will vanish, and the remaining Terms will be divisible by E' and A^2 ; consequently the Formula will be reduced to

$$A^2 E'^3 - 2ACC'E'^2 - ABD'E'^2 + 3ADD'C'E' + ACD'^2 E' + B^2 C'E'^2 - 2BDC'E' - ADD'^3 + D^2 C'^3 - CD C'^2 D' + BDC'D'^2 + C^2 C'^2 E' - BCC'D'E' = 0.$$

In which substituting for E' , D' , C' ; A , B , C , D , their Values,
 $-a^3 y^6 - 8a^4 y^5 + 2a^3 y^6 + 3a^2 y^7 - 4a^2 y^7 + 4a^5 y^4 + 4a^4 y^5 + y^9 + a^3 y^6 + 4a^3 y^6 + 2a^2 y^7 - 16a^5 y^4 - 8a^4 y^5$
 which after the Terms are reduced and ordered becomes,
 $y^9 + a^2 y^7 + 6a^3 y^6 - 12a^4 y^5 - 12a^5 y^4 = 0 = y^5 + a^2 y^3 + 6a^3 y^2 - 12a^4 y - 12a^5.$

LX.

The foregoing Method for exterminating an unknown Quantity of several Dimensions in two Equations, each involving two unknown Quantities, also serves to determine the Degree of the resulting Equation.

How the degree of the resulting equation is determined.

Let the highest Dimensions of y in A be p , of y in B , $p + 1$, of y in C , $p + 2$, &c. In like Manner let the highest Dimension of y in A' be p' , of y in B' be $p' + 1$, of y in C' , $p' + 2$, &c. Then the highest Dimensions of y in the Coefficients of each of the unknown Quantities, included in the Equations arising from the Supposition, that the Terms affected by the Powers of x in the Sum of the two Products of the proposed Equations into the undetermined Multinomials should be destroyed, form so many arithmetical Progressions, (it is thus, a Series of Quantities, that increase or decrease by the same constant Difference, is called) having the same common Difference.

ELEMENTS OF

*	*	*	*	*	*	*	*	p	p'
*	*	*	*	*	*	p	p'	$p+1$	$p'+1$
*	*	*	*	p	p'	$p+1$	$p'+1$	$p+2$	$p'+2$
*	*	p	p'	$p+1$	$p'+1$	$p+2$	$p'+2$	$p+3$	$p'+3$
p	p'	$p+1$	$p'+1$	$p+2$	$p'+2$	$p+3$	$p'+3$	$p+4$	$p'+4$
&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.

As will appear by the Inspection of those Equations, (Art. LIV.)

It will also appear by the Inspection of the Formulas of Art. LXXXVII. Chap. I. that each of the Terms of the resulting Equation after the unknown Quantities are exterminated, consists of the same Number $m+n+1$ of the Coefficients of those unknown Quantities, and that two Coefficients of the same unknown Quantity do not enter the same Term: Whence it is easy to conclude, that the highest Dimension of y in the resulting Equation will be expressed by the Sum of $m+n+1$ Terms of $m+n+1$ the foregoing Progressions, of which Terms two are not in the same Column or upon the same Line. Now upon Examination it will be found that $m+n+1$ Terms of $m+n+1$ of those Progressions taken in this Manner will always make the same Sum. For Example, If we take five Terms of five of those Progressions in such a Manner that two of those Terms are not contained in the same Column, or are upon the same Line, their Sum will be found to be $3p' + 2p + 6$; and the general Expression of

this Sum will be $S + (m+n+1) \times \left(\frac{m+n}{2}\right)$, S denoting the Sum of the Terms which compose the first Line, if the Progressions were continued so far; that is, of $p, p-1, p-2, p-3$, continued to $n+1$ Terms, and of $p', p'-1, p'-2, p'-3$, &c. continued to $n'+1$ Terms; wherefore $S = (2p-n) \frac{n+1}{2} + (2p'-n') \times \left(\frac{n'+1}{2}\right)$; consequently, if the highest Dimension of y in the resulting Equation be expressed by G , then

$$G = (m+n+1) \left(\frac{m+n}{2}\right) + (2p+n) \frac{n+1}{2} + (2p'+n') \frac{n'+1}{2};$$

in which substituting for n and n' their Values $m'-1$ and $m-1$ there results $G = mm' + pm' + p'm$.

LXX.

Limit which
the degree
of the result
ing equation
can not ex-
ceed.

It is easy to perceive, that the Degree of the resulting Equation when an unknown Quantity of several Dimensions in each Equation is exterminated, can never exceed $mm' + pm' + p'm$, p and p' expressing the Excess of the greatest Sum of the Exponents of the unknown Quantities in any one Term above the greatest Exponent of the Quantity to be exterminated in each Equation; for Example, if there were given the two Equations $a^3 x^5 y - 2a^4 y^2 x^3 + y^3 x - a^2 = 0$, and

A M
A' M'
A'' M''

$a^3x^3 - 3a^3xy^2 + y^5x - y^6 - a^4 = 0$; the greatest Sum of the Exponents of x and y in any one Term of the first Equation is 9, which subtracted from 5 the greatest Exponent of x gives $p = 4$; and the greatest Sum of the Exponents of x and y in any one Term of the second Equation which is 6 subtracted from 3 the greatest Exponent of x gives $p' = 3$, now x being exterminated out of the Equations $a y^4 x^5 - 2 a^4 y^6 x^3 + y^8 x - a^9 = 0$, $a^3 y^3 x^3 - 3 a^3 y^5 x + y^5 x - y^6 - a^4 = 0$, the Degree of the resulting Equation by the foregoing Art. will not exceed $5 \times 3 + 4 \times 3 + 5 \times 3$, that is, the 42d. Degree, much less then will the Equation in y deduced from the two proposed Equations ascend to the 42d. Degree.

LXII.

$$\text{Let } \begin{cases} A x^m + B x^{m-1} + C x^{m-2} + D x^{m-3} + \dots V = 0 \\ A' x^{m'} + B' x^{m'-1} + C' x^{m'-2} + D' x^{m'-3} + \dots V' = 0 \\ A'' x^{m''} + B'' x^{m''-1} + C'' x^{m''-2} + D'' x^{m''-3} + \dots V'' = 0 \end{cases}$$

be three Equations; in which A, B, C , &c. A', B', C' , &c. A'', B'', C'' , &c. denote Functions of two unknown Quantities y and z , and known Quantities, and let the Dimensions of A, B, C , &c. be expressed by $p, p+1, p+2$, &c. those of A', B', C' , &c. by $p', p'+1, p'+2$, &c. and those of A'', B'', C'' , &c. by $p'', p''+1, p''+2$, &c. to obtain the two Equations which result when x is exterminated, it suffices to find three Functions of x as simple as possible, by which the three Equations being multiplied respectively, in the Sum of the three Products the Terms affected by the Powers of x will be destroyed, and three other Functions of x different from the first, as simple also as possible endued with the same Property.

$$\text{Let } \begin{cases} M x^n + N x^{n-1} + P x^{n-2} + Q x^{n-3} + \dots T \\ M' x^{n'} + N' x^{n'-1} + P' x^{n'-2} + Q' x^{n'-3} + \dots T' \\ M'' x^{n''} + N'' x^{n''-1} + P'' x^{n''-2} + Q'' x^{n''-3} + \dots T'' \end{cases}$$

be the three first Functions, performing the Multiplications, and adding the three Products there will result,

$$\begin{aligned} & A M x^{m+n} + B M x^{m+n-1} + C M x^{m+n-2} + D M x^{m+n-3} \dots V T = 0 \\ & A' M' x^{m'+n'} + A' N x^{m'+n-1} + B' N x^{m'+n-2} + C' N x^{m'+n-3} \dots V' T' \\ & A'' M'' x^{m''+n''} + B' M' x^{m'+n'-1} + A' P x^{m'+n'-2} + B' P x^{m'+n'-3} \dots V'' T'' \\ & \quad + A' N' x^{m'+n'-1} + C' M' x^{m'+n'-2} + A' Q x^{m'+n'-3} \\ & \quad + B'' M'' x^{m''+n''-1} + B' M' x^{m'+n'-2} + D' M' x^{m'+n'-3} \\ & \quad + A'' N'' x^{m''+n''-1} + A' P' x^{m'+n'-2} + C' N' x^{m'+n'-3} \\ & \quad + C'' M'' x^{m''+n''-2} + B' P' x^{m'+n'-3} \\ & \quad + B'' N'' x^{m''+n''-2} + A' Q' x^{m'+n'-3} \\ & \quad + A'' P'' x^{m''+n''-2} + D'' M'' x^{m''+n''-3} \\ & \quad + C'' N'' x^{m''+n''-3} \\ & \quad + B'' P'' x^{m''+n''-3} \\ & \quad + A'' Q'' x^{m''+n''-3} \end{aligned}$$

Let us suppose first $m + n = m' + n'$ and $m + n > m'' + n''$, or at furthest equal. It is obvious that in Order to destroy all the Terms affected with x , the Number of undetermined Coefficients must be $m + n + 1$, wherefore $m + n + 1 = n + 1 + n' + 1 + n'' + 1$, wherefore $n' = m - n'' - 2$ and $n = m' - n'' - 2$. This being supposed, putting the Coefficients of each Power of x equal to Nothing, there will result a Series of Equations; in which it is easy to perceive, that the highest Dimensions of the same unknown Quantity, M or N &c. form an arithmetical Progression, and that all those Progressions have the same common Difference; whence by a similar Reasoning to that employed in Article LX. it will appear, that the Number G expressing the highest Dimension of the Equation in y and z resulting from the Combination of $m + n + 1$ Equations, will be $G = S + (m + n + 1) \frac{m + n}{2}$, S expressing the Sum of the Terms which compose the first Line if the Progressions were continued so far. Now it is easy to perceive

1°. That the Numbers which denote the Dimensions which the Coefficients M , N , P , &c. should have in the first Equation if they were found there, are p , $p - 1$, $p - 2$, $p - 3$, &c. the Number of Terms

being $n + 1$, whose Sum is expressed by $(2p - n) \times \left(\frac{n + 1}{2}\right)$.

2°. That the Numbers which in the same Supposition denote the Dimensions of M' , N' , P' , &c. are p' , $p' - 1$, $p' - 2$, &c. continued to a Number of Terms $= n' + 1$, whose Sum is expressed by

$(2p' - n') \times \left(\frac{n' + 1}{2}\right)$. 3°. In Order to find the Numbers which

denote in the first Equation the Dimensions of M'' , N'' , P'' , &c. if those Quantities were found there, I observe, that as the Place of the Term $A'' M'' x^{m''} + n''$ in the Order of the Powers of x is denoted by $m + n - m'' - n'' + 1$; so will its Place in the Order of the Equations be denoted by the same Number; consequently, to discover what should be the Dimension of the Coefficient M'' in the first Equation, we must retrograde $m + n - m'' - n''$ Equations; consequently, the Number which denotes this Dimension will be $p'' - m - n + m'' + n''$; and consequently, the Numbers which denote in the first Equation the Dimensions of M'' , N'' , P'' , &c. if those Quantities were found there, are $p'' - m - n + m'' + n''$, $p'' - m - n + m'' + n'' - 1$, $p'' - m - n + m'' + n'' - 2$, &c. continued to a Number of Terms $= n'' + 1$,

whose Sum will be $(2p'' - 2m - 2n + 2m'' + 2n'') \times \left(\frac{n'' + 1}{2}\right)$.

Consequently, $G = (m + n + 1) \left(\frac{m + n}{2}\right) + (2p - n) \left(\frac{n + 1}{2}\right) + (2p' - n') \left(\frac{n' + 1}{2}\right) + (2p'' - 2m - 2n + 2m'' + 2n'') \left(\frac{n'' + 1}{2}\right)$.

Or by substituting for n and n' their Values found above,
 $G = m m' + p m' + p' m - m - m' + m'' - p - p' + p'' + 1$
 $-(p + p' - p'' + m + m' - m'' - n'' - 2) n''$. A Quantity which
 is undetermined since n'' has not been determined.

LXIV.

Hence the Equation in y and z may be obtained different Ways, but it is manifest that that which gives for G the least Quantity is only to be chosen; and it is from this Condition that n'' is to be determined. To determine therefore this extreme Value of G we are to find the limiting Ratio of the Cotemporary Increments of the two undetermined Quantities G and n'' in the Equation $G = m m' + p m' + p' m - m - m' + m'' - p - p' + p'' + 1 - (p + p' - p'' + m + m' - m'' - n'' - 2) n''$, and to put this Ratio equal to Nothing.

Let dG and dn'' denote the cotemporary Increments of G and n'' , then $G + dG = m m' + p m' + p' m - m - m' + m'' - p - p' + p'' + 1 - p n'' - p d n'' - p' n'' - p' d n'' + p'' n'' + p'' d n'' - m n'' - m d n'' - m' n'' - m' d n'' + m'' n'' + m'' d n'' + n'' n'' + 2 n'' d n'' + d n'' d n'' + 2 n'' + 2 d n''$; and since $G = m m' + p m' + p' m - m - m' + m'' - p - p' + p'' + 1 - p n'' - p' n'' + p'' n'' - m n'' - m' n'' + m'' n'' + n'' n'' - 2 n''$.
 $dG = -p d n'' - p' d n'' + p'' d n'' - m d n'' - m' d n'' + m'' d n'' + 2 n'' d n'' + d n'' d n'' + 2 d n''$.

And $\frac{dG}{dn''} = -p - p' + p'' - m - m' + m'' + 2 n'' + d n'' + 2$;

wherefore, $-p - p' + p'' - m - m' + m'' + 2 n'' + d n'' + 2$, expresses the Ratio of any cotemporary Increments of G and n'' , and this Ratio is always greater than $-p - p' + p'' - m - m' + m'' + 2 n'' + 2$; but $d n''$ decreasing, this Quantity will also decrease, and as we may take $d n''$ as small as we please, we may make $-p - p' + p'' - m - m' + m'' + 2 n'' + d n'' + 2$ approach as near as we please to $-p - p' + p'' - m - m' + m'' + 2 n'' + 2$; wherefore, $-p - p' + p'' - m - m' + m'' + 2 n'' + 2$ is the Limit of $-p - p' + p'' - m - m' + m'' + 2 n'' + d n'' + 2$; that is, of the Ratio $\frac{dG}{dn''}$; which gives

$$n'' = \frac{m + m' + m'' + p + p' + p''}{2} - m'' - p'' - 1. \text{ when made equal}$$

to Nothing. But it will not always be possible to employ this Value,

1°. Because n'' should be a positive Quantity. 2°. Because

$\frac{m + m' + m'' + p + p' + p''}{2}$ cannot always be reduced to an Integer.

3°. Because $m + n$ should be greater than $m'' + n''$, or at least equal to it.

4°. Because m' being supposed less than m or $= m$, it follows, that $m' - 2$ should be greater than n'' , or at least $= n''$. We shall therefore

ELEMENTS OF

assume $n'' = \frac{m + m' + m'' + p + p' + p'' + a}{2} - m'' - p'' - 1$

supposing, a to be the least Quantity that can answer those Conditions.

This Value of n'' being substituted in G there will result,

$$G = m m' + m m'' + m p' + m p'' + m' m'' + m' p + m' p'' + m'' p + m'' p' + p p'' + p' p'' + \frac{a^2}{4} - \left(\frac{p + p' + p'' + m + m' + m''}{2} \right)^2.$$

LXV.

Observations which serve to find the values of n, n', n'' .

Though there cannot be found a Value for a which will answer those four Conditions, we are not to conclude, that from the Combination of the three Equations a more simple Equation cannot be derived, than if they were combined two by two; for we are to observe, that those Conditions arise from the Supposition, that $m + n = m' + n'$, and $m + n > m'' + n''$ &c. now there was no Reason why we should suppose $m + n = m' + n'$, rather than $m + n = m'' + n''$ or $m' + n' = m'' + n''$. It will therefore be proper to make three different Tryals. And if they all three give positive Values for n, n', n'' , (as small a Quantity as possible being put for a) we are to employ the two Results which give the least Value for G . and if there result positive Values for n, n', n'' , only two different Ways, they are to be employed, provided they give a less Value for G than would result from the Combination of the Equations two by two, and if there can be found positive Values for n, n', n'' , but one Way, we are to conclude, that the unknown Quantity is once to be exterminated by combining only two of the three Equations together. Finally, if there be no Possibility of obtaining for n, n', n'' , positive Values without rendering the Value of G greater than it would be if the Equations were combined two by two, we are to have Recourse to this last Method, but this last Case will scarcely ever happen, if it happens at all: For Example, it can never occur when $p = p' = p'' = 0$, one Equation at least will be found more simple than by combining the Equations two by two, except when some of the Quantities m, m', m'' , are equal to 1.

LXVI.

The foregoing Principles are sufficient for finding in all Cases the two Equations in y and z . We shall now proceed to illustrate them by some Examples.

$$1^o. \text{ Let } p = p' = p'' = 0, \text{ and } m = m' = m'', \text{ then } n'' = \frac{3m - a}{2} - m - 1 \\ = \frac{m - a - 2}{2}, \quad n' = \frac{m + a - 2}{2}, \quad n = \frac{m + a - 2}{2}.$$

If m is even, then a may be put $= 0$, and the Value of G will be found $= \frac{3}{4} m^2$; consequently, less than would result from the Combination of the Equations two by two, we may also put $a = 2$, except when $m = 2$, and then $G = \frac{3m^2 + 1}{4}$.

If m is odd, put $a = 1$, and there will result $G = \frac{3m^2 + 1}{4}$ which Value is less than would result from the Combination of the Equations two by two; now as there is no Reason why the equal Values of n and n' should be employed with Respect to two of three Equations, rather than with Respect of the two others, we are to make Use of them both Ways, and there will result two Equations in y and z , each of a Degree expressed by $\frac{3m^2 + 1}{4}$.

Hence, if there are three Equations given each of a Degree m in which the highest Dimension of x is expressed by m , if m be an even Number we are to multiply each Equation by an undetermined Multinomial of a Degree expressed by $\frac{m-2}{2}$, and by Means of the Equations resulting from the Supposition that the Powers of x in the Sum of the three Products should be destroyed, an Equation in y and z will be obtained of a Degree expressed by $\frac{3}{4}m^2$. To obtain another Equation, we are to multiply two of the three proposed Equations by a Multinomial of a Degree expressed by $\frac{m}{2}$, and the third by a Multinomial of

Rules derived from the precedent principles for exterminating an unknown quantity of several dimensions in three equations.

a Degree expressed by $\frac{m-4}{2}$, and by putting in the Sum of the three Products the Terms including the Powers of $x = 0$, there will result an Equation in y and z of a Degree expressed by $\frac{3}{4}m^2 + 1$.

But if m be an odd Number, we are to multiply the first and second Equation by a Multinomial of a Degree expressed by $\frac{m-1}{2}$, and the third by a Multinomial of a Degree expressed by $\frac{m-3}{2}$, we are to multiply also the first and third Equations, each by a Multinomial of a Degree expressed by $\frac{m-1}{2}$, and the second by a Multinomial of a Degree expressed by $\frac{m-3}{2}$, or else the second and third by a Multinomial of a Degree $\frac{m-1}{2}$, and the first by a Multinomial of a Degree $\frac{m-3}{2}$, and there will result two Equations in y and z , each of a Degree expressed by $\frac{3m^2 + 1}{4}$.

LXVII.

From whence we may perceive, that by combining the Equations two by two, there will result an Equation in y and z of higher Dimensions

Inconveni-
ency to
which the
method of
exterminat-
ing an un-
known quan-
tity by com-
paring the
equations
two by two
is liable.

than when the three Equations are combined together; for by the first Method the resulting Equation will rise in general to a Number of Dimensions expressed by m^4 , and by the second to $\frac{3}{4} m^2 (\frac{3}{4} m^2 + 1)^2$ when m is an even Number, and to $(\frac{3 m^2 + 1}{4})^2$ Dimensions, when m is odd. This Difference is rendered more sensible by the following Table.

	$m =$	2	...	3	...	4	...	5	...	6	...	7
By 1st. Method,	$G =$	16	...	81	...	256	...	625	...	1296	...	2401
By 2d. Method,	$G =$	12	...	49	...	156	...	361	...	756	...	1369

LXVIII.

2°. Let $m = 6, m' = 5, m'' = 3, p = 4, p' = 5, p'' = 19$, there will result $n'' = \frac{-4 - a}{2}, n' = \frac{12 + a}{2}, n = \frac{10 + a}{2}$.

The least Value that can be given to a is $a = -4$, and then $G = 83$. But if we invert the Order of the Equations, and write down $m = 6, m' = 3, m'' = 5, p = 4, p' = 19, p'' = 5$, there will result $n'' = \frac{20 - a}{2}, n' = \frac{a - 12}{2}, n = \frac{a - 18}{2}$, the least Value that can be given to a is $a = 18$, and then $G = 104$.

Inverting again the Order of the Equations, and writing down $m = 5, m' = 3, m'' = 6, p = 5, p' = 19, p'' = 4$, there will result $n'' = \frac{20 - a}{2}, n' = \frac{a - 14}{2}, n = \frac{a - 18}{2}$, the least Value that can be given to a is still $a = 18$, and then G will be found $= 85$; but this Combination cannot take Place because it gives $m + n < m'' + n''$ &c. and if the Equations be combined two by two, there will be found $G = 80, G = 144$, and $G = 125$. Wherefore, in this Case, one of the Equations in x and y is found by combining the Equations together, and the other by combining two of them together; we are to multiply the Equations indicated by $m = 6, m' = 5, m'' = 3$; I say, multiply those Equations by the Multinomials indicated by $n = 3, n' = 4, n'' = 0$, and the two Equations to be combined to obtain the second Equation in y and z , are those indicated by $m = 6, m' = 5$.

LXIX.

3°. Let $m = 7, m' = 6, m'' = 5, p = 3, p' = 4, p'' = 2$, there will result $n'' = \frac{n - a}{2}, n' = \frac{a - 1}{2}, n = \frac{a - 3}{2}$, the least Value therefore that can be given to a is $a = 3$; but it is useless, because it gives $m + n < m'' - n''$; let then $a = 5$, and G will be found $= 52$; changing the Order of the Equations, and putting $m = 7, m' = 5, m'' = 6, p = 3, p' = 2, p'' = 4$, there will result $n'' = \frac{5 - a}{2}$,

$n' = \frac{5+a}{2}$, $n = \frac{1+a}{2}$, the least Value that can be given to a is $a = -1$, but it is useless, because it gives $m+n < m''+n''$. Let $a = 1$ there will result $G = 52$, as it should be, since the Multipliers of the Equations will be the same in this Case as in the foregoing. Let then $a = 3$ there will result $G = 54$, changing again the Order of the Equations, and writing down $m = 6$, $m' = 5$, $m'' = 7$, $p = 4$, $p' = 2$, $p'' = 3$; there will result $n'' = \frac{5-a}{2}$, $n' = \frac{3+a}{2}$, $n = \frac{1+a}{2}$; the Suppositions of $a = -1$, $a = 1$ cannot take Place. Let then $a = 3$, then $G = 52$, which give the same Multipliers as were found above, and for the same Equations. If a be put equal to 5, then $G = 56$, combining the Equations two by two, there results $G = 88$, $G = 64$, $G = 62$. Wherefore, the two Equations in y and z will be found by combining the three Equations together, the one being of 52 Dimensions, and the other of 54.

LXX.

If four Equations are given involving four unknown Quantities, to exterminate one of them as x ; let m, m', m'', m''' , denote the Exponents of x in the four Equations, let the Dimensions of the Coefficients of the several Powers of x in the first Equation be expressed by $p, p+1, p+2, p+3$, &c. in the second by $p', p'+1, p'+2, p'+3$, &c. in the third by $p'', p''+1, p''+2$, &c. and in the fourth by $p''', p''' + 1, p''' + 2, p''' + 3$, &c.

How an unknown quantity of several dimensions is exterminated out of four equations by the foregoing method.

Let n, n', n'', n''' , denote the Exponents of the undetermined Multipliers by which the Equations being multiplied, in the Sum of the four Products the Powers of x will be destroyed.

By a similar Reasoning to that employed above, it will appear, that $m+n$ should be put equal to $m'+n'$, and $m+n+1 = n+1+n'+1+n''+1+n''' + 1$; from whence is deduced $n = m' - n' - n'' - 3$ and $n' = m - n'' - n''' - 3$; also $m+n$ should be $> m''+n''$ or at furthest equal, and $m+n > m''' + n'''$, or at furthest equal.

It will appear likewise that $S = (2p - n) \left(\frac{n+1}{2} \right) + (2p' - n') \left(\frac{n'+1}{2} \right) + (2p'' - 2m - 2n + 2m'' + n'') \left(\frac{n''+1}{2} \right) + (2p''' - 2m - 2n + 2m''' + n''') \left(\frac{n''' + 1}{2} \right)$,
Consequently, $G = (m+n+1) \left(\frac{m+n}{2} \right) + (2p-n) \left(\frac{n+1}{2} \right) + (2p' - n') \left(\frac{n'+1}{2} \right) + (2p'' - 2m - 2n + 2m'' + n'') \left(\frac{n''+1}{2} \right) + (2p''' - 2m - 2n + 2m''' + n''') \left(\frac{n''' + 1}{2} \right)$

3 P

$$+ (2p''' - 2m - 2n + 2m''' + n''') \left(\frac{n''' + 1}{2} \right),$$

or substituting for n and n' their Values,

$$G = m m' + p m' + p' m - 2m - 2p - 2m' - 2p' + m'' + p'' + m''' + p''' + 3 - n'' n''' - (m + m' - m'' + p + p' - p'' - n'' - n''' - 3)n'' - (m + m' - m''' + p + p' - p''' - n' - n'' - 3)n''.$$

To find the Values of n' and n''' which substituted in the preceding Expression of G will render it the least possible, we are to find, 1°. The limiting Ratio of the cotemporary Increments of G and n'' , considered as the only Quantities undetermined in the preceding Equation, and to put this Limit equal to Nothing; then to find the limiting Ratio of the cotemporary Increments of G and n''' , supposed to be the only Quantities undetermined in the same Equation, and to put this Limit also equal to Nothing. These Operations being performed you will find,

$$n'' = \frac{m + m' + m'' + m''' + p + p' + p'' + p'''}{3} - m'' - p'' - 1, \text{ and}$$

$$n''' = \frac{m + m' + m'' + m''' + p + p' + p'' + p'''}{3} - m''' - p''' - 1,$$

but because n, n', n'', n''' , should be positive integer Numbers, and answer the Conditions mentioned above.

$$\text{We must put } n'' = \frac{m + m' + m'' + m''' + p + p' + p'' + p''' - a}{3} - m'' - p'' - 1,$$

$$\text{and } n''' = \frac{m + m' + m'' + m''' + p + p' + p'' + p''' - b}{3} - m''' - p''' - 1,$$

General expression of the degree of the resulting equations after an unknown quantity of several dimensions is exterminated out of four equations.

a and b expressing the least Numbers that can answer those Conditions.

Substituting in G those Values of n'' and n''' , there will result

$$G = m m' + m m'' + m m''' + m p' + m p'' + m p''' + m' m'' + m' m''' + m' p + m' p'' + m' p''' + m'' m''' + m'' p + m'' p'' + m'' p''' + m''' p + m''' p'' + m''' p''' + p p'' + \frac{a^2 + ab + b^2}{9} - 3 \left(\frac{m + m' + m'' + m''' + p + p' + p'' + p'''}{3} \right)^2.$$

LXXI.

Let $p = p' = p'' = p''' = 0, m = m' = m'' = m'''$, then

$$n''' = \frac{m - b - 3}{3}, \quad n'' = \frac{m - a - 3}{3}, \quad n' = \frac{m + a + b - 3}{3},$$

$$n = \frac{m + a + b - 3}{3}.$$

It is easy to perceive first, that whatever Values positive or negative are given to a and b , if m be less than 3, it will not be possible to render n, n', n'', n''' , at the same Time positive. In effect if $m = 2$, as in this Case, x^2 and x are the only Quantities to be exterminated, the

Equations are to be combined three by three two different Ways, and the third Equation will be obtained by combining two of them together.

If $m = 3$, then by putting $a = 0$, and $b = 0$, positive Values will be obtained for n, n', n'', n''' , consequently the four Equations may be combined together; the two other Equations without x will be obtained by combining the Equations three by three, two different Ways.

But when m is greater than 3, there are three Cases to be considered, either m is exactly divisible by 3, or after the Division there remains 1, or there remains 2.

FIRST CASE. To obtain in this Case the first Equation without x , put $a = 0, b = 0$; to obtain the second, put $a = 3, b = 0$; and to obtain the third, put $a = 0, b = 3$. The corresponding Values of G will be $G = \frac{2}{3}m^2, G = \frac{2}{3}m^2 + 1, G = \frac{2}{3}m^2 + 1$. From whence it is easy to conclude that the Degree of the resulting Equation after all the unknown Quantities are exterminated, will be expressed by $(\frac{2}{3}m^4 + \frac{2}{3}m^2)(\frac{2}{3}m^4 + \frac{2}{3}m^2 + 1)$.

Rules derived from the preceding principles for exterminating an unknown quantity of several dimensions in four equations.

SECOND CASE. To obtain the first Equation without x , put $a = 1, b = 1$, and because by changing the Order of the Equations, there will result a different Factor for the same Equation, the Supposition of $a = 1$ and $b = 1$ alone will furnish the three Factors to be applied to each Equation to obtain the three Equations without x . For Example, if $m = 4$ then $n''' = 0, n'' = 0, n' = 1, n = 1$, and $m, m', &c.$ being all equal, it is indifferent to which of the four Equations the Values of n or of n' are referred, consequently there may be formed three Combinations, each of which will furnish a different Equation. The Value of G will be the same in each Case and $= \frac{2m^2 + 1}{3}$, from whence it is easy to conclude that the Degree of the resulting Equation after all the unknown Quantities are exterminated will be expressed by $(\frac{m^4 + m^2 + 1}{3})^2$.

THIRD CASE. The three Equations without x will be obtained by putting $a = 2, b = 2$, and by applying the Factors three different Ways to the proposed Equations, each of which will rise to a Degree expressed by $\frac{2m^2 + 4}{3}$, from whence it is easy to conclude that the Degree of the resulting Equation will be $(\frac{m^4 + 4m^2 + 4}{3})(\frac{m^4 + 4m^2 + 4}{3})$.

Let now $p = 2, p' = 0, p'' = 1, p''' = 1, m = 7, m' = 6, m'' = 5, m''' = 4$; we will have $n''' = \frac{8-b}{3}, n'' = \frac{5-a}{3}, n' = \frac{a+b-1}{3}, n = \frac{a+b-4}{3}$. Let $b = -1$ and $a = 5$, we will have $n''' = 3, n'' = 0, n' = 1, n = 0$, and $G = 26$. If we put $a = -1, b = 5$,

there will result $n''' = 1$, $n'' = 2$, $n' = 1$, $n = 0$, and $G = 26$. If we put $b = 2$, $a = 2$, then $n''' = 2$, $n'' = 1$, $n' = 1$, $n = 0$, and $G = 25$. Now whatever Alteration is made in the Order of the Equations, there will result the same Values for G , if not greater; it remains therefore to examine whether more simple Values may not be obtained for G , by combining the Equations three by three, or two by two. The Calculation being performed, it will appear that the Equations indicated by $m' = 6$, $m'' = 5$, $m''' = 4$, $p' = 0$, $p'' = 1$, $p''' = 1$, will give $G = 24$, and those indicated by $m = 7$, $m'' = 5$, $m''' = 4$, $p = 2$, $p'' = 1$, $p''' = 1$, will give $G = 25$; wherefore in order to obtain the three simplest Equations that can result after x is exterminated, we are to compare the four proposed Equations three by three, the two different Ways, just mentioned; and the four together after the Manner indicated by the Combination which gave $G = 25$. The Combinations of the Equations two by two, all give Values for G greater than those already found.

LXXII.

If five Equations are given involving five unknown Quantities, to exterminate one of them as x , let m, m', m'', m''', m'''' , denote the Exponents of the Degree of x in the proposed Equations, and let p, p', p'', p''', p'''' express the Dimensions of the first Coefficient of those Equations, the Dimensions of the other Coefficients being supposed to observe the same Law as heretofore; by a similar Reasoning to that employed for 2, 3, 4 unknown Quantities, it will appear that $m + n = m' + n'$, $m + n + 1 = n + 1 + n' + 1 + n'' + 1 + n''' + 1 + n'''' + 1$, from whence is deduced $n = m' - n'' - n''' - n'''' - 4$, and $n' = m - n'' - n''' - n'''' - 4$. It will be demonstrated in like Manner that

$$G = (m + n + 1) \left(\frac{m + n}{2} \right) + (2p - n) \left(\frac{n + 1}{2} \right) + (2p' - n') \left(\frac{n' + 1}{2} \right) \\ + (2p'' - 2m - 2n + 2m'' + n'') \left(\frac{n'' + 1}{2} \right) + (2p''' - 2m - 2n + 2m''' + n''') \left(\frac{n''' + 1}{2} \right) \\ + (2p'''' - 2m - 2n + 2m'''' + n'''') \left(\frac{n'''' + 1}{2} \right) = mm' + pm' + p'm - 3m - 3p - 3m' \\ - 3p' + m'' + p'' + m''' + p''' + m'''' + p'''' - n'n'' - n'n''' - n'n'''' + 6 - (p + p' \\ - p'' + m + m' - m'' - n'' - n''' - n'''' - 4)n'' - (p + p' - p'' + m + m' - m'' - n'' \\ - n''' - n'''' - 4)n''' - (p + p' - p'' + m + m' - m'' - n'' - n''' - n'''' - 4)n''''.$$

The Condition that G should be the least Number possible, requires that we should put equal to nothing, 1°. The limiting Ratio of the cotemporary Increments of G and n'' , considered as the only Quantities undetermined in the preceding Expression. 2°. The limiting Ratio of the cotemporary Increments of G and n''' , considered as the only undetermined Quantities in the foregoing Expression. 3°. The limiting

How an unknown quantity of several dimensions is exterminated out of five equations by the foregoing method.

Ratio of the cotemporary Increments of G and n'''' , considered as the only undetermined Quantities in the aforesaid Expression. These Operations being performed, we will have

$$n'' = \frac{m+m'+m''+m''' + p+p'+p''+p'''+1}{4} - m'' - p'' - 1,$$

$$n''' = \frac{m+m'+m''+m''' + p+p'+p''+p'''+1}{4} - m''' - p''' - 1,$$

$$n'''' = \frac{m+m'+m''+m''' + p+p'+p''+p'''+1}{4} - m'''' - p'''' - 1,$$

but because n'' , n''' , n'''' should be integer Numbers and positive, and such that $m+n > m''+n''$, $m+n > m''' + n'''$, $m+n > m'''' + n''''$, or at the furthest equal.

$$\text{I put } n'' = \frac{m+m'+m''+m''' + p+p'+p''+p'''+a}{4} - m'' - p'' - 1,$$

$$n''' = \frac{m+m'+m''+m''' + p+p'+p''+p'''+b}{4} - m''' - p''' - 1,$$

$$n'''' = \frac{m+m'+m''+m''' + p+p'+p''+p'''+c}{4} - m'''' - p'''' - 1,$$

a , b , c being the least Numbers positive or negative that can answer those Conditions.

If we substitute in G those Values of n'' , n''' , n'''' , we will have

$$\begin{aligned} G = & m m' + m m'' + m m''' + m m'''' + m p' + m p'' + m p''' + m p'''' \\ & + m' p'''' + m' m'' + m' m''' + m' m'''' + m' p' + m' p'' + m' p''' + m' p'''' \\ & + m'' p'''' + m'' m''' + m'' m'''' + m'' p' + m'' p'' + m'' p''' + m'' p'''' \\ & + m''' p'''' + m''' p' + m''' p'' + m''' p''' + m''' p'''' + m''' p' + m''' p'' + m''' p''' + m''' p'''' \\ & + p' p'''' + p' p' + p' p'' + p' p''' + p' p'''' + p'' p' + p'' p'' + p'' p''' + p'' p'''' \\ & + p''' p'''' + p''' p' + p''' p'' + p''' p''' + p''' p'''' + p'''' p' + p'''' p'' + p'''' p''' + p'''' p'''' \\ & + \frac{a^2 + b^2 + c^2 + ab + ac + bc}{16} \\ & - 6 \left[\frac{m+m'+m''+m''' + p+p'+p''+p'''+1}{4} \right]^2. \end{aligned}$$

LXXIII.

$$\text{Let } p = p' = p'' = p''' = p'''' = 0, m = m' = m'' = m''' = m'''' , \text{ then}$$

$$n'''' = \frac{m-c-4}{4}, n''' = \frac{m-b-4}{4}, n'' = \frac{m-a-4}{4}, n' = \frac{m+a+b+c-4}{4},$$

$$n = \frac{m+a+b+c-4}{4}.$$

Whence if m is less than 4, there are no Values which substituted for a , b , c will render n , n' , n'' , n''' , n'''' positive, that is, the unknown Quantity is to be exterminated by the Rules laid down for a lesser Number of unknown Quantities.

If $m = 4$, the unknown Quantity may be exterminated by combining the five Equations together but only one Way, by putting $a = b = c = 0$.

General expression of the degree of the resulting equations after an unknown quantity of several dimensions is exterminated out of five equations.

Rules derived from the

preceding principles for exterminating an unknown quantity of several dimensions in five equations.

If m be greater than 4, there are four Cases to be considered, either m is exactly divisible by 4, or after the Division there remains 1 or 2, or in fine 3.

FIRST CASE. The first Equation without x will be obtained by putting $a = 0, b = 0, c = 0$, the second by putting $a = 0, b = 0, c = 4$, the third by putting $a = 0, b = 4, c = 0$, in fine, the fourth by putting $a = 4, b = 0, c = 0$. The corresponding Values of G will be $G = \frac{5}{8} m^2, G = \frac{5}{8} m^2 + 1, G = \frac{5}{8} m^2 + 1, G = \frac{5}{8} m^2 + 1$.

SECOND CASE. Put $a = 1, b = 1, c = 1$, and the Values which result for $n, n', \&c.$ being not all of the same Degree, they may be applied four different Ways to the five Equations, whereby the four Equations without x will be obtained each of a Degree, $G = \frac{5m^2 + 3}{8}$.

THIRD CASE. Put $a = 2, b = 2, c = 2$; proceeding as in the second Case, the four Equations without x will be obtained; which will rise each to a Degree expressed by $\frac{5m^2}{8} + \frac{3}{2}$.

FOURTH CASE. Put $a = 3, b = 3, c = 3$, and proceeding as in the two foregoing Cases, the four Equations without x will be obtained, each of a Degree expressed by $\frac{5}{8} m^2 + \frac{27}{8}$.

LXXIV.

In general if there are N unknown Quantities and N Equations given; from all that precedes it is easy to conclude,

1^o. That if A denotes the Sum of the Quantities $m, m', m'', \&c.$ and $p, p', p'', \&c.$ Then

$$n'' = \frac{A - a}{N - 1} - m'' - p'' - 1, n''' = \frac{A - b}{N - 1} - m''' - p''' - 1,$$

$$n'''' = \frac{A - c}{N - 1} - m'''' - p'''' - 1, n''''' = \frac{A - d}{N - 1} - m''''' - p''''' - 1,$$

$$\text{and so on; likewise } n = \frac{A + a + b + c + d + \&c.}{N - 1} - m - p - p' - 1,$$

$$n' = \frac{A + a + b + c + d + \&c.}{N - 1} - m' - p - p' - 1.$$

2^o. That if B denotes the Sum of the Squares together with the Sum of the Products of the Quantities $a, b, c, \&c.$ taken two by two, and C the Sum of the Products of the Quantities $m, m', m'', \&c. p, p', p'', p''', \&c.$ multiplied two by two, omitting the Products $mp, m'p', m''p'', \&c.$ and the Product pp' of the two Quantities, which belong to the Equations in which $m + n$ was supposed equal to $m' + n'$, there will result in general

$$G = C + \frac{B}{(N-1)^2} - (N-1) \frac{N-2}{2} \left(\frac{A}{N-1} \right)^2 = C + \frac{B}{(N-1)^2} - \frac{N-2}{2(N-1)} \times A^2.$$

General expression of the degree of the result of the equations after an unknown quantity of several dimensions is exterminated out of N equations.

C. H. A. P. III.

Of the Resolution of Equations of all Degrees which consist of two Terms only, of those consisting of three Terms that can be reduced by the Method of Equations of the second Degree, to Equations consisting of two Terms: with different Operations relative to those Equations, such as the raising of Quantities to any Power, the Extraction of Roots, the Reduction of radical Quantities, &c. and of the Reduction of Equations by surd Divisors.

HAVING fully treated of Problems of the second Degree, we shall now proceed to explain how to solve Problems of higher Degrees, beginning with those which produce Equations consisting of two Terms; of which Kind is the following, which in its full Extent produces Equations of every Degree consisting of two Terms.

A young Gentleman borrowed a Sum of Money a , for the Loan of which he was to allow Interest payable at stated Terms, but it not suiting his Convenience to pay it at the End of the first Term, the Lender agreed that it should be continually added to the Principle after it came due, so that at the End of each Term, the Sum or Amount should become a new Principle for the succeeding Term, until the Whole should be discharged at one Payment. Now after a certain Time t , the Debt amounted to a Sum r . The Rate of Interest allowed is required?

Problem
which in its
full extent
produces e-
quations of
every degree
consisting of
two terms.

Let the arbitrary Number (but commonly 100) that the Rate of Interest is computed from, be expressed by d , and the Interest of this Number for one Term by i . Now the Principal of the first Term being a , its Interest will be $\frac{ai}{d}$; to which a or $\frac{ad}{d}$ being added, the Sum due at the End of the first Term will be $\frac{ad+ai}{d} = a \times \frac{d+i}{d}$,

the Principal of the second Term being $\frac{ad+ai}{d}$, its Interest will be

$\frac{a(d+i)i}{d}$, to which the Principal reduced to the same Denominator, d^2 being added, the Sum due at the End of the second Term will be $\frac{ad^2+2aid+ai^2}{d^2} = a \times \left(\frac{d+i}{d}\right)^2$. In like Manner it will

appear that the Sum due at the End of the third Term will be $\frac{ad^3+3aid^2+3ai^2d+ai^3}{d^3} = a \times \left(\frac{d+i}{d}\right)^3$. Without pro-

ceeding any further it is easy to perceive that those different Results form a Series of Quantities which increase by one common Multiplier, or in

the Language of the Analysts, a geometrical Progression; the first Term being a , and the common Multiplier $\frac{d+i}{d}$, which for brevity Sake I shall denote by x . The Term of the Progression where x is raised to the Power whose Exponent is 1, expresses the Sum due at the End of the first Term, that where x is raised to the Power whose Exponent is 2, the Sum due at the End of the second Term, and in general the Term of the Progression where x is raised to the Power whose Exponent is t , expresses the Sum r , due at the Expiration of the Time t ; whence we have $r = a x^t$, which Equation being solved, the Value of x , and consequently the Rate of Interest $i = x d - d$ will be obtained.

II.

Equations of the third degree consisting of two terms.

The figure 3 is put over the character $\sqrt{}$ to denote the cube root.

Let us first suppose the Equation to be of the third Degree, that is, $a x^3 = r$, to solve this Equation, it is easy to perceive that it suffices to take away from x^3 its Coefficient, and extract the cube Root of the two Members. The Character employed by the Analysts to express the cube Root, is the same as for the square Root, with 3 put over it.

Thus to express the Value of x deduced from the Equation $a x^3 = r$, or $x^3 = \frac{r}{a}$, they write down $x = \sqrt[3]{\frac{r}{a}}$. If, for Example, the Sum lent a be £1000 and remained unpaid 3 Years amounted to £1331, then the Amount of £1 in one Year, or $x = \sqrt[3]{1,331}$.

III.

A radical cube can have but one sign prefixed to it.

It is to be observed that we are not at Liberty to prefix the Sign $+$ or $-$ to the radical Sign of the cube Root, as to that of the square Root, but that the Sign of the cube Root is the same as the Sign of the Quantity whose Root is required, because the Cube of a positive Quantity is positive, and that of a negative Quantity negative. However, though a Cube has only one Root, and this Root has only one Sign, we are not to conclude that an Equation as $a x^3 = r$, will give only one Value for x . That this may appear, let the Equation $a x^3 = r$ be reduced to this Form $x^3 - \frac{r}{a} = 0$, and its Root $x = \sqrt[3]{\frac{r}{a}}$, also to this Form $x - \sqrt[3]{\frac{r}{a}} = 0$, then dividing $x^3 - \frac{r}{a}$ by $x - \sqrt[3]{\frac{r}{a}}$, the Quotient will be an Equation of the second Degree, which will contain the other two Roots.

To perform this Operation with less Trouble, I put $\frac{r}{a} = c^3$, whereby the preceding Equations are transformed into $x^3 - c^3 = 0$, and $x - c = 0$, dividing the first by the second, the Quotient will be found to be $x^2 + c x + c^2 = 0$, whose two Roots are expressed by $x = -\frac{1}{2}c \pm \sqrt{-\frac{3}{4}c^2}$, and will become the second and third Values

of x in the Equation $x^3 = \frac{b}{a}$ when for c is substituted its Value $\sqrt[3]{\frac{b}{a}}$.

IV.

The Substitution being made, the two Values of x will be found to be $-\frac{1}{2}\sqrt[3]{\frac{b}{a}} \pm \sqrt{\left(-\frac{3}{4}\sqrt[3]{\frac{bb}{aa}}\right)}$ for it is easy to per-

ceive that the Square of $\sqrt[3]{\frac{b}{a}}$ that is, the product of $\sqrt[3]{\frac{b}{a}}$ into

$\sqrt[3]{\frac{b}{a}}$ should be $\sqrt[3]{\frac{bb}{aa}}$, and that in general the Multiplication of How cubi-
cube Roots, as that of square Roots, is performed by multiplying first the cal radicals
Quantities which are under the radical Sign, and afterwards prefixing this are multi-
Sign to their Product. plied into
one another

V.

The three Roots of the proposed Equation $ax^3 = b$, are therefore
 $x = \sqrt[3]{\frac{b}{a}}$, $x = -\frac{1}{2}\sqrt[3]{\frac{b}{a}} + \sqrt{\left(-\frac{3}{4}\sqrt[3]{\frac{bb}{aa}}\right)}$, $x =$
 $-\frac{1}{2}\sqrt[3]{\frac{b}{a}} - \sqrt{\left(-\frac{3}{4}\sqrt[3]{\frac{bb}{aa}}\right)}$ the first real, and the two
others imaginary, which however may be said to resolve the proposed
Equation.

Roots of an
equation of
the third de-
gree consist-
ing of two
terms.

VI.

In like Manner an Equation consisting of two Terms of any Degree may
be solved, by employing a radical having for Exponent the Exponent of
the unknown Quantity in the proposed Equation. For Example, the

Equation $ax^t = r$ or $x^t = \frac{r}{a}$ being proposed, there would result

$$x = \sqrt[t]{\frac{r}{a}}.$$

Resolution
of equations
consisting of
two terms of
every de-
gree.

If t is an odd Number, this Quantity is necessarily negative or positive,
according as $\frac{r}{a}$ is negative or positive: if t is an even Number the Root will

have as the square Root the Sign $\pm$, and will be reel when $\frac{r}{a}$ is posi-

tive, but when $\frac{r}{a}$ is negative (t being an even Number) the two Roots

expressed by $\pm \sqrt[t]{\frac{r}{a}}$ will be both imaginary, whence all the Equa-

tions expressed in general by $x^t = \frac{r}{a}$ can have no more than two reel

Roots, the other Roots being necessarily imaginary, consequently the Pro-
blems which produce those Equations can have but two Solutions.

Those equa-
tions can on-
ly have two
real roots.

Example.

For Example, if the Debt was £10000. and remained unpaid for four Years amounted to £20736. the Equation to be solved would be $x^4 = 2,0736$, the two real Roots of which are 1,2 and $-1,2$, the two imaginary ones $+\sqrt{-144}$ and $-\sqrt{-144}$; however only one solves the Question as it is proposed, and gives 20 *per Cent.* for the Rate of Interest required.

Another example.

If the Debt was £100000. and remained unpaid for five Years amounted to £161051. the Equation to be solved in this Case would be $x^5 = 1,61051$, which has only one real Root $+1,1$, the other Roots are found by resolving the Equation $x^4 + 1,1x^3 + 1,21x^2 + 1,331x + 1,4641 = 0$ which arises by dividing the Equation $x^5 - 1,61051 = 0$ by $x - 1,1 = 0$; it is easy to perceive without resolving this Equation that its Roots are all imaginary, for if any of them was real they would resolve also the Equation $x^5 = 1,61051$, consequently some other Number besides 1,1 raised to the fifth Power would give 1,61051.

VII.

Observations on the involution of quantities

To compleat the Resolution of Equations consisting of two Terms, it remains to shew how the radical Quantities they produce are reduced to their lowest Terms, or how their Roots are extracted when they are perfect Powers.

To enable the young Analysts to distinguish those Cases it will be necessary to make some Observations on the Converse of this Operation, that is, on the raising Quantities to any Power.

If a Quantity as $a b c d$ was proposed to be raised to a Power m , it is easy to perceive that this Power would be $a^m b^m c^m d^m$.

If it was proposed to raise to any Power a Quantity as $\frac{a b c}{d e}$ having a Divisor, it is manifest that the Divisor and Dividend must be raised to the proposed Power, and if the Quantities had Coefficients they must likewise be raised to the same Power also; if the Factors of the given Quantity are raised to any Powers, they will then be raised to a new Power, whose Exponent will be the Product of the Exponent they had already into that of the Power to which they are proposed to be raised, thus $\frac{2 a^2 b^3}{3 c^4}$ raised to the

Rule for involving single quantities.

Power 3 will give $\frac{8 a^6 b^9}{27 c^{12}}$; $\frac{a^m b^r}{c^n}$ raised to the Power q will give $\frac{a^{mq} b^{rq}}{c^{nq}}$; all which is very easy to conceive, and is a necessary Consequence of this Principle: that a Quantity raised to any Power is the Result of the Multiplication of this Quantity into itself as many Times less one, as there are Units in the Exponent of the Power.

Example.

If the Sum of £1000. was lent at 6 *per Cent. per Annum*, and remained unpaid 3 Years, and it were required to find the Sum due at the Ex-

piration of that Time, in this Case $a = 1000$, $d = 100$, $i = 6$, $t = 3$,

$$\frac{d+i}{d} = x = \frac{106}{100} = \frac{53}{50}; \text{ consequently the Sum due } r = ax^t$$

$$= 1000 \times \left(\frac{53}{50}\right)^3 = 1000 \times \frac{148877}{125000} = \frac{148877}{125} = £1191. \frac{2}{125}.$$

VIII.

It is now easy to perceive that the Converse of this Operation, that is the Extraction of Roots is performed by dividing the Exponents of the Parts or Factors of the Quantity by the Number that denominates the Root required, whether those Factors are in the Numerator or in the Denominator. Let it be proposed for Example to extract the cube Root of $\frac{a^3 b^6}{c^9}$, I divide the Exponents 3, 6, 9 by 3, and assume the Quotients

Evolution of single quantities.

1, 2, 3 for the new Exponents of the same Letters, whence $\frac{a b^2}{c^3}$ will be the cube Root sought.

If the Quantity had a Coefficient its cube Root is also to be extracted; thus the cube Root of $\frac{8 a^3 b^6}{c^9}$ will be $\frac{2 a b^2}{c^3}$.

In like Manner the Root of the fourth Power extracted out of $\frac{16 a^4 b^8}{d^4 c^{12}}$ will be $\frac{2 a b^2}{d c^3}$.

In general, the Root n of the Quantity $\frac{a^m b^r}{c^q}$ will be expressed by

$$\frac{a^{\frac{m}{n}} b^{\frac{r}{n}}}{c^{\frac{q}{n}}}$$

let the particular Values of m, r, q, n be what they will,

whence in the Place of any Quantity that has a fractional Exponent we may substitute the Root of a Quantity whose Exponent is an Integer, by placing above the radical Sign the Denominator of this Fraction, and placing within the radical Sign the Quantity having for Exponent the Numerator of this Fraction, that is $\sqrt[n]{\frac{a^m b^r}{c^q}} = \frac{a^{\frac{m}{n}} b^{\frac{r}{n}}}{c^{\frac{q}{n}}}$.

What fractional powers denote.

If for Example the Sum of £1000. was lent at $7\frac{1}{2}$ per Cent. per Annum and remained unpaid one Year, two Months and thirteen Days, and the Sum due at the Expiration of that Time be required. Example.

$$\text{In this Case } a = 1000, d = 100, i = 7\frac{1}{2}, \frac{d+i}{d} = x = \frac{107\frac{1}{2}}{100}$$

$$= \frac{215}{200} = \frac{43}{40}, \text{ and } \frac{438}{365} = \frac{6}{5} \text{ Years, consequently the Sum}$$

$$\text{due } r = a x^t = 1000 \times \left[\frac{43}{40} \right]^5 = 1000 \sqrt[5]{\frac{43}{40}}$$

IX.

If it was required to find what principal Sum a , let out at a certain Rate of Interest i , would amount to a Sum r at the Expiration of a certain Time t , then $a = \frac{r}{x^t} = r x^{-t}$, those two Expressions being equal, for if they be multiplied by the same Power of x there will result the same Product, for Example, $x^{2t} \times r x^{-t} = r x^t = \frac{x^{2t} \times r}{x^t}$, and in general any Quantity placed in the Denominator of a Fraction may be transposed to the Numerator if the Sign of its Exponent be changed.

What negative powers denote.

Example

what the power denotes.

To apply this general Solution to an Example, let it be required to find what principal Sum of Money lent out at 6 per Cent per Annum would amount to £385 13s. 7½d. at the Expiration of 7 Years, here $r = 385.6811$ $x = 1.06$ and $t = 7$, consequently $a = \frac{r}{x^t} = £256.10s.$ When $t = 0$ then $\frac{d+i}{d}$ or $x = 1$ is equal Unity, consequently $x^0 = 1$.

X.

As the foregoing Reductions and Transformations are necessary in an infinite Number of Operations, to render the Application of them easy to Beginners, here follow some Examples.

Examples of the foregoing Reductions and Transformations.

$$\begin{aligned} \frac{a^2 b^{-3}}{c a^{-1}} &= a^3 b^{-3} c^{-1} = \frac{a^3}{b^3 c}, \quad \sqrt{\frac{a^{-2} b^2}{a b^{-1}}} = \sqrt{a^{-3} b^3} \\ &= \sqrt{\frac{b^3}{a^3}} = \frac{b^{\frac{3}{2}}}{a^{\frac{3}{2}}}, \quad \frac{a^{\frac{1}{3}} \sqrt{b^{-3}}}{a^{\frac{1}{3}} b^{\frac{3}{2}}} = \frac{a^{\frac{1}{3}} b^{-\frac{3}{2}}}{a^{\frac{1}{3}} b^{\frac{3}{2}}} = \frac{1}{b^3} \\ &= a^{-\frac{1}{3}} b^{-\frac{3}{2}} = \frac{1}{a^{\frac{1}{3}} b^{\frac{3}{2}}}, \quad \frac{\sqrt[3]{\frac{a^3 c^5}{b}} \times \sqrt[4]{a c^3}}{\sqrt[5]{\frac{a b^2}{a c^3}}} \\ &= \frac{a^{\frac{1}{3}} b^{-\frac{1}{3}} c^{\frac{5}{3}} \times a^{\frac{1}{4}} c^{\frac{3}{4}}}{a^{\frac{1}{5}} b^{\frac{2}{5}} c^{-\frac{3}{5}}} = \frac{a^{\frac{1}{3} + \frac{1}{4}} b^{-\frac{1}{3}} c^{\frac{5}{3} + \frac{3}{4}}}{a^{\frac{1}{5}} b^{\frac{2}{5}} c^{-\frac{3}{5}}} = \frac{a^{\frac{7}{12}} b^{-\frac{1}{3}} c^{\frac{35}{12}}}{a^{\frac{1}{5}} b^{\frac{2}{5}} c^{-\frac{3}{5}}} \\ &= \frac{a^{\frac{7}{12}} b^{-\frac{1}{3}} c^{\frac{35}{12}}}{a^{\frac{1}{5}} b^{\frac{2}{5}} c^{-\frac{3}{5}}} = \frac{a^{\frac{7}{12}} b^{-\frac{1}{3}} c^{\frac{35}{12}}}{a^{\frac{1}{5}} b^{\frac{2}{5}} c^{-\frac{3}{5}}} = \frac{a^{\frac{7}{12}} b^{-\frac{1}{3}} c^{\frac{35}{12}}}{a^{\frac{1}{5}} b^{\frac{2}{5}} c^{-\frac{3}{5}}} \end{aligned}$$

XI.

When there is only a Part or Factor of the proposed Quantity whose Root can be extracted, the Root of this Part is placed before the Sign and the other Factors under the Sign, let it be proposed, for Example, to extract the Root of the fifth Power out of $\frac{32 a^{10} b^8}{486 c^7}$, which is composed of the Product of $\frac{32 a^{10} b^5}{243 c^5}$ into $\frac{b^3}{2 c^2}$, the first Factor being the fifth Power of $\frac{2 a^2 b}{3 c}$ and the other having no Root of the fifth Power I set down $\frac{2 a^2 b}{3 c} \sqrt[5]{\frac{b^3}{2 c^2}}$ for the Root required.

Extraction
of the roots
of imperfect
powers.

In like Manner the cube Root of $\frac{8 a^3 b + 16 a^3 c}{54 d}$ will be $\frac{2}{3} a \sqrt[3]{\frac{b + 2 c}{2 d}}$, because the Quantity $\frac{8 a^3 b + 16 a^3 c}{54 d}$ is the Product of $\frac{8 a^3}{27}$ into $\frac{b + 2 c}{2 d}$, the first of which Quantities is a perfect Cube, that of $\frac{2}{3} a$, and the second has no cube Root. In like Manner $\sqrt[5]{\frac{32 a^9 + 128 a^6 b^3 - 160 a^5 b^4}{3 b^6}} = \frac{2 a}{b} \sqrt[5]{\frac{a^4 + 4 a b^3 - 5 b^4}{3 b}}$.

XII.

When the Quantity whose Root is proposed to be extracted is composed as the foregoing ones of several Terms, and after having separated from those Terms such Quantities as they all include that are perfect Powers of the same Name with the Root, it is suspected that the Remainder is a perfect Power of some commensurable Quantity composed of several Terms; the Operation to be performed to discover when this happens is somewhat more difficult, to find the Method to be pursued in this Operation, we shall make some Observations on the Converse, that is, on the raising of compound Quantities to any Power, from whence we will derive Rules for extracting the Roots of those kind of Quantities beginning by the cube Root.

XIII.

Let the simplest compound Quantity $u + z$ be assumed, and let it be raised first to the Cube, or third Power. First by multiplying $u + z$ into itself we shall have $u u + z u z + z z$, which is the Square or second Power of $u + z$, and this again multiplied by $u + z$ gives $u^3 + 3 u u z + 3 u z z + z^3$ or the Cube of $u + z$, whence it appears, that a Quantity consisting of two Parts when raised to the Cube gives, 1°. The Cube of the first Part, 2°. The Triple of the Square of this first Part multiplied into the second Part, 3°. The Triple of the first

In what the
cube of a bi-
nomial con-
sists.

Part multiplied into the Square of the second, 4°. The Cube of the second.

XIV.

Method of
extracting
the cube
root of com-
pound quan-
tities.

The cube Root therefore of any compound Quantity will be discovered after this Manner, first look out for a Term which is a Cube and this Cube corresponding to u^3 place its Root in the Quotient, which will correspond to u , then subtract its Cube from the proposed Quantity, and divide the Remainder by the Triple of the Square of this Root, the Quotient will be the second Member of the Root, and will correspond to z ; then multiply this Term by the Quantity corresponding to $3uu + 3uz + zz$, that is, by the Triple of the Square of the first Member, together with the Triple of the Product of the first Quantity into the second, and the Square of the second, subtract the Product from the aforesaid Remainder, and if nothing remains then the cube Root of the Binomial corresponding to $u + z$ is obtained: If there still remain several Terms, to discover whether the proposed Quantity is not the Cube of a Trinomial, those two Terms must be employed in the same Manner as the first was, to find the second.

XV.

Some Examples will illustrate this Method. Let the cube Root of the Quantity $8y^6 + 60b^2y^4 + 150b^4y^2 + 125b^6$ be required.

$$\begin{array}{r} 8y^6 + 60b^2y^4 + 150b^4y^2 + 125b^6 \\ - 8y^6 \end{array} \left\{ \begin{array}{l} 12y^4 + 30b^2y^2 + 25b^4 \\ 2y^2 + 5b^2 \end{array} \right.$$

$$\begin{array}{r} 60b^2y^4 + 150b^4y^2 + 125b^6 \\ - 60b^2y^4 - 150b^4y^2 - 125b^6 \\ \hline 0 \quad 0 \quad 0 \end{array}$$

I extract the cube Root of the first Term $8y^6$, and set down this Root $2y^2$ beside the proposed Quantity, and then write down the Cube of $2y^2$ under this first Term, observing to change its Sign, that is, prefixing to it the Sign —, the Subtraction or Reduction being performed, I set down the Remainder $60b^2y^4 + 150b^4y^2 + 125b^6$, and place above $2y^2$ the Triple of its Square, that is, $12y^4$, I then divide the first Term $60b^2y^4$ by $12y^4$, and place the Quotient $5b^2$ beside $2y^2$, I then add to $12y^4$ the Product $30b^2y^2$ of the Triple of $2y^2$ into $5b^2$, and I add to those two first Terms $25b^4$ the Square of $5b^2$, I then multiply $5b^2$ by those three Terms, and I set down their Products with their Signs changed under the Quantity $60b^2y^4 + 150b^4y^2 + 125b^6$, and after the Reduction there being no Remainder left, I conclude that the proposed Quantity is a perfect Cube, and that its Root is $2y^2 + 5b^2$.

XVI.

$$\begin{array}{r}
 x^6 + 6bx^5 + 21b^2x^4 + 44b^3x^3 + 63b^4x^2 + 54b^5x + 27b^6 \\
 -x^6 \\
 \hline
 6bx^5 + 21b^2x^4 + 44b^3x^3 \\
 -6bx^5 - 12b^2x^4 - 8b^3x^3 \\
 \hline
 9b^2x^4 + 36b^3x^3 + 63b^4x^2 + 54b^5x + 27b^6 \\
 -9b^2x^4 - 36b^3x^3 - 63b^4x^2 - 54b^5x - 27b^6 \\
 \hline
 0 \quad 0 \quad 0 \quad 0 \quad 0
 \end{array}$$

$\left\{ \begin{array}{l} 3x^4 + 12bx^3 + 21b^2x^2 + 18b^3x + 9b^4 \\ 3x^4 + 6bx^3 + 4b^2x^2 \\ x^2 + 2bx + 3b^2 \end{array} \right.$

Second example.

If the cube Root of $x^6 + 6bx^5 + 21b^2x^4 + 44b^3x^3 + 63b^4x^2 + 54b^5x + 27b^6$ was required, which it is easy to perceive should consist of more than two Terms, by an Operation similar to that employed in the foregoing Example, the two first Terms of the Root will be found to be $x^2 + 2bx$; but as there is a Remainder $9b^2x^4 + 36b^3x^3 + 63b^4x^2 + 54b^5x + 27b^6$, I divide the first Term $9b^2x^4$ of this Remainder by $3x^4$ triple of the Square of x^2 , because this Term is the first of those which arises by tripling the Square of the Quantity $xx + 2bx$ which actually corresponds to the first Part (called μ Art. XIII.) of the cube Root sought, having divided $9b^2x^4$ by $3x^4$, I set down the Quotient $3b^2$ beside $xx + 2bx$, I then add together the Triple of the Square of $xx + 2bx$, the Triple of the Product of $xx + 2bx$ into $3b^2$ and the Square of $3b^2$ and there results $3x^4 + 12bx^3 + 21b^2x^2 + 18b^3x + 9b^4$, which I multiply by $3b^2$, and I set down the Terms with their Signs changed under the Quantity $9b^2x^4 + 36b^3x^3 + 63b^4x^2 + 54b^5x + 27b^6$, and as after the Reduction there is no Remainder left I conclude that $x^2 + 2bx + b^2$ is the cube Root of the proposed Quantity.

XVII.

The cube Root of any Number may be found after the same Manner, if it be a Number under 1000 its nearest cube Root is found by the following Table.

Cubes	1	8	27	64	125	216	343	512	729	1000
Roots	1	2	3	4	5	6	7	8	9	10

Method of extracting the cube root of numbers.

where it is easy to observe that the Cube of a simple Number cannot consist of more than three Places of Figures, since the Cube of 10 the least Number consisting of two Places is 1000, the least Number consisting of four Places of Figures, that the Cube of a Number consisting of two Places of Figures cannot consist of more than six Places, since the Cube of 100 the least Number consisting of three Places of Figures is 1000000, the least Number consisting of seven Places: Proceeding in this Manner it will appear that the Cube of any Number cannot consist

of more than the Triple of its Places, and in general, *that the Power p of a Number consisting of n Places cannot consist of more than p n Places*; from whence it follows, that a Number consisting of less than four Places of Figures can have but one Figure in its Root, that a Number consisting of more than four but less than seven Places can have but two Figures in its Root, that a Number consisting of more than seven but less than ten Places of Figures can have but three Places of Figures in its Root and so on, taking for Limits the Numbers 1, 4, 7, 10, 13, &c. whose common Difference is 3, consequently the Number of Places which the cube Root of any Number consists of is discoverable by Inspection: How the Numbers corresponding to those Places are found we shall now proceed to explain.

XVIII.

The foregoing method explained by an example.

Let it be proposed to extract the cube Root of 5,305,472 as it consists of seven Places of Figures, I conclude that the Root will consist of three, to determine the Numbers corresponding to those Places, I express them by x , y & z , consequently $x^3 + 3x^2y + 3xy^2 + y^3 + 3[x^2 + 2xy + y^2]z + 3[a + y]z^2 + z^3 = 5,305,472$: Hence the Difficulty is reduced to find the Terms of this Quantity in the given Number. It has been proved that the Product of two Numbers multiplied into one another has as many Places of Figures to the right Hand of it as there are to the right Hand both of the Multiplicand and Multiplier, consequently the Product of three Numbers multiplied into one another will always have as many Places of Figures to the right Hand of it, as there are to the right Hand of the three Numbers multiplied together. Now the first Number x being hundreds its Cube x^3 will have six Places of Figures to the right Hand of it, whence separating by a Line the six Figures 305472 to the right Hand of the proposed Number, The Cube of the unknown Number x of Hundreds of the cube Root will be found in the Part 5 to the left Hand of the separating Line, and will be the greatest Cube contained in this Part 5, but 1 is the greatest Cube it contains, consequently 1 the cube Root of 1 is the Number of Hundreds of the Root which I therefore set down in the Quotient and deducting 1 from 5 the proposed Number is reduced to 4305472.

$$\begin{array}{r}
 5,305,472(174,41, \&c. \\
 \underline{1} \\
 4305 \\
 \underline{300} \\
 2100 \\
 \underline{1470} \\
 343 \\
 \underline{3913} \\
 392472 \\
 \underline{86700} \\
 346800 \\
 \underline{8160} \\
 64 \\
 \underline{355024} \\
 37448000 \\
 \underline{9082800} \\
 36331200 \\
 \underline{83520} \\
 64 \\
 \underline{36414784} \\
 1033216000 \\
 \underline{912460800} \\
 912460800 \\
 \underline{52320} \\
 1 \\
 \underline{912513121} \\
 120702879, \&c.
 \end{array}$$

The Square of a Number of Hundreds which has four Places of Figures to the right Hand of it, being multiplied by a Number of Tens which has one Place of Figures to the right Hand of it, the Product will have five Places of Figures to the right Hand of it: wherefore the Term $3x^2y$, will be contained in the Part 43 which has five Figures to the right Hand of it. But the Product arising from the Multiplication of three times the Square of the Number of Hundreds multiplied into a Number of Tens, being divided by three Times the Square of the Number of Hundreds, will manifestly give the Number of Tens, whence the Number of Tens of the Root will be found by dividing 43 by 3, triple the Square of the Number 1 Hundred of the Root; to determine the Part of this Quotient to be employed, I observe that $3x^2y + 3xy^2 + y^3$, or three Times the Square of the Number of Hundreds of the Root multiplied into the Number of Tens, together with three Times the Square of the Number of Tens into the Number of Hundreds, and the Cube of the Number of Tens should not exceed 4305, since the Cube of the Number of Tens should have three Places of Figures to the right Hand of it: dividing therefore 43 by 3 or 4305 by 300, instead of the whole Quotient 14 resulting from this Division I find only 7 to answer the above Condition; I therefore set down 7 in the Root to the right Hand of the 1 Hundred, and deduct $300 \times 7 + 7 \times 7 \times 10 \times 3 + 7 \times 7 \times 7$ or $2100 + 1470 + 343$ or 3913 from 4305, and to the Remainder 392 I annex the remaining Figures of the proposed Number, and there results 392472, which should contain the Terms $3(x^2 + 2xy + y^2)z + 3(x+y)z^2 + z^3$, or three Times the Square of the 17 Tens multiplied by the unknown Number of Units, together with three Times the Square of the Number of Units, multiplied into the Number of Tens, and the Cube of the Number of Units: Now because three Times the Square of the 17 Tens multiplied into the Number of Units, should have two Figures to the right Hand, it will be contained in the Part 3924, which has two Figures to the right Hand of it, whence dividing 3924 by 867 three Times the Square of 17, or dividing 392472 by 86700 the Quotient 4 will be the Number of Units of the Root, which I set down to the right Hand of the 17 Tens already found, and deduct $3 \times 170 \times 170 \times 4 + 3 \times 170 \times 16 + 64$, or $346800 + 8160 + 64$, that is, 355024 from 392472 and there remains 37448, whence 174 is the cube Root required of the greatest Cube contained in the proposed Number 5305472, which exceeds this Cube by 37448.

XIX.

When a Number is not an exact Cube, as the foregoing one, we may however approximate to its cube Root to any Degree of Exactness; for Example, to extract the cube Root of 5305472, instead of 5305472

3 R

Method of approximating to the cube root of numbers.

it is easy to perceive that we may take $\frac{5305472000000}{1000000}$, which is equal to it; observing that the Denominator should be a cube Number, then extracting the cube Root of the Numerator which may be found true to an Unit, and dividing this Root by 100 the cube Root of the Denominator; the cube Root of $\frac{5305472000000}{1000000}$ or of 5305472 will be obtained true to $\frac{1}{100}$: If we had wrote $\frac{5305472000000000}{1000000000}$, it easy to perceive that the Root would be found true to $\frac{1}{1000}$, and so on, whence in extracting the cube Root after the proposed Number is gone through, if there is a Remainder, the Operation being continued by adding Periods of Cyphers to that Remainder, the true Root will be obtained in Decimals to any Degree of Exactness, as in the Example.

XX.

Process of
the extracti-
on of the
cube root of
numbers.

It is easy to perceive, that the cube Root of any Number may be discovered by the same Method of reasoning employed in the foregoing Example, the Process of which may be expressed thus; point every third Figure beginning with the Units Place, and by these Points the Number will be distinguished into as many Periods as there are Figures in the Root; then find by the Table of Cubes the greatest Cube in the first pointed Period of the proposed Number, the cube Root of which place in the Quotient to the right Hand, and subtract the Cube thereof from the first pointed Period, bringing down to the Remainder the second Period; then divide this Resolvend by thrice the Square of the first Quotient Figure, considered as expressing a Number of Tens, whereby the second Quotient Figure of the required Root will be found; continue this Method till all the Periods are brought down, and then, if the proposed Number be a perfect Cube, there will be no Remainder: but if something should remain, annex Cyphers three at a Time, and carry on the Extraction decimally to any proposed Degree of Exactness.

XXI.

As by raising the Binomial $u + z$ to the third Power Rules are obtained for extracting the cube Root of Quantities; in like Manner, by raising it to the fourth, fifth, &c. Powers, Rules may be derived for extracting the Roots of the fourth, fifth, &c. Powers of Quantities: But to comprehend under one general Rule, the Extraction of Roots, the Analysts have sought a general Expression for the Power m of $u + z$.

Investigati-
on of Sir Isaac
Newton's
theorem for
raising a bi-
nomial to
any power.

To find this general Value of $(u + z)^m$, or of $u + z$ multiplied into itself as often as there are Units less one in m ; let us first examine the Product arising from the Multiplication of several binomial Factors $x + a, x + b, x + c, x + d, \&c.$ and endeavour to discover the Law observed in the Formation of all the Terms of this Product.

Multiplying all those Factors into one another there results,
 $x^5 + (a+b+c+d+e)x^4 + (ab+ac+ad+ae+bc+bd+be+cd+ce+de)x^3$
 $+ (abc+abd+abe+acd+ace+ade+bcd+bce+bde+cde)x^2$
 $+ (abcd+abce+abde+acde+bcde)x + abcde.$

From the Inspection of this Product it is plain, 1°. That the first Term is x raised to a Power equal to the Number of Factors.

2°. That the second Term contains x raised to a Power less by an Unit, having for Coefficient the Sum of all the Letters ($a, b, c, d, e,$).

3°. That the third Term is composed of x raised to a Power less by two Units, with a Coefficient equal to the Sum of all the Products that can be made by multiplying any two of the Letters ($a, b, c, d,$ &c.) by one another.

4°. That the 4th Term includes x raised to a Power less by three Units, with a Coefficient equal to the Sum of all the Products that can be made by multiplying into one another any three of the Letters ($a, b, c, d, e,$ &c.) and after the same Manner all the other Terms are formed.

Applying therefore those Observations to the present Case, in which all the Factors are equal, and their Number is expressed in general by m , it will appear that the first Term will be x^m ; that the second Term will be x^{m-1} multiplied by ma , since $b, c,$ &c. are equal to a , and their Number is m ; that the third Term will be x^{m-2} , with a Coefficient equal to a^2 , repeated as often as there are Rectangles $ab, ac, bc,$ &c. in the Coefficient of the third Term of the Product of the Factors $x+a, x+b, x+c,$ &c. the Number of which is supposed to be m ; since all the Products $ab, ac, bc,$ &c. should be all equal each to a^2 , when $b, c,$ &c. are equal to a : that the fourth Term will be x^{m-3} , with a Coefficient equal to a^3 , repeated as often as there are Products $abc, abd, acd, bcd,$ &c. in the Coefficient of the fourth Term of the Product of the Factors $x+a, x+b, x+c, x+d,$ &c. whose Number is m ; and after the same Manner all the other Terms are formed.

The Question is therefore reduced to know how many different Products a Number m of Letters will admit of, when taken two by two, three by three, &c. for supposing that those Numbers were found, and were expressed by $A, B, C, D,$ &c. there would result $x^m + ma x^{m-1} + Aa^2 x^{m-2} + Ba^3 x^{m-3} + Ca^4 x^{m-4} + Da^5 x^{m-5},$ &c. for the Value sought of $(x+a)^m$.

To find first how many different Products a Number m of Letters $a, b, c, d,$ &c. taken two by two will admit of, I observe that when all those Products are formed, there will be twice as many Letters set down as there are Terms.

I observe next, that each of the Letters $a, b, c,$ &c. should be repeated the same Number of Times, and that each Letter, being multiplied by all the rest and not by itself, can not be repeated more than $m-1$ Times;

wherefore the Number of Letters to be set down in forming all those Products should be $m \times (m - 1)$; wherefore the Number of all those Products should be $\frac{m \times (m-1)}{2}$, and this is the Value of A , or of the Coefficient of the third Term of the Formula sought.

To find the Coefficient of the fourth Term, that is, the Number of different Products of three Letters $a b c$, $a b d$, $a c d$, $b c d$, &c. that a Number m of Letters a , b , c , d , &c. will admit of, taken three by three, I observe first that this Number should be the one third of the Letters set down in forming those Products.

I observe next, that each of those Letters should be repeated the same Number of Times, and that this Number is the same as that which expresses how many different Products all the other Letters, taken two by two, will admit of; for it is manifest, that each Letter a for Example, should be joined to all the Products $b c$, $b d$, $c d$, &c. of the other Letters, taken two by two.

The number of Times therefore that each of the Letters a , b , c , d , &c. should be repeated, is the same as that which expresses how many different Products a Number $m - 1$ of Letters b , c , d , &c. taken two by two, will admit of. Now we have seen that when the Number of Letters was m , the Number of their Products, when taken two by two, was expressed by the one Half of the Number m , multiplied by the Number $m - 1$. Wherefore, when the Number of Letters is $m - 1$, we should

take the one Half $\frac{m-1}{2}$ of this Number, and multiply it by $m - 2$, that is, $\frac{(m-1) \times (m-2)}{2}$ expresses the Number of Times that each of the Letters a , b , c , &c. should be repeated in all the Products in Question; and as the Number of those Letters is m , consequently, $\frac{m \times (m-1) \times (m-2)}{2}$ will be the Number of all the Letters set

down; wherefore the Number sought of the Products of three Letters $a b c$, $a b d$, &c. will be $\frac{m \times (m-1) \times (m-2)}{2 \times 3}$, and this is the Value of B , or of the Coefficient of the fourth Term.

To find the Coefficient C of the fifth Term, that is, the Number of Products of four Letters, that a Number m of Letters will admit of; I observe, that this Number should be the one fourth of all the Letters set down in the Products; that each of those Letters should be repeated the same Number of Times, and should be combined with all the Products of three Letters, that a Number $m - 1$ of Letters will admit of: and in fine, that those Products of three Letters, that a Number $m - 1$ of Let-

ters will admit of, should be expressed by $\frac{(m-1) \times (m-2) \times (m-3)}{2 \times 3}$

for the same Reason that $\frac{m \times (m-1) \times (m-2)}{2 \times 3}$ expresses the Number of Products of three Letters, that a Number m Letters will admit of; hence the Number sought of the Products of four Letters $abcd, abce, \&c.$ will be expressed by $\frac{m \times (m-1) \times (m-2) \times (m-3)}{2 \times 3 \times 4}$.

Forming after the same Manner the other Coefficients, and substituting in the foregoing Formula in the Room of $A, B, C, D, E, \&c.$ their Values thus found, there will result at length $x^m + m a x^{m-1}$

$$+ m \times \frac{m-1}{2} a^2 x^{m-2} + \frac{m \times (m-1) \times (m-2)}{2 \times 3} a^3 x^{m-3}$$

$$+ \frac{m \times (m-1) \times (m-2) \times (m-3)}{2 \times 3 \times 4} a^4 x^{m-4}$$

$$+ \frac{m \times (m-1) \times (m-2) \times (m-3) \times (m-4)}{2 \times 3 \times 4 \times 5} a^5 x^{m-5}, \&c.$$

for the Power m of $x + a$.

$$\begin{aligned} \text{In like Manner, the Value of } \overline{u+z^m} \text{ will be } u^m + m z u^{m-1} \\ + \frac{m \times (m-1)}{2} z^2 u^{m-2} + \frac{m \times (m-1) \times (m-2)}{2 \times 3} z^3 u^{m-3} \\ + \frac{m \times (m-1) \times (m-2) \times (m-3)}{2 \times 3 \times 4} z^4 u^{m-4}, \&c. \end{aligned}$$

General formula for raising $u+z$ to the power m

As to the Value of $(u-z)^m$, it is manifest, in order to obtain it, it suffices to make z negative in this Formula, which will transform it into

$$u^m - m z u^{m-1} + \frac{m \times (m-1)}{2} z^2 u^{m-2} + \frac{m \times (m-1) \times (m-2)}{2 \times 3} z^3 u^{m-3}$$

$$+ \frac{m \times (m-1) \times (m-2) \times (m-3)}{2 \times 3 \times 4} z^4 u^{m-4}, \&c.$$

XXII.

To raise any Binomial to a given Power by means of the foregoing Formula, we have no more to do but to substitute in the foregoing Value of $(u+z)^m$ in the Place of u the first Term of the given Binomial, in the Place of z the second, and in the Place of m the Exponent of the Power to which the proposed Binomial is to be raised.

Application of the foregoing formula to an example.

$$\begin{aligned} \text{Let it be proposed, for Example, to raise } 3ac - 2bd \text{ to the fifth Power, I put } 3ac = u, -2bd = z, 5 = m, \text{ and there results} \\ u^m = (3ac)^5 = 243 a^5 c^5, m z u^{m-1} = 5 \times -(2bd) \times (3ac)^4 \\ = -810 a^4 b c^4 d, \frac{m \times (m-1)}{2} z^2 u^{m-2} = 10 \times 4 b b d d \times (3ac)^3 \\ = 1080 a^3 b b c^3 d d, \frac{m \times (m-1) \times (m-2)}{2 \times 3} z^3 u^{m-3} \\ = 10 \times -(2bd)^3 \times (3ac)^2 = -720 a^2 b^3 c^2 d^3 \end{aligned}$$

$$\frac{m \times (m-1) \times (m-2) \times (m-3)}{2 \times 3 \times 4} z^4 u^{m-4} = 5 \times - (2bd)^4 \times 3ac$$

$$= 240 a b^4 c d^4, \frac{m \times (m-1) \times (m-2) \times (m-3) \times (m-4)}{2 \times 3 \times 4 \times 5} z^5 u^{m-5}$$

$$= 1 \times - (2bd)^5 \times (3ac)^0 = - 32 b^5 d^5.$$

As to the other Terms, their Coefficients including among their Factors $m-5$, which is equal to nothing when $m=5$, they should all vanish; whence the Value of $(3ac - 2bd)^5$ will be $243 a^5 c^5 - 810 a^4 b c^4 d + 1080 a^3 b^2 c^3 d^2 - 720 a^2 b^3 c^2 d^3 + 240 a b^4 c d^4 - 32 b^5 d^5$.

XXIII.

How the foregoing formula may be applied to quantities consisting of more than two terms.

If a Quantity consisting of more than two Terms is to be involved, it may be easily effected by the same Method. If it be a Trinomial, for Example, calling u the first Term of this Trinomial, z the Sum of the two other Terms, the Difficulty of raising a Trinomial to any Power will be reduced to that of raising a Binomial to the same Power, since each of the Terms $mu^{m-1}z$, $\frac{m \times (m-1)}{2} u^{m-2} z^2$, &c. does not include any Quantity to be involved more composed than the Binomial.

XXIV.

To shew by an Example how the foregoing Formula is to be employed for raising to any Power a Quantity consisting of more than two Terms, let it be proposed to raise $a + 2b - c$ to the fourth Power. I put $m=4$, $u=a$, $z=2b-c$, and substituting those Values in the Formula, there results $u^m = a^4$, $mz u^{m-1} = 4 \times (2b-c) \times a^3 = 8a^3b - 4a^3c$.

Example.

$$\frac{m \times (m-1)}{2} z^2 u^{m-2} = 6 \times (2b-c)^2 \times a^2 = 24a^2b^2 - 24a^2bc$$

$$+ 6a^2c^2, \frac{m \times (m-1) \times (m-2)}{2 \times 3} z^3 u^{m-3} = 4 \times (2b-c)^3 \times a$$

$$= 32a^3b^3 - 48a^3b^2c + 24a^3b^2c^2 - 4a^3c^3,$$

$$\frac{m \times (m-1) \times (m-2) \times (m-3)}{2 \times 3 \times 4} z^4 u^{m-4} = (2b-c)^4$$

$$= 16b^4 - 4 \times (2b)^3 \times c + \frac{4 \times 3}{2} \times (2b)^2 \times c^2 - \frac{4 \times 3 \times 2}{2 \times 3} \times 2b \times c^3$$

$$+ \frac{4 \times 3 \times 2 \times 1}{2 \times 3 \times 4} c^4 = 16b^4 - 32b^3c + 24b^2c^2 - 8bc^3 + c^4,$$

and consequently, $(a + 2b - c)^4 = a^4 + 8a^3b - 4a^3c + 24a^2b^2 - 24a^2bc + 6a^2c^2 + 32a^3b^3 - 48a^3b^2c + 24a^3b^2c^2 - 4a^3c^3 + 16b^4 - 32b^3c + 24b^2c^2 - 8bc^3 + c^4$.

XXV.

To find the Roots of Numbers by means of the foregoing Theorem, every fourth, fifth, and in general every m^{th} Figure, beginning from the Units Place, is to be pointed, according as it is the Root of the fourth, fifth, or in general of the m^{th} Power that is required; and if there be

any Decimals annexed to the Number, they are to be pointed after the same Manner, proceeding from the Units Place towards the right Hand, whereby it is easy to perceive, that the Number will be divided into so many Periods as there are Figures in the Root required. Then inquire which is the greatest m Power in the first Period to the Left of the given Number, and the Root of that Power will be the first Figure of the Root required, which call u , and subtract u^m from this first Period. To find the second Figure z of the Root to the Remainder annex the first Figure of the second Period, and this Number will contain the Term $m u^{m-1} z$ of the Binomial $u + z$ raised to the Power m , whence if $m u^{m-1} z$ be divided by $m u^{m-1}$, and after the Division is performed, if what remains of the two first Periods does not exceed $\frac{m \times m - 1}{2} u^{m-2} z^2$

General rule for extracting the roots of numbers derived from the foregoing theorem.

$$+ \frac{m \cdot m - 1 \cdot m - 2}{2 \times 3} u^{m-3} z^3 + \frac{m \cdot m - 1 \cdot m - 2 \cdot m - 3}{2 \times 3 \times 4} u^{m-4} z^4 +, \&c.$$

the Quotient will be the second Figure of the Root. The third Figure will be found by Means of the two first, as the second was found by the first, and afterwards the fourth Figure (if there be a fourth Period) after the same Manner from the three first.

XXVII

Let the fifth Root of 6436343 be required, I first point it 64,36343, the Formula of the fifth Power is $u^5 + 5 u^4 z + 10 u^3 z^2 + 10 u^2 z^3 + 5 u z^4 + z^5$; wherefore 64 is either equal to, or contains u^5 , it is not equal to it, since no Root raised to the fifth Power will give 64; consequently it contains it, but the greatest fifth Power in 64 is 32, whose fifth Root is 2; wherefore $2 = u$. I subtract 32 from 64, and to the Remainder 32 I annex 3, the first Figure of the second Period, and there results 323, now $u^4 = 16$, and $5 u^4 = 80$; I therefore divide 323 by 80, and the Quotient is 4, leaving a Remainder 3, which annexed to the remaining Figures in the proposed Number, there results 36343, which should not exceed $10 u^3 z^2 + 10 u^2 z^3 + 5 u z^4 + z^5$, but $10 u^3 z^2$ is equal to 1280 Thousands, $10 u^2 z^3$ to 2560 Hundreds, which added to the 1280 Thousands already found, makes 15360 Hundreds, and as this Number exceeds 363 Hundreds that remain in the proposed Power, the second Part of the Root 4 has not been rightly determined: Let the second Figure of the Root be 3, then $3 \times 80 = 240$, and $323 - 240 = 83$, wherefore there will remain 836343 which should not exceed $10 u^3 z^2 + 10 u^2 z^3 + 5 u z^4 + z^5$; now $10 u^3 z^2 = 720$ Thousands, and $10 u^2 z^3 = 1080$ Hundreds, which added to 720 Thousands, makes 8280 Hundreds, and $5 u z^4 = 810$ Tens, which added to 8280 Hundreds, makes 83610 Tens; lastly, $z^5 = 243$, which added to the 83610 Tens, makes 836343; wherefore 3 is the second Figure of the Root required.

Application of this method to an example.

To approximate to the n Root of any Number N , let r denote the nearest less Root in Integers, and $r + z$ the true Root: Then will

$$N = r^n + n r^{n-1} z + n \cdot \frac{n-1}{2} \cdot r^{n-2} z^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^{n-3} z^3 \\ + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} r^{n-4} z^4, \&c. \text{ or } - \frac{m}{n r^{n-1}} + z \\ + \frac{n-1}{2} \cdot \frac{1}{r} \cdot z^2 + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{1}{r^2} \cdot z^3 \\ + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{1}{r^3} \cdot z^4, \&c. = 0.$$

General method of approximating to the roots of numbers.

(putting $N - r^n = m$). Multiplying this Equation by $1 + A z$, there will result,

$$- \frac{m}{n r^{n-1}} + \left(1 - \frac{m A}{n r^{n-1}}\right) z + \left(\frac{n-1}{2} \cdot \frac{1}{r} + A\right) z^2 \\ + \left(\frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{1}{r^2} + \frac{n-1}{2} \cdot \frac{A}{r}\right) z^3 \\ + \left(\frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{1}{r^3} + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{A}{r^2}\right) z^4, \&c. = 0$$

in which the third Term will be destroyed by determining A by Means of the Equation $\frac{n-1}{2} \cdot \frac{1}{r} + A = 0$; from which Equation A is found $= - \frac{n-1}{2} \cdot \frac{1}{r}$, and by substituting this Value above, we have

$$- \frac{m}{n r^{n-1}} + \left(1 + \frac{n-1 \cdot m}{2 n r^n}\right) z - \frac{n+1}{4} \cdot \frac{n-1}{3} \cdot \frac{1}{r^2} \cdot z^3 \\ - \frac{n+1 \cdot n-1 \cdot n-2}{4 \times 3 \times 2} \cdot \frac{z^4}{r^3} - \frac{n+1 \cdot n-1 \cdot n-2 \cdot n-3}{5 \times 4 \times 3 \times 2} \cdot \frac{z^5}{r^4}, \&c. = 0$$

the Terms of which Series decreases very swiftly, neglecting, therefore, all the Terms after the second as exceeding small, it will be reduced to

$$- \frac{m}{n r^{n-1}} + \left(1 + \frac{n-1 \cdot m}{2 n r^n}\right) z = 0, \text{ whence } z = \frac{r m}{n r^n + \frac{n-1}{2} \cdot m}.$$

To obtain a more approximate Value of z , it is sufficient to multiply

$$- \frac{m}{n r^{n-1}} + z + \frac{n-1}{2} \cdot \frac{1}{r} \cdot z^2 + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{1}{r^2} z^3 \\ + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{1}{r^3} \cdot z^4, \&c. = 0, \text{ by } 1 + A z + B z^2, \\ \text{whence } - \frac{m}{n r^{n-1}} + \left(1 - \frac{m A}{n r^{n-1}}\right) z + \left(\frac{n-1}{2 r} + A - \frac{m B}{n r^{n-1}}\right) z^2 \\ + \left(\frac{n-1 \cdot n-2}{2 \times 3 \times r^2} + \frac{n-1 \cdot A}{2 \times r} + B\right) z^3 + \left(\frac{n-1 \cdot n-2 \cdot n-3}{2 \times 3 \times 4 \times r^3} \right. \\ \left. + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{A}{r^2} + \frac{n-1}{2} \cdot \frac{B}{r}\right) z^4, \&c. = 0.$$

In which Equation the third and fourth Terms will be destroyed, by determining A and B by Means of the Equations

$$\frac{n-1}{2 \times r} + A - \frac{mB}{nr^{n-1}}, = 0, \quad \frac{n-1}{2} \cdot \frac{n-2}{3 \times r^2} + \frac{n-1}{2} \cdot \frac{A}{r} + B = 0$$

Let the latter of those Equations be multiplied by $\frac{m}{nr^{n-1}}$, and then add the two Products together, so shall

$$\frac{n-1}{2} \cdot \frac{1}{r} + A + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{m}{nr^{n+1}} + \frac{n-1}{2} \cdot \frac{mA}{nr^n} = 0.$$

$$\text{whence is deduced } \frac{\frac{n-1}{2} \cdot \frac{1}{r} + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{m}{nr^{n+1}}}{1 + \frac{n-1}{2} \cdot \frac{m}{nr^n}} = -A.$$

To render this Value of $-A$ more simple, I observe, that if two Quantities be decreased by two other small Quantities nearly in the same Ratio with the two first, that their Difference will still be in the same Ratio with the two first Quantities very near; wherefore, seeing the Numerator of this Fraction is in Proportion to the Denominator nearly as

$$\frac{\frac{n-1}{2} \cdot \frac{1}{r}}{\frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{m}{nr^{n+1}}} : 1, \text{ or as } \frac{\frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{m}{nr^{n+1}}}{\frac{n-2}{3} \cdot \frac{m}{nr^n}},$$

let $\frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{m}{nr^{n+1}}$ be therefore taken from the Numerator, and $\frac{n-2}{3} \cdot \frac{m}{nr^n}$, from the Denominator; by this Means the

$$\text{Fraction itself will be reduced to } \frac{\frac{1}{2} \cdot (n-1) \cdot \frac{1}{r}}{1 + (n+1) \cdot \frac{1}{6} \cdot \frac{m}{nr^n}}$$

$$\text{or to } \frac{\frac{1}{2} \cdot (n-1)}{r + (n+1) \cdot \frac{1}{6} \cdot \frac{m}{nr^{n-1}}}, \text{ whence putting } \frac{N-r^n}{nr^n} = p$$

we will have $z = \frac{p + (n+1) \times \frac{1}{6} p^2}{1 + (2n-1) \times \frac{1}{3} p}$ for an Approximation of the third Degree.

If more Terms of the Series $1 + Az + Bz^2 + Cz^3$, &c. are taken the Root will be found exacter in Proportion; for Example, if four Terms of this Series were employed, there would result for an Approximation of the fourth Degree — $A = \frac{\frac{1}{2} \times (n-1) + (n-1) \times (2n-1) \times \frac{1}{12} p}{1 + \frac{1}{2} n p}$

$$\text{and } z = \frac{p + \frac{1}{2} n p^2}{1 + \frac{2n-1}{2} p + \frac{(2n-1) \times (n-1)}{12} p^2}.$$

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But both those Expressions, in Cases where p is a proper Fraction, will be better adapted to Practice by making $\frac{n r^n}{N - n r^n} = v = \frac{1}{p}$, and substituting $\frac{1}{v}$ for p its Equal; whence after proper Reductions $x = r + \frac{r}{v} \times \frac{6v + n + 1}{6v + 4n - 2}$ nearly; or $x = r + \frac{r \times (2v + n)}{v \times (2v + 2n - 1) + \frac{1}{6} \times (2n - 1) \times (n - 1)}$

XXVIII.

Application of the foregoing approximations to an example.

To shew the Use and great Exactness of those Approximations by an Example, let the Equation $x^3 = 500$ be given, then assuming $r = 8$ we have $z = -\frac{32}{508}$, and consequently $r + z = 8 - \frac{32}{508} = 7.937$ by the first Approximation.

By the second we have $v = \frac{n r^n}{N - r^n} = \frac{3 \times 512}{-12}$, hence by the second Approximation $x = 8 - \frac{1}{16} \times \frac{-768 + 4}{-768 + 10} = 8 - \frac{191}{3032} = 7.93700527$ nearly. And by the third $x = 8 - \frac{8 \times 253}{128 \times 251 + \frac{5}{3}} = 8 - \frac{6072}{96389} = 7.9370052599$.

When a near Value of z is determined, then adding it to r , substitute the Aggregate in the Place of N in the Formulas, and you will by a new Operation obtain a more correct Value of the Root required; and by thus proceeding you may arrive at any Degree of Exactness.

XXIX

Use of the foregoing rules in the solution of problems. First example.

To shew the Use of the foregoing Rules in the Solution of Problems, here follow some Examples.

Suppose that out of a Cask holding c Gallons of Wine when full, a certain Quantity x was drawn and the Cask filled up with Water, and that the same Quantity of the Mixture was afterwards drawn, and supplied by Water several Times, and then it appeared that besides Water there were but r Gallons of Wine left in the Cask: How much Wine was drawn out each Time?

$c - x$ expresses the Quantity of Wine left after the first Drawing; but $c : c - x = x : \frac{(c - x) \times x}{c}$ the Quantity of Wine drawn out at second Drawing, and $c - x - \frac{(c - x) \cdot x}{c} = \frac{(c - x)^2}{c}$ the Quantity of Wine left after the third Drawing:

Also $c : \frac{(c - x)^2}{2} = x : \frac{(c - x)^2 \cdot x}{c^2}$ the Quantity of Wine drawn out at the fourth Drawing, and $\frac{(c - x)^2}{c} - \frac{(c - x)^2 \cdot x}{c \cdot c} = \frac{(c - x)^3}{c \cdot c}$

equal the Quantity of Wine left after the fourth Drawing; then
 $c : \frac{(c-x)^{t-1}}{c^{t-2}} = x : \frac{(c-x)^{t-1} \cdot x}{c^{t-1}}$ equal the Quantity of Wine drawn
 out at the t Drawing, and $\frac{(c-x)^{t-1}}{c^{t-2}} - \frac{(c-x)^{t-1} \cdot x}{c^{t-1}} = \frac{(c-x)^t}{c^{t-1}}$
 the Quantity of Wine left after the t Drawing. Now $\frac{(c-x)^t}{c^{t-1}} = r$

by the Question, or $(c-x)^t = r c^{t-1}$ and $c-x = r c^{t-1}]^{\frac{1}{t}}$ and $c - r c^{t-1}]^{\frac{1}{t}} = x$.

To apply this general Solution to an Example, let the Contents of the Cask be 81 Gallons, the Wine remaining 16 Gallons, the Number of Times Liquor was drawn out 4, then $x = 88 - \sqrt[4]{8503056} = 27$ Gallons.

xxx.

If every Century the Number of Men were doubled, what would be the annual Increase? Second example.

Let the primitive Number of Men be expressed by n , and let their Number be increased annually by their $\frac{1}{x}$ Part.

Their Number at the End of the first Year will be expressed by $n + \frac{n}{x}$, at the End of the second Year by $\frac{n + nx}{x} + \frac{n + nx}{x^2}$, or by $\left(\frac{1+x}{x}\right)^2 n$; at the End of the third Year by $\left(\frac{1+x}{x}\right)^3 n$; and by $\left(\frac{1+x}{x}\right)^{100} n$ at the End of the 100th Year. Now by the Que-

tion $\left(\frac{1+x}{x}\right)^{100} n = 2n$, wherefore $\frac{1+x}{x} = 2^{\frac{1}{100}}$.

Whence $\frac{1+x}{x} = \frac{10070240}{10000000}$, wherefore $x = \frac{10000000}{70240} = 144$ nearly, which shews that the Number of Men should be annually increased by their $\frac{1}{144}$ Part, in order that every Century their Number should be doubled: It is therefore no way surprizing that the World should have been peopled by one Pair.

xxxI.

After the Annalysts had found the general Theorem for the Involution of Binomials, they soon perceived that it might be extended to other Powers besides those whose Exponents are whole and positive Numbers;

having found, for Example, that $a^{\frac{1}{n}}$ might be substituted for $\sqrt[n]{a}$; they concluded, without Doubt, that to extract the Root n of any compound

ELEMENTS OF

Quantity represented by $u + z$; it sufficed to suppose $m = \frac{1}{n}$ in the foregoing Formula, or what amounted to the same thing that

$$u^{\frac{1}{n}} + \frac{1}{n} u^{\frac{1}{n}-1} z + \frac{\frac{1}{n} \times \frac{1}{n} - 1}{2} u^{\frac{1}{n}-2} z^2 + \&c. = (u + z)^{\frac{1}{n}},$$

or the Root n of $u + z$.

In like Manner, that $\frac{1}{(u+z)^n}$, or $(u+z)^{-n} = u^{-n} - n u^{-n-1} z$
 $- \frac{n \times -n - 1}{2} u^{-n-2} z^2 - \&c.$ In one Word the Order and Generality that the Analysts always found in the analitick Operations, easily suggested to them, that though the foregoing Formula had been investigated only in the Supposition that m was a whole and positive Number, however that it might be applied to all the other Values of m .

Let it first be proposed to shew that the Formula in Question serves to raise a Quantity to a Power whose Exponent is a Fraction, or what amounts to the same thing, let it be proposed to prove that

$$(u+z)^{\frac{r}{n}} = u^{\frac{r}{n}} + \frac{r}{n} u^{\frac{r}{n}-1} z + \frac{\frac{r}{n} \times \frac{r}{n} - 1}{2} u^{\frac{r}{n}-2} z^2 + \&c.$$

which by dividing the two Members by $u^{\frac{r}{n}}$ and putting $\frac{z}{u} = p$ becomes

$$(1+p)^{\frac{r}{n}} = 1 + \frac{r}{n} p + \frac{\frac{r}{n} \times \frac{r}{n} - 1}{2} p^2 + \frac{\frac{r}{n} \times \frac{r}{n} - 1 \times \frac{r}{n} - 2}{2 \times 3} p^3 + \&c.$$

To prove this Equation it suffices to shew that its two Members, being raised to the same Power n , there will result equal Quantities, that is, putting

$$s = \frac{r}{n} p + \frac{\frac{r}{n} \times \frac{r}{n} - 1}{2} p^2 + \frac{\frac{r}{n} \times \frac{r}{n} - 1 \times \frac{r}{n} - 2}{2 \times 3} p^3 + \&c.$$

$$(1+p)^r = (1+s)^n.$$

Now as r and n are two whole Numbers, the Members of the foregoing Equation may be raised to the Powers indicated by their Exponents, by Means of the general Formula $(u+z)^m$. Whence

$$1+rp+\frac{r(r-1)}{2} p^2+\frac{r(r-1)(r-2)}{2 \times 3} p^3+\frac{r(r-1)(r-2)(r-3)}{2 \times 3 \times 4} p^4+\&c.$$

$$=1+ns+\frac{n(n-1)}{2} s^2+\frac{n(n-1)(n-2)}{2 \times 3} s^3+\frac{n(n-1)(n-2)(n-3)}{2 \times 3 \times 4} s^4+\&c.$$

The Problem being reduced to prove this Equation, we must find by Means of the Value of s those of $s^2, s^3, s^4, \&c.$ and then multiply the Va-

Value of r by n , that of r^2 by $\frac{n(n-1)}{2}$, that of r^3 by $\frac{n(n-1)(n-2)}{2 \times 3}$, that of r^4 by $\frac{n(n-1)(n-2)(n-3)}{2 \times 3 \times 4}$, &c. to obtain the Values of all the Terms of the second Member of the foregoing Equation, those Values being found, and set down under one another, there results

$$\begin{aligned}
 1 + nr &= 1 + n \cdot \frac{r}{n} + n \cdot \frac{\frac{r}{n} \cdot \frac{r}{n} - 1}{2} p^2 + n \cdot \frac{\frac{r}{n} \cdot \frac{r}{n} - 1 \cdot \frac{r}{n} - 2}{2 \times 3} p^3 + n \cdot \frac{\frac{r}{n} \cdot \frac{r}{n} - 1 \cdot \frac{r}{n} - 2 \cdot \frac{r}{n} - 3}{2 \times 3 \times 4} p^4 +, \&c. \\
 + \frac{n \cdot n - 1}{2} r^2 &= + \frac{n \cdot n - 1}{2} \cdot \frac{r^2}{n^2} p^2 + \frac{n \cdot n - 1}{2} \cdot \frac{r^2}{n^2} \cdot \frac{r}{n} - 1 \cdot p^3 + \frac{n \cdot n - 1}{2} \cdot \frac{r^2}{n^2} \cdot \frac{r}{n} - 1 \cdot \frac{n}{12} p^4 +, \&c. \\
 + \frac{n \cdot n - 1 \cdot n - 2}{2 \times 3} r^3 &= + \frac{n \cdot n - 1 \cdot n - 2}{2 \times 3} \cdot \frac{r^3}{n^3} p^3 + \frac{n \cdot n - 1 \cdot n - 2}{2 \times 3} \cdot \frac{3r^3}{2n^3} \cdot \frac{r}{n} - 1 p^4 +, \&c. \\
 + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{2 \times 3 \times 4} r^4 &= + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{2 \times 3 \times 4} \cdot \frac{r^4}{n^4} p^4 +, \&c. \\
 &+ , \&c.
 \end{aligned}$$

Reducing the Terms of the second Member of this Equation, there results, $1 + nr + \frac{n \cdot n - 1}{2} r^2 + \frac{n \cdot n - 1 \cdot n - 2}{2 \times 3} r^3 + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{2 \times 3 \times 4} r^4 +, \&c.$

$$= 1 + r p + \frac{r \cdot r - 1}{2} p^2 + \frac{r \cdot r - 1 \cdot r - 2}{2 \times 3} p^3 + \frac{r \cdot r - 1 \cdot r - 2 \cdot r - 3}{2 \times 3 \times 4} p^4 +, \&c.$$

which is precisely the foregoing Equation. Wherefore the general Theorem found for the Involution of Binomials will serve also for their Evolution.

The general formula found for the involution of binomials, serves also for their evolution.

XXXI.

Let it now be proposed to prove that the same Formula serves to raise Quantities to negative Powers, either Integral or Fractional, or that

$$(u+z)^{-\frac{r}{n}} = u^{-\frac{r}{n}} - \frac{r}{n} u^{-\frac{r}{n}-1} z + \frac{\frac{r}{n} \cdot \frac{r}{n} + 1}{2} u^{-\frac{r}{n}-2} z^2 + \frac{\frac{r}{n} \cdot \frac{r}{n} + 1 \cdot \frac{r}{n} + 2}{2 \times 3} u^{-\frac{r}{n}-3} z^3 +, \&c.$$

which Equation by multiplying the two Members by $u^{\frac{r}{n}}$, and putting

$p = \frac{z}{u}$ is reduced to

$$(1+p)^{-\frac{r}{n}} = 1 - \frac{r}{n} p + \frac{\frac{r}{n} \cdot \frac{r}{n} + 1}{2} p^2 - \frac{\frac{r}{n} \cdot \frac{r}{n} + 1 \cdot \frac{r}{n} + 2}{2 \times 3} p^3 + \frac{\frac{r}{n} \cdot \frac{r}{n} + 1 \cdot \frac{r}{n} + 2 \cdot \frac{r}{n} + 3}{2 \times 3 \times 4} p^4 +, \&c.$$

It is manifest that instead of the Quantity $(1+p)^{-\frac{r}{n}}$ we may set

down $\frac{1}{(1+p)^{\frac{r}{n}}}$, consequently the foregoing Equation is transformed into

$$\frac{1}{(1+p)^{\frac{r}{n}}} = 1 - \frac{r}{n}p + \frac{\frac{r}{n} \cdot \frac{r}{n} + 1}{2}p^2 - \frac{\frac{r}{n} \cdot \frac{r}{n} + 1 \cdot \frac{r}{n} + 2}{2 \times 3}p^3 +, \&c.$$

But as we can substitute for $(1+p)^{\frac{r}{n}}$ its Value deduced from the Formula of the foregoing Article, it is manifest that it suffices to prove that

$$\begin{aligned} & 1 + \frac{r}{n}p + \frac{\frac{r}{n} \cdot \frac{r}{n} - 1}{2}p^2 + \frac{\frac{r}{n} \cdot \frac{r}{n} - 1 \cdot \frac{r}{n} - 2}{2 \times 3}p^3 + \frac{\frac{r}{n} \cdot \frac{r}{n} - 1 \cdot \frac{r}{n} - 2 \cdot \frac{r}{n} - 3}{2 \times 3 \times 4}p^4 +, \&c. \\ & = 1 - \frac{r}{n}p + \frac{\frac{r}{n} \cdot \frac{r}{n} + 1}{2}p^2 - \frac{\frac{r}{n} \cdot \frac{r}{n} + 1 \cdot \frac{r}{n} + 2}{2 \times 3}p^3 + \frac{\frac{r}{n} \cdot \frac{r}{n} + 1 \cdot \frac{r}{n} + 2 \cdot \frac{r}{n} + 3}{2 \times 3 \times 4}p^4 -, \&c. \end{aligned}$$

Now to prove this Equation, it suffices to multiply the second Member by the Denominator of the first, and to make it appear that the Product that results is Unity, the Numerator of the first Member: which happens in effect; for performing the Multiplication, there results

$$\begin{aligned} & 1 - \frac{r}{n}p + \frac{\frac{r}{n} \cdot \frac{r}{n} + 1}{2}p^2 - \frac{\frac{r}{n} \cdot \frac{r}{n} + 1 \cdot \frac{r}{n} + 2}{2 \times 3}p^3 + \frac{\frac{r}{n} \cdot \frac{r}{n} + 1 \cdot \frac{r}{n} + 2 \cdot \frac{r}{n} + 3}{2 \times 3 \times 4}p^4 -, \&c. \\ & + \frac{r}{n}p - \frac{r}{n} \cdot \frac{r}{n}p^2 + \frac{\frac{r}{n} \cdot \frac{r}{n} \cdot \frac{r}{n} + 2}{2}p^3 - \frac{\frac{r}{n} \cdot \frac{r}{n} \cdot \frac{r}{n} \cdot \frac{r}{n} + 1 \cdot \frac{r}{n} + 2}{2 \times 3 \times 4}p^4 +, \&c. \\ & + \frac{\frac{r}{n} \cdot \frac{r}{n} - 1}{2}p^2 - \frac{\frac{r}{n} \cdot \frac{r}{n} \cdot \frac{r}{n} - 1}{2}p^3 + \frac{\frac{r}{n} \cdot \frac{r}{n} \cdot \frac{r}{n} + 1 \cdot \frac{r}{n} - 1}{2 \times 3}p^4 -, \&c. \\ & + \frac{\frac{r}{n} \cdot \frac{r}{n} - 1 \cdot \frac{r}{n} - 2}{2 \times 3}p^3 - \frac{\frac{r}{n} \cdot \frac{r}{n} \cdot \frac{r}{n} - 1 \cdot \frac{r}{n} - 2}{2 \times 3}p^4 +, \&c. \\ & + \frac{\frac{r}{n} \cdot \frac{r}{n} - 1 \cdot \frac{r}{n} - 2 \cdot \frac{r}{n} - 3}{2 \times 3 \times 4}p^4 -, \&c. \\ & +, \&c. \end{aligned}$$

whose first Term, which is Unity, is the only one that remains after the Reduction.

XXXIII.

Since to extract any Root of a given Quantity, is the same Thing as to raise that Quantity to a Power, whose Exponent is a Fraction that has its

Denominator equal to the Number that expresses what Kind of Root is to be extracted, it is plain that the Root of any Quantity may be found by the foregoing Formula, by putting $m = \frac{1}{2}$ or $\frac{1}{3}$ or $\frac{1}{4}$, &c. according as it is the square Root, cube Root, or the Root of the fourth Power, &c. that is required; and if the proposed Quantity is an exact Square, or an exact Cube, &c. the Series will stop of itself, the Numerator of one of the Terms, in this Case vanishing; if not, it may be carried on to any Number of Terms, as will appear by the following Examples.

Application of the foregoing formula for finding the roots of perfect powers of compound quantities.

Let it be proposed to extract the square Root of $1 + 2b + b^2$, putting $u = 1$, $z = 2b + b^2$, and $m = \frac{1}{2}$ in the foregoing Formula, I find the first Term $u^m = 1$, the second $m u^{m-1} z = b + \frac{1}{2} b^2$, the third

$$\frac{m \times (m-1)}{2} u^{m-2} z^2 = -\frac{1}{2} b^2 - \frac{1}{2} b^3 - \frac{1}{8} b^4; \text{ the 4th. Term.}$$

$$\frac{m \times (m-1) \times (m-2)}{2 \times 3} u^{m-3} z^3 = \frac{1}{2} b^3 + \frac{3}{4} b^4 + \frac{3}{8} b^5 + \frac{1}{16} b^6$$

$$\text{the 5th. } \frac{m \times (m-1) \times (m-2) \times (m-3)}{2 \times 3 \times 4} u^{m-4} z^4 = -\frac{5}{8} b^4$$

$-\frac{5}{4} b^5 - \frac{15}{16} b^6 - \frac{15}{16} b^7 - \frac{5}{128} b^8$. The 6th., &c. and the Sum of all those Terms will be the Root required.

Adding all those Terms, I observe that the first 1 remains entirely, that the 2d. $b + \frac{1}{2} b^2$ is reduced to b , because the Part $\frac{1}{2} b^2$ of this second Term is destroyed by the same Quantity with a negative Sign in the third Term $-\frac{1}{2} b^2 - \frac{1}{2} b^3 - \frac{1}{8} b^4$; that what remains of the third Term $-\frac{1}{2} b^3 - \frac{1}{8} b^4$ after $\frac{1}{2} b^2$ is destroyed, should be likewise entirely destroyed by the fourth $\frac{1}{2} b^3 + \frac{3}{4} b^4 + \frac{3}{8} b^5 + \frac{1}{16} b^6$, of which after the Reduction there only remains $\frac{5}{8} b^4 + \frac{3}{8} b^5 + \frac{1}{16} b^6$, this Remainder of the fourth Term is destroyed in like Manner by the fifth Term, and continuing the Reduction further on, it will appear that the Whole vanishes except $1 + b$, which is the Root required.

XXXIV.

Let the Root of the 5th. Power of the Quantity $a + b$ be required, substituting the greatest of the two Parts of this Quantity, which I suppose to be a in the Room of u , the least Quantity b in the Room of z , and $\frac{1}{5}$ in the Room of m , there will result for the Root required

Application of the foregoing formula for finding the roots of perfect powers of compound quantities.

ing the roots
of imperfect
powers of
compound
quantities.

$$a^{\frac{1}{5}} + \frac{1}{5} a^{-\frac{4}{5}} b - \frac{1 \times 4}{2 \times 25} a^{-\frac{9}{5}} b^2 + \frac{1 \times 4 \times 9}{2 \times 3 \times 125} a^{-\frac{14}{5}} b^3 - \frac{1 \times 4 \times 9 \times 14}{2 \times 3 \times 4 \times 625} a^{-\frac{19}{5}} b^4 + \frac{1 \times 4 \times 9 \times 14 \times 19}{2 \times 3 \times 4 \times 5 \times 3125} a^{-\frac{24}{5}} b^5, \&c.$$

$$\text{or } a^{\frac{1}{5}} \times \left(1 + \frac{b}{5a} - \frac{2bb}{25aa} + \frac{6b^3}{125a^3} - \frac{21b^4}{625a^4} +, \&c. \right)$$

Though it be requisite to carry on this Series to an infinite Number of Terms, in order that it may express exactly the Root required, however, as this Series will converge sooner, the greater a is in Respect of b , by taking a Number of its Terms we shall approximate to the Root to any assigned Degree of Exactness.

Let, for Example, $b = \frac{1}{10} a$ the six first Terms of the Series will be

$$a^{\frac{1}{5}} \times \left(1 + \frac{1}{50} - \frac{2}{1250} + \frac{3}{62500} - \frac{21}{6210000} + \frac{399}{1561500000} \right),$$

which converges very swiftly.

XXXV.

The foregoing
formula
applied to
raising quan-
tities to ne-
gative pow-
ers.

Let it be proposed to approximate to the Quotient of $\frac{aa}{x \pm b}$, I bring the Quantity $x \pm b$ from the Denominator to the Numerator, by changing the Sign of the Exponent 1 into its opposite -1 , so that the Expression stands thus, $aa \times (x \pm b)^{-1}$: I then reduce $(x \pm b)^{-1}$ into a Series, by substituting the greatest of the two Parts x in the Room of u , and the least $\pm b$ in the Room of z , and -1 in the Room of m , and I find

$$aa \times (x \pm b)^{-1} = \frac{aa}{x} + \frac{aab}{x^2} + \frac{aab^2}{x^3} + \frac{a^2b^3}{x^4} + \frac{a^2b^4}{x^5} + \&c.$$

• XXXVI.

If in the proposed Quantity $\frac{aa}{b \pm x}$, b was $= x$, then we will have

Difficulty
that occurs
when the
two terms
of the deno-
minator are
equal.

1°. $\frac{aa}{b-b} = \frac{aa}{b} \times (1 + 1 + 1 + 1 + 1 + 1 + 1 +, \&c.)$, whose Value is infinite, which is no more than what we knew already, since $\frac{aa}{b-b} = \frac{aa}{0} = \infty$. The Sign ∞ being employed by the Analysts to express an infinite Number.

$$2°. \frac{aa}{b+b} = \frac{aa}{b} \times (1 - 1 + 1 -, \&c.) = \frac{aa}{b} \times (0 + 0 +, \&c.)$$

whence $\frac{aa}{b+b} = 0$, which is not true, consequently a Fraction whose

Denominator consists of two equal Terms cannot in that State be reduced into a Series: To remedy this Inconveniency, the Analysts divide the Sum of the two Parts of the Denominator into other unequal Parts: For Example, in the proposed Fraction $\frac{a a}{b + b}$ let $c = 2b + x$, and the Fraction will be transformed into $\frac{a a}{c - x}$, whose Denominator being $= 2b$, the Series arising from it will be equal $\frac{a a}{b + b}$.

XXXVII.

The Utility of the foregoing Formula is not confined to the finding by Approximation all Sorts of fractional or negative Powers; it is of infinite Use for reducing complicated Radicals: thus, if it was proposed to reduce $\frac{\sqrt{[c^4 - (cc + rr)x^2]}}{\sqrt{(c^4 - ccx^2)}}$ to an infinite Series, I bring the Quantity

All kinds of quantities may be reduced into Series by the foregoing formula.

$\sqrt{(c^4 - ccx^2)}$ from the Denominator to the Numerator, by changing the Sign of the Exponent $\frac{1}{2}$ into its opposite, so that the Expression stands

thus, $[c^4 - (cc - rr)x^2]^{\frac{1}{2}} \times (c^4 - ccx^2)^{-\frac{1}{2}}$, which by substituting for Brevity sake, a for cc and b for $-cc + rr$ will become

$(a^2 + bx^2)^{\frac{1}{2}} \times (a^2 - ax^2)^{-\frac{1}{2}}$, reducing those two Factors into infinite Series, and afterwards multiplying them into one another, there results,
 $1 + \frac{a+b}{2a^2}x^2 + \frac{3a^2+2ab-b^2}{8a^4}x^4 + \frac{5a^3+3a^2b-ab^2+b^3}{16a^6}x^6$
 $+ \frac{35a^4+20a^3b-6a^2b^2+4ab^3-5b^4}{128a^8}x^8$, &c. for the Value
 of $\frac{\sqrt{(a^2 + bx^2)}}{\sqrt{(a^2 - ax^2)}}$.

XXXVIII.

We have seen (Chap. II. Art. xxxix, xl, xli.) how the Analysts perform upon Radicals of the second Degree the Operations of Addition, Subtraction, Multiplication, and Division; those Operations being equally necessary for radical Quantities of higher Degrees, we shall proceed to explain what those new Radicals require.

Addition & subtraction of all sorts of radical quantities.

As to Addition and Subtraction, they require no more than what has been said already of those Operations upon Radicals of the second Degree: it suffices to reduce each Radical to its most simple Expression, and to add and subtract them as commensurable Quantities.

Let it be proposed, for Example, to subtract $\sqrt[3]{(b^4 + 2ab^3)}$ from $\sqrt[3]{(8a^3b + 16a^4)}$; I transform the first Quantity into $b\sqrt[3]{(b + 2a)}$, and the second into $2a\sqrt[3]{(b + 2a)}$, and then subtracting one from the

the other, the Difference is $(2a - b)\sqrt[3]{(b + 2a)}$: In like Manner, $3a\sqrt[4]{(16b^8 + 32b^4a^4)}$ being added to $4b\sqrt[4]{(a^4b^4 + 2a^8)}$ their Sum will be $10ab\sqrt[4]{(b^4 + 2a^4)}$.

XXXIX.

Multiplication and Division of radical quantities that have the same exponent.

As to Multiplication and Division, if the radical Quantities have the same Exponent, the Method is the same as for Radicals of the second Degree; it suffices to operate upon the Quantities to which the radical Sign is prefixed, and set the same radical Sign over their Product or Quotient.

Thus, $\sqrt[3]{5ayy} \times \sqrt[3]{7ayz} = \sqrt[3]{35aay^3z}$, or $y\sqrt[3]{35a^2z}$

$$\sqrt[5]{\frac{9a^2b^3}{4}} \times \sqrt[5]{27a^3b^6} = \sqrt[5]{\frac{243a^5b^9}{4}} = 3ab\sqrt[5]{\frac{b^4}{4}}$$

$$\sqrt[3]{(a^2b^2 + b^4)} \text{ divided by } \sqrt[3]{\frac{a^2 - b^2}{8b}} \text{ gives for Quotient } \sqrt[3]{\frac{8a^2b^3 + 8b^5}{a^2 - b^2}}$$

$$= 2b\sqrt[3]{\frac{a^2 + b^2}{a^2 - b^2}}, \frac{\sqrt[3]{(a^2x^3 - x^5)}}{\sqrt[3]{(a^4b^4 + b^4x)}} = \frac{x}{b}\sqrt[3]{\frac{a - x}{b}}$$

XL.

To perform those operations upon radical quantities of different exponents, they are first to be reduced to the same exponent.

But to perform those Operations on radical Quantities of different Exponents, they must first be reduced to others of equivalent Value that shall have the same radical Sign.

Let, for Example, $\sqrt[3]{ab}$ and $\sqrt[5]{abb}$ be proposed to be reduced to the same radical Sign, I raise ab to the fifth Power, and set down $\sqrt[15]{}$ in the Place of $\sqrt[3]{}$ and there results $\sqrt[15]{a^5b^5} = \sqrt[3]{ab}$, I likewise raise abb to the third Power, and set down $\sqrt[15]{}$ instead of $\sqrt[5]{}$, whence $\sqrt[15]{a^3b^6}$.

Now if it was proposed to multiply $\sqrt[3]{ab}$ by $\sqrt[5]{abb}$ the Product would be $\sqrt[15]{a^8b^{11}}$, if the first was to be divided by the second the Quotient would be $\sqrt[15]{\frac{a^5b^5}{a^3b^6}} = \sqrt[15]{\frac{a^2}{b}}$.

In like Manner the Product of $\sqrt[3]{a^2b^4}$ into $\sqrt[5]{a^4b^5}$ would be $\sqrt[15]{a^{16}b^{23}} = a^2b^3\sqrt[15]{a^4b^6}$.

Method of performing this reduction.

In general to reduce two radical Quantities $\sqrt[m]{a^p b^q}$ and $\sqrt[n]{a^r b^s}$ to the same Exponent, the first must be transformed into $\sqrt[mn]{a^{pn} b^{qn}}$, and the second into $\sqrt[mn]{a^{rm} b^{sm}}$. If they are to be multiplied, their Product would be $\sqrt[mn]{a^{pn+rm} b^{qn+sm}}$, and if the first is to be divided by the second, the Quotient will be $\sqrt[mn]{a^{pn-rm} b^{qn-sm}}$.

It is easy to perceive that when the Exponents of the radical Quantities have a common Divisor, there is no Necessity of reducing each Radical into another, whose Exponent is the Product of the two first Exponents; for Example, if $\sqrt{ab}$ and $\sqrt[4]{ab^3}$ be proposed, the first will be transformed into $\sqrt[4]{a^2b^2}$; in like Manner if $\sqrt[4]{a^3b}$, $\sqrt[6]{a^6b}$ are transformed into $\sqrt[12]{a^9b^3}$ and $\sqrt[12]{a^{12}b^2}$.

XLI.

The foregoing Operations may be performed after another Manner, by considering the radical Quantities as Powers with fractional Exponents.

Thus to multiply $\sqrt[p]{a^p b^q}$ by $\sqrt[r]{a^r b^s}$ is the same Thing as to multiply $a^{\frac{p}{p}} b^{\frac{q}{p}}$ by $a^{\frac{r}{r}} b^{\frac{s}{r}}$, whose Product is $a^{\frac{p}{p} + \frac{r}{r}} b^{\frac{q}{p} + \frac{s}{r}}$, or $a^{\frac{pn+rm}{pn}} b^{\frac{qm+rs}{qn}}$. Another Method of performing the foregoing operations.

If it was proposed to divide the first Quantity $\sqrt[p]{a^p b^q}$ by the second $\sqrt[r]{a^r b^s}$ the Quotient would be $a^{\frac{p}{p} - \frac{r}{r}} b^{\frac{q}{p} - \frac{s}{r}}$ or $a^{\frac{pn-rm}{pn}} b^{\frac{qm-rs}{qn}}$.

If it was proposed to divide $\sqrt[p]{a^m b^{3n} c^{\frac{1}{p}}}$ by $\sqrt[p]{a^{2m} b^n c^{\frac{1}{p}}}$, transforming the first Quantity into $a^{\frac{m}{2p}} b^{\frac{3n}{2p}} c^{\frac{1}{2p}}$, and the second into $a^{\frac{2m}{2p}} b^{\frac{n}{2p}} c^{\frac{1}{2p}}$, and subtracting the Exponents $\frac{2m}{2p}$, $\frac{n}{2p}$, $\frac{1}{2p}$, of the Letters a, b, c in the second Quantity from the Exponents $\frac{m}{2p}$, $\frac{3n}{2p}$, $\frac{1}{2p}$ of the same Letters in the first Quantity, the Quotient will be found to be

$$a^{-\frac{3m}{2p}} b^{\frac{n}{2p}} \times 1 = \frac{b^{\frac{n}{2p}}}{a^{\frac{3m}{2p}}}$$

XLII.

Compound Surds are such as consist of two or more joined together; simple Surds by being multiplied into themselves give at length rational Quantities, yet compound Surds multiplied into themselves, commonly give still irrational Products, but when any compound Surd is proposed, there is another compound Surd, which multiplied into it, gives a rational Product; thus $\sqrt{a} + \sqrt{b}$ multiplied by $\sqrt{a} - \sqrt{b}$ gives $a - b$. whence Fractions affected with surd Quantities, may be reduced to a more simple Expression, by multiplying the Denominator of the Fraction by that Surd which will give a rational Product, and multiplying the Numerator by the same Surd.

XLIII.

Let $a^n + b$ express any binomial Surd, and let it be proposed to transform it into another Binomial, in which the Quantity a shall have a given Exponent n , and the Quantity b shall not be multiplied by a , that is,

let it be proposed to find a multinomial, by which the proposed Binomial being multiplied, the first Term of the Product, shall be a^n , and all the other Terms except the last, shall destroy each other.

Investigation of the Surd which multiplied into a proposed Surd gives a rational Product.

Let $a^x by + a^z bu$ be any two Terms of this Multinomial immediately succeeding each other, and let them be multiplied by the proposed Binomial, the Product thence arising will be $a^{m+x} by + a^x bl + y + a^{m+z} bu + a^z bl^u$, and the first Term of the Product should be a^n , $m+x = n$, $x = n-m$ and $y = 0$, wherefore the first Term of the Multiplier is a^{n-m} .

To find the second Term, I substitute this first Term in the assumed Product, whence there results $a^n + a^{n-m} bl + a^{m+x} bu + a^z bl^u$ but $+ a^{n-m} bl + a^{m+x} bu = 0$. Consequently the second Term of the Multiplier should be negative, and because $n-m = m+z$ & $l = u$, Consequently $z = n-2m$ and $2l = u+l$, this second Term will be $-a^{n-2m} bl$.

To find the third Term of the Multiplier, I substitute this second Term in the assumed Product, and there results $a^n - a^{n-m} bl + a^{n-m} bl + a^{n-2m} b^2 l$. Let now the last Term of this Product, be put equal to the first Term of the assumed Product, because those Terms should destroy each other. $+ a^{n-2m} b^2 l + a^{m+x} by = 0$, whence $n-2m = m+x$, $x = n-3m$ and $y = 2l$ and $l+y = 3l$, Consequently the third Term of the Multiplier will be $+ a^{n-3m} b^2 l$. To find the fourth Term, I substitute the foregoing Values in the assumed Product, and there results $a^n - a^{n-m} bl + a^{n-m} bl - a^{n-2m} b^2 l + a^{n-2m} b^2 l + a^{n-3m} b^3 l + a^{m+x} bu + a^z bl^u$ whence $n-3m = m+z$ & $z = n-4m$ and $u = 3l$ and $l+u = 4l$, Consequently the fourth Term of the Multiplier will be $-a^{n-4m} b^3 l$, and in General, the p Term will be $a^n - p^m b (p-1)^l$.

Now let the p Term be the last Term of the Multiplier, since the last Term of the Product results from the Multiplication of the last Term of the Multiplier into the last Term of the proposed Binomial, and that it contains none of the Powers of a , it will be expressed by

$$b(p-1)^l + l = b^{\frac{n}{m}} \text{ as } n - p \cdot m = 0 \text{ and of Course, } p = \frac{n}{m} \text{ and its}$$

Sign is positive only, when $\frac{n}{m}$ is an odd Number, since all the Terms of the Multiplier expressed by even Numbers, are Negative.

XLIV.

Application of the foregoing Theorem for reducing Fractions involving surd Quant.

Whence in General, when any Quantity is divided by a Binomial Surd $a^m \pm b^l$ where l and m represent any Fractions whatsoever, by taking n , the least integer Number, such that $\frac{nl}{m}$ will be an Integer, and multiplying both Numerator and Denominator by $a^{n-m} \mp a^{n-2m} b^l + a^{n-3m} b^2 l \mp a^{n-4m} b^3 l$, &c. the Denominator of the Product will become rational

and equal to $a \pm b^{\frac{n}{m}}$ then dividing all the Members of the Numerator by this rational Quantity, the Quote arising, will be that of the proposed Quantity divided by the Binomial Surd expressed in it's last Terms. ties to a more simple Expression.

$$\begin{aligned} \text{Thus } \frac{3}{\sqrt{5}-\sqrt{2}} &= \frac{3\sqrt{5}+3\sqrt{2}}{3} = \sqrt{5}+\sqrt{2}; \quad \frac{\sqrt{6}}{\sqrt{7}-\sqrt{3}} = \frac{\sqrt{42}+\sqrt{18}}{4} \\ \frac{3\sqrt{20}}{3\sqrt{4}-3\sqrt{2}} &= \frac{3\sqrt{20}}{3\sqrt{4}-3\sqrt{2}} \times \frac{3\sqrt{16}+2+3\sqrt{4}}{3\sqrt{16}+2+3\sqrt{4}} = \frac{3\sqrt{20}}{3\sqrt{4}-3\sqrt{2}} \times \frac{2\sqrt{2}+2+3\sqrt{4}}{2\sqrt{2}+2+3\sqrt{4}} \\ &= \frac{2\sqrt{40}+2\sqrt{20}+3\sqrt{80}}{2} = 2\sqrt{5} \times \sqrt{20} \times \sqrt{10}, \text{ also } \frac{\sqrt{10}}{\sqrt{2}-\sqrt{3}} \\ &= \frac{4\sqrt{20}+4\sqrt{10} \times \sqrt{3}+2\sqrt{20} \times \sqrt{9}+6\sqrt{10}+3\sqrt{20} \times \sqrt{3}+3\sqrt{10} \times \sqrt{9}}{-1} \\ &= -8\sqrt{5}-4\sqrt{10} \times \sqrt{3}-8\sqrt{5} \times \sqrt{9}-6\sqrt{10}-6 \times \sqrt{5} \times \sqrt{3}-3\sqrt{10} \times \sqrt{9} \end{aligned}$$

XLV.

To compleat the Resolution of Equations, consisting of two Terms; it remains to explain how the unknown Quantity is disengaged when it enters the Exponent.

Thus if it was required to find in what Time x a Sum a lent out at Interest at a given Rate *per Cent.* would amount Principle and Interest to a Sum r , the Equation to be solved would be $r = a p^x$ or $\frac{r}{a} = p^x$, in which Equation the unknown Quantity enters the Exponent.

XLVI.

Since the Exponent x includes all determined Numbers if in the Room of x be substituted successively all the integer Numbers there will result $p^{-7}, p^{-6}, p^{-5}, p^{-4}, p^{-3}, p^{-2}, p^{-1}, p^0, p^1, p^2, p^3, p^4$, &c. which forms a geometrical Progression. Now, if between the Terms of this Progression, there be inserted a Number of Terms n , it is easy to perceive that they will be expressed by the Powers of p with fractional Exponents, and that there will result a new Progression, the Multiplier being $q^{\frac{1}{n+1}}$ and that $n+1$ may be conceived so great and consequently the Progression may be conceived to increase so slowly, that all the Numbers from 1 to ∞ , and from 1 to 0, will be found there, p being supposed to be an Integer, and greater than Unity.

XLVII.

Let for Example $p = 10$, then the Progression will be transformed into

$$1, 10, 100, 1000, 10000, 100000, \&c.$$

ELEMENTS OF

Now, between 1 & 10, there are eight whole Numbers, 2, 3, 4, 5, 6, 7, 8, 9, and between 10 and 100, there are 89 whole Numbers, viz. 11, 12 &c. but if between each of the Terms of this Progression, there be inserted a Number of Terms n , there will result a new Progression, the

common Multiplicator being $10^{\frac{1}{n+1}}$ among whose Terms those intermediate Numbers will be found.

To find the Term of this Progression for Example, which is equal to 3, I proceed thus, the Number 3 required being one of the Terms of a geometrical Progression, of which 1 & 10 are the Extreames; I investigate the middle Term of this Progression; now, this Term being equally distant from the Extreames 1 and 10, those three Terms the first 1 the middle Term and the last Term 10 will be in continued Proportion, so that the middle Term will be equal to $\sqrt{1 \times 10}$ or $10^{.5}$, which I find to be 3,16227766. The Number 3, 16227766 being greater than 3, I investigate a new Term that will be the Mean of a continued geometrical Proportion, having for Extreames 1 & 3,16227766, which will be $\sqrt{1 \times 3,16227766}$ or $10^{.25} = 1,77827941$. As the Number 3 is greater than 1,77827941, or $10^{.25}$ and less than 3,16227766 or $10^{.5}$, I investigate a new Term, a mean proportional between 1,77827941, and 3,16227766, which will be the square Root of $10^{.25} \times 10^{.5}$ that is of $3,16227766 \times 1,77827941$, or of 5,6234132514809806, which is $10^{.375}$ or 2,37137370. As the Number 3 is greater than the new Term, 2,37137370 and less than 3,16227766, I investigate a mean Proportional between those two Terms by extracting the square Root of $10^{.375} \times 10^{.5}$ that is of $2,37137370 \times 3,16227766$ which is 2,73841962. As the Number 3 is greater than the new Term 2,73841962 and less than 3,16227766, I find a mean Proportional between those two Terms by extracting the square Root of $10^{.4375} \times 10^{.5}$ that is of $2,73841962 \times 3,16227766$ or of 8,6596431880316892 which will be $10^{.46875}$ or 2,94272717. The Number 3 being still greater than the last Term 2,94272717 and less than the first 3, 16227766, I investigate a mean Proportional between these two Terms, by extracting the square Root of $10^{.46875} \times 10^{.5}$, or of $2,94272717 \times 3,16227766$, which will be $10^{.484375}$, or 3,05052789.

Proceeding in this Manner to investigate new Mean proportionals between two Terms, one greater and the other less than 3, after having performed 26 of those Operations, I find $10^{.47712125} = 3,00000000$, which does not differ from 3 by a decimal Unit of the eighth Order, these 26 mean Proportionals are set down in the following Table, each of them being placed between the Extreames, and to indicate the Order of the Operations, the two first Extreames are marked by the Letters *A* and *B*, and the mean Term placed between them by *C*, the other Means are denoted by the following Letters of the Alphabet.

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A	100,00000000	=	1,00000000
C	100,50000000	=	3,16227766
B	101,00000000	=	10,00000000
A	100,00000000	=	1,00000000
D	100,25000000	=	1,77827941
C	100,50000000	=	3,16227766
D	100,25000000	=	1,77827941
E	100,37500000	=	2,37137370
C	100,50000000	=	3,16227766
E	100,37500000	=	2,37137370
F	100,43750000	=	2,73841962
C	100,50000000	=	3,16227766
F	100,34750000	=	2,73841962
G	100,46875000	=	2,94272717
C	100,50000000	=	3,16227766
G	100,46875000	=	2,94272717
H	100,48437500	=	3,05052789
C	100,50000000	=	3,16227766
H	100,48437500	=	3,05052789
I	100,47656250	=	2,99614286
G	100,46875000	=	2,94272717
I	100,47656250	=	2,99614286
K	100,48046875	=	3,02321308
H	100,48437500	=	3,05052789
K	100,48046875	=	3,02321308
L	100,47851563	=	3,00964753
I	100,47656250	=	2,99614286
L	100,47851563	=	3,00964753
M	100,47753906	=	3,00288762
I	100,47656250	=	2,99614286
M	100,47753906	=	3,00288762
N	100,47705078	=	2,99951334
O	100,47729492	=	3,00120000
M	100,47753906	=	3,00288762
O	100,47729492	=	3,00120000
P	100,47717285	=	3,00035655
N	100,47705078	=	2,99951334

P	100,47717285	=	3,00035655
Q	100,47711182	=	2,99993491
N	100,47705078	=	2,99951334
Q	100,47711182	=	2,99993491
R	100,47714233	=	3,00014572
P	100,47717285	=	3,00035655
R	100,47714233	=	3,00014572
S	100,47712708	=	3,00004031
Q	100,47711182	=	2,99993491
S	100,47712708	=	3,00004031
T	100,47711945	=	2,99998751
Q	100,47711182	=	2,99993491
T	100,47711945	=	2,99998751
V	100,47712326	=	3,00001390
X	100,47712135	=	3,00000070
T	100,47711945	=	2,99998751
X	100,47712135	=	3,00000070
T	100,47712040	=	2,99999410
T	100,47711945	=	2,99998751
T	100,47712040	=	2,99999410
Z	100,47712088	=	2,99997739
X	100,47712135	=	3,00000070
Z	100,47712088	=	2,99997739
AA	100,47712112	=	2,99999908
X	100,47712135	=	3,00000070
AA	100,47712112	=	2,99999908
BB	100,47712123	=	2,99999989
X	100,47712135	=	3,00000070
BB	100,47712123	=	2,99999989
CC	100,47712129	=	3,00000029
X	100,47712135	=	3,00000070
CC	100,47712129	=	3,00000029
DD	100,47712126	=	3,00000009
BB	100,47712123	=	2,99999989
DD	100,47712126	=	3,00000009
EE	100,47712125	=	3,00000000
BB	100,47712123	=	2,99999989

By the same Method of Proceeding, the Terms corresponding to the other intermediate Numbers will be found, by investigating mean Proportionals between 1 and 10, for the Numbers less than 10; between 10 & 100, for the Numbers less than 100, but greater than 10; between 100 & 1000, for the Numbers less than 1000 and greater than 100, &c.

XLVIII.

What is meant by the Logarithm of a Number.

Whence if p be given, and if $p^x = b$, we can find the Value of x such that $p^x = b$: this Value of x is called the Logarithm of b , and p the Base of the Logarithm.

How the Logarithms of Numbers of one System are deduced from those of another.

There are therefore as many different Systems of Logarithms, as there are different Numbers p which may be assumed for the Base, but in two Systems the Logarithms of the same Number are always in a constant Ratio. Let the Base of one System $= p$, and of the other $= q$, and the Logarithm of the Number b in the first System $= x$, & in the second $= y$,

then $p^x = b$ and $q^y = b$, wherefore $p^x = q^y$ consequently $p = q^{\frac{x}{y}}$ whence the Fraction $\frac{x}{y}$ is always the same whatever Number is assumed for b .

Whence if one System of Logarithms of all Numbers were computed, the Logarithms of any other System may be found by the Rule of Three. Thus if the Logarithms corresponding to the Base 10, were calculated, the Logarithms corresponding to any other Base for Example 3 may be found. Let the Logarithm y of the Number b corresponding to the Base 3 be required, the Logarithm x of the same Number b corresponding to the Base 10 being given. Since $\text{Log. } 3$, corresponding to the Base 10 $= 0,47712125$ and $\text{Log. } 3$, corresponding to Base 3 $= 1$. $0,47712125 : 1 = x : y$, wherefore $y = \frac{x}{0,47712125} = 2,095933 \times x$, if therefore all the Logarithms corresponding to the Base 10, be multiplied by 2,095933, there will result the Logarithms corresponding to the Base 3.

XLIX.

In every System the Logarithm of Unity is 0.

In every System of Logarithms, $\text{Log. } 1 = 0$, for if in the Equation $p^x = b$, we put $b = 1$, then $x = 0$.

The Logarithms of negative Numbers are imaginary.

The Logarithms of Numbers greater than Unity, are positive, thus, $\text{Log. } p = 1$; $\text{Log. } p^2 = 2$; $\text{Log. } p^3 = 3$, &c. But the Logarithms of Numbers less than Unity, but positive, are negative, for $\text{Log. } \frac{1}{p} = -1$;

$\text{Log. } \frac{1}{p^2} = -2$, $\text{Log. } \frac{1}{p^3} = -3$, &c. and the Logarithms of negative Numbers are imaginary.

In like Manner if $\text{Log. } p = x$, then $\text{Log. } p^2 = 2x$ $\text{Log. } p^3 = 3x$ and in general $\text{Log. } p^n = nx$ or $\text{Log. } p^n = n \text{Log. } p$, on Account of $x = \text{Log. } p$.

whence the Logarithm of any Power of p is equal to the Logarithm of p multiplied by the Exponent of the Power; thus $\text{Log. } \sqrt{p} = \frac{1}{2} x = \frac{1}{2} \text{Log. } p$, $\text{Log. } \frac{1}{\sqrt{p}} = \text{Log. } p^{-\frac{1}{2}} = -\frac{1}{2} \text{Log. } p$; and so on. But if $\text{Log. } a = y$ and $\text{Log. } b = x$: because $a = p^y$, and $b = p^x$, then $\text{Log. } a b = y + x$, whence the Logarithm of the Product of two Numbers, is equal to the Sum of the Logarithms of the Factors. In like Manner it will appear that $\text{Log. } \frac{a}{b} = y - x = \text{Log. } a - \text{Log. } b$, hence the Logarithm of a Fraction is equal to the Logarithm of the Numerator less the Logarithm of the Denominator.

Whence from the Equation $p^x = \frac{r}{a}$ is deduced $x = \frac{\text{Log. } r - \text{Log. } a}{\text{Log. } p}$ it being indifferent what System of Logarithms is employed (Art. XLVIII).

L.

The Resolution therefore of Exponential Equations, or of those in which the unknown Quantity enters the Exponent, is reduced to the Investigation of a general Formula for finding the Logarithm of any given Number.

Let y, y', y'' express any three Terms of the Progression of Art. XLVI. immediately succeeding each other; we will have $y : y' = y' : y''$, or $y' : y = y'' : y'$, Consequently $y' - y : y = y'' - y' : y'$, or (putting $y' - y = dy$ and $y'' - y' = dy'$) $dy : y = dy' : y'$, wherefore $\frac{dy}{y} = \frac{dy'}{y'}$, whence $\frac{dy}{y}$ is a permanent Quantity.

Investigation of a general formula for finding the logarithm of any given number.

Let the Exponent of the Power of p which is equal y be expressed by x , the Exponent of the Power of p , which is equal y' be expressed by $x + dx$, the Exponent of the Power of p which is equal y'' will be expressed by $x + 2dx$, because the Exponents of the Terms of this Progression increase by the same constant Difference dx , whence assuming M for a determined Quantity, $dx = \frac{M dy}{y}$, or $\frac{dx}{dy} = \frac{M}{y}$, consequently $\frac{M}{y}$ is the Value of the Ratio of the cotemporary Increments of any Number y and of its Logarithm x .

To find the Ratio of the Quantities y and x from that of their Increments, I put $y = 1 + z$, and there results $dy = dz$, and $\frac{dx}{dy} = \frac{dx}{dz} = \frac{M}{1+z} = M \times (1+z)^{-1} = M(1 - z + z^2 - z^3 + z^4 - z^5 + \&c.)$.

I multiply this Value of $\frac{dx}{dz}$ by z , and consequently there will arise $M(z - z^2 + z^3 - z^4 + z^5 - z^6 + \&c.)$ which Terms I divide severally by their Number of Dimensions, and the Result

By the same Method of Proceeding, the Terms corresponding to the other intermediate Numbers will be found, by investigating mean Proportionals between 1 and 10, for the Numbers less than 10; between 10 & 100, for the Numbers less than 100, but greater than 10; between 100 & 1000, for the Numbers less than 1000 and greater than 100, &c.

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The logarithm of the product of two numbers is equal to the Sum of the logarithms of the factors, and the logarithm of their quotient, to their difference.

Whence from the Equation $px = \frac{r}{a}$ is deduced $x = \frac{\text{Log. } r - \text{Log. } a}{\text{Log. } p}$ it being indifferent what System of Logarithms is employed (Art. XLVIII).

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The Resolution therefore of Exponential Equations, or of those in which the unknown Quantity enters the Exponent, is reduced to the Investigation of a general Formula for finding the Logarithm of any given Number.

Let y, y', y'' express any three Terms of the Progression of Art. XLVI. immediately succeeding each other; we will have $y : y' = y' : y''$, or $y' : y = y'' : y'$, Consequently $y' - y : y = y'' - y' : y'$, or (putting $y' - y = dy$ and $y'' - y' = dy'$) $dy : y = dy' : y'$, wherefore $\frac{dy}{y} = \frac{dy'}{y'}$, whence $\frac{dy}{y}$ is a permanent Quantity.

Investigation of a general formula for finding the logarithm of any given number.

Let the Exponent of the Power of p which is equal y be expressed by x , the Exponent of the Power of p , which is equal y' be expressed by $x + dx$, the Exponent of the Power of p which is equal y'' will be expressed by $x + 2dx$, because the Exponents of the Terms of this Progression increase by the same constant Difference dx , whence assuming M for a determined Quantity, $dx = \frac{M dy}{y}$, or $\frac{dx}{dy} = \frac{M}{y}$, consequently $\frac{M}{y}$ is the Value of the Ratio of the cotemporary Increments of any Number y and of its Logarithm x .

To find the Ratio of the Quantities y and x from that of their Increments, I put $y = 1 + z$, and there results $dy = dz$, and $\frac{dx}{dy} = \frac{dx}{dz} = \frac{M}{1+z} = M \times (1+z)^{-1} = M(1 - z + z^2 - z^3 + z^4 - z^5 + \&c.)$.

I multiply this Value of $\frac{dx}{dz}$ by z , and consequently there will arise $M(z - z^2 + z^3 - z^4 + z^5 - z^6 + \&c.)$ which Terms I divide severally by their Number of Dimensions, and the Result

$$M \times (z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \frac{1}{4} z^4 + \frac{1}{5} z^5 - \frac{1}{6} z^6 +, \&c).$$

I put $= x$, to which Equation no constant Quantity is to be added, because when $z = 0$ it is reduced to 0, as it really should, for when $z = 0$ $y = 1 + z = 1$, whose Logarithm $= 0$. And by this Equation is expressed the Ratio between x & y , since from this Equation we can return back to the proposed Equation.

LI.

Inconveniency to which the foregoing method is liable.

Now it may be removed.

By this Formula, the Logarithm of any Number greater than Unity, but less than 2 may be found; I say less than 2, for if $1 + z$ is equal 2, then the two Terms of the Denominator will be equal, and the Series will be erroneous, and much more so when z exceeds 2, however, the foregoing Formula may serve for finding the Logarithm of any Number greater than Unity, by calculating the Logarithms of such Numbers that are less than 2, which multiplied into one another, or divided by each other, produce the proposed Number.

For Example, finding that

$$\frac{\frac{12}{10} \times \frac{12}{10}}{\frac{8}{10} \times \frac{9}{10}} = \frac{1,2 \times 1,2}{0,8 \times 0,9} = 2,$$

I calculate by the foregoing Formula the Logarithm of 1, 2, putting $z = 0, 2$; the Logarithm of 0, 8 supposing $z = -0, 2$, and the Logarithm of 0, 9 by putting $z = -0, 1$. I add together the Logarithms of 0, 8 & 0, 9, and deduct their Sum from the Double of the Logarithm of 1, 2, and there results $M \times 0,693147180559$, &c. for the Logarithm of 2 required.

To determine the Value of M corresponding, for Example, to the System of Logarithms whose Base is 10, I first investigate the Logarithm of 10, finding that $\frac{2 \times 2 \times 2}{0,8} = 10$, I triple the Logarithm already found, and deduct from it the Logarithm of 0, 8, the Remainder will be $M \times 2,302585092994$, &c. is the Logarithm of 10; now the Logarithm of 10 corresponding to the Base 10 is 1, and of Course we will have $1 = M \times 2,302585092994$, wherefore the Value of M corresponding to the System of Logarithms, whose Base is 10, will be 0,43429448, &c.

LII.

Another method for finding the logarithm of any number

The foregoing Method of finding the Logarithms of Numbers being not very ready, we shall proceed to explain how the Analysts have remedied this Inconveniency.

Let the Sum of any two Numbers be expressed by z , and their Difference by v , consequently the greatest will be $\frac{1}{2} z + \frac{1}{2} v$, and the least $\frac{1}{2} z - \frac{1}{2} v$. Now let $y = \frac{\frac{1}{2} z + \frac{1}{2} v}{\frac{1}{2} z - \frac{1}{2} v} = \frac{z + v}{z - v}$, and supposing x

to be a permanent Quantity, we will have $dy = \frac{2z \cdot dv}{(z-v)(z-v-dv)}$;

Wherefore the Equation. $\frac{dx}{dy} = \frac{M}{y}$ will be transformed into

$$\frac{dx \times (z-v)(z-v-dv)}{2z \cdot dv} = \frac{M \times (z-v)}{z+v} \text{ or into } \frac{dx}{dv} = \frac{2z \cdot M}{(z+v) \times (z-v-dv)}$$

Wherefore $\frac{2z \cdot M}{(z+v) \times (z-v-dv)}$ expresses the Ratio of any Cotemporary Increments of x and v , and this Ratio is always greater than $\frac{2z \cdot M}{(z+v)(z-v)}$, but dv decreasing, this Quantity will also decrease, and as we may take dv as small as we please, we may make $\frac{2z \cdot M}{(z+v) \times (z-v-dv)}$ approach as near as we please to $\frac{2z \cdot M}{(z+v)(z-v)}$ wherefore $\frac{2z \cdot M}{(z+v)(z-v)}$ is the Limit of the Ratio $\frac{dx}{dv}$.

To find the Ratio of the Quantities x and v , from the limiting Ratio of their cotemporary Increments $\frac{2z \cdot M}{(z+v)(z-v)} = \frac{2Mz}{z^2 - v^2} =$

$$2Mz \cdot (z^2 - v^2)^{-1} = 2M \left(\frac{1}{z} + \frac{v^2}{z^3} + \frac{v^4}{z^5} + \frac{v^6}{z^7} + \&c \right),$$

I multiply this Value of $\frac{dx}{dv}$ by v , and consequently there will arise

$2M \left(\frac{v}{z} + \frac{v^3}{z^3} + \frac{v^5}{z^5} + \frac{v^7}{z^7} + \&c \right)$, which Terms I divide severally by the Number of their Dimensions, and I put the Result

$$2M \left(\frac{v}{z} + \frac{v^3}{3z^3} + \frac{v^5}{5z^5} + \frac{v^7}{7z^7} + \&c \right) \text{ equal } x,$$

to which Equation no constant Quantity is to be added, because when $v = 0$ it is reduced to 0, as it really should, for when $v = 0$, the Fraction

$$\frac{z+v}{z-v} \text{ is reduced to } \frac{z}{z} = 1, \text{ whose Logarithm} = 0.$$

LIII.

Whence if the Sum of two Numbers be expressed by z , and their Difference by v , and if $\frac{2Mv}{z}$ be put $= A$, $\frac{Av^2}{z^2} = B$, $\frac{Dv^2}{z^2} = C$,

$\frac{Cv^2}{z^2} = D$, &c. the Logarithm of the Quotient of the greatest divided

by the least will be expressed by $A + \frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D + \&c.$

ELEMENTS OF

For Example, putting $M=1$, the Logarithm p of the Quotient of 126 divided by 125 may be found, the Logarithm q of the Quotient of 225 divided by 224, the Logarithm r of the Quotient of 2401 divided by 2400, and the Logarithm f of the Quotient of 4375 divided by 4374.

$$\text{But } p = \text{Log. } 126 - \text{Log. } 125 = \text{Log. } 2 \cdot 3^2 \cdot 7 - \text{Log. } 5^3 = \text{Log. } 2 + 2 \text{ Log. } 3 - 3 \text{ Log. } 5 + \text{Log. } 7$$

$$q = \text{Log. } 225 - \text{Log. } 224 = \text{Log. } 3^2 \cdot 5^2 - \text{Log. } 2^5 \cdot 7 = -5 \text{ Log. } 2 + 2 \text{ Log. } 3 + 2 \text{ Log. } 5 - \text{Log. } 7.$$

$$r = \text{Log. } 2401 - \text{Log. } 2400 = \text{Log. } 7^4 - \text{Log. } 2^5 \cdot 3 \cdot 5^2 = -5 \text{ Log. } 2 - \text{Log. } 3 - 2 \text{ Log. } 5 + 4 \text{ Log. } 7.$$

$$f = \text{Log. } 4375 - \text{Log. } 4374 = \text{Log. } 5^4 \cdot 7 - \text{Log. } 2 \cdot 3^7 = -\text{Log. } 2 - 7 \text{ Log. } 3 + 4 \text{ Log. } 5 + \text{Log. } 7.$$

$$\text{Whence } ap = a \text{ Log. } 2 + 2a \text{ Log. } 3 - 3a \text{ Log. } 5 + a \text{ Log. } 7.$$

$$bq = -5b \text{ Log. } 2 + 2b \text{ Log. } 3 + 2b \text{ Log. } 5 - b \text{ Log. } 7.$$

$$cr = -5c \text{ Log. } 2 - c \text{ Log. } 3 - 2c \text{ Log. } 5 + 4c \text{ Log. } 7.$$

$$df = -d \text{ Log. } 2 - 7d \text{ Log. } 3 + 4d \text{ Log. } 5 + d \text{ Log. } 7.$$

Let a, b, c, d be such, that $ap + bq + cr + df = \text{Log. } 2 + \text{Log. } 5$, whence arises the following Equations for determining those Quantities, $a - 5b - 5c - d = 1$; $2a + 2b - c - 7d = 0$; $-3a + 2b - 2c + 4d = 1$; $a - b + 4c + d = 0$; which give $a = 239, b = 90, c = -63, d = 103$; wherefore $239p + 90q - 63r + 103f$, or $2, 302585092924$, $= \text{Log. } 10$, the same as before, and the Value of M corresponding to the System whose Base is 10, will be found to be 0,434294481903, and substituting for M this Value, there will result

$$M(202p + 76q - 53r + 87f) = \text{Log. } 7.$$

$$M(167p + 63q - 44r + 72f) = \text{Log. } 5.$$

$$M(114p + 43q - 30r + 49f) = \text{Log. } 3.$$

$$1 - M(167p + 63q - 44r + 72f) = \text{Log. } 2.$$

Having M , and the Logarithms of the prime Numbers 2, 3, 5, 7 below 10, to obtain the Logarithms of the prime Numbers 11, 13, 17, 19 &c. Above 10, let A be any prime Number, I suppose that the Logarithm of all the prime Numbers that precede A have been found, then

$$\text{Log. } A = \frac{1}{2} \left[\text{Log. } \frac{A^2}{A^2 - 1} + \text{Log. } (A - 1) + \text{Log. } (A + 1) \right].$$

If the Logarithms of all the prime Numbers that precede A , are not known, let the Logarithm of any other Number B be given. Find the Logarithm of the Fraction $\frac{A}{B}$ if A be greater than B , or of $\frac{B}{A}$ if it is less: This Logarithm being found, add it, in the first Case, to the given Lo-

garithm of B , and there will result the Logarithm of $\frac{A}{B} \times B$, that is, of A .

In the second Case, deduct the Logarithm of the Fraction $\frac{B}{A}$ from the given Logarithm of B , and there will result the Logarithm of $\frac{B}{\frac{B}{A}} = B \times \frac{A}{B} = A$.

LIV.

To shew the Use of these Formulas by an Example, let it be required to find in what Time 575 *l.* will raise a Stock of 756 *l.* 13 *s.* 2 *d.* $\frac{1}{4}$ at 4 *per Cent.*

Application of the foregoing formulas to an example.

In this Case we have $p = 1, 04$, $a = 575$, and $r = 756, 66$. Whence

$$x = \frac{\text{Log. } 756, 66 - \text{Log. } 575}{\text{Log. } 1, 04} = 7 \text{ the Number of Years required.}$$

LV.

The Logarithms corresponding to the Base 10, besides their Utility which they have in common with other Systems, are the best adapted to numeral Arithmetick. For since the Logarithms of all Numbers except of the Powers of 10 are expressed in decimal Fractions, the Logarithms of Numbers between 1 and 10, will be contained within the Limits 0 and 1, the Logarithms of the Numbers between 10 & 100 will be contained within the Limits 1 and 2, and so on. Every Logarithm therefore is made up of an Integer and a decimal Fraction, the Integer is called the Characteristick, and the decimal Fraction the Mantissa. The Characteristick consists of as many Units less one as there are Places of Figures in the Number to which it corresponds, the Characteristick, for Example, of the Logarithm of the Number 78509 will be 4, because it consists of five Figures. Whence, from the Logarithm of any Number it is perceivable, how many Places of Figures it consists of: Thus the Number corresponding to the Logarithm 7,5804631 will consist of 8 Places of Figures.

The logarithms of numbers corresponding to the base 10 the best adapted to numeral arithmetick and why.

LVI.

If the Logarithms differ only by their Characteristicks, the Numbers to which they correspond, will be to each other as a Power of 10 to 1, and consequently will be expressed by the same significative Figures, thus the Numbers corresponding to the Logarithms 4,9130187, and 6,9130187 will be 81850 & 8185000, the Number corresponding to the Logarithm 3,9130187 will be 8185, and the Number corresponding to the Logarithm 0,9130187 will be 8,185: The Mantissa therefore indicates the Figures by which the Number is expressed, and the Characteristick shews how many Figures to the right Hand are to be separated by a Comma. Thus, if the Lo-

garithm 2, 7603429 was found, the Mantissa indicates those Figures 5758945, and the Characteristick 2 determines the Number corresponding to this Logarithm to be 575,8945. if the Characteristick was 0, the Number corresponding would be 5,758945, if the Characteristick was — 1, the Number corresponding would be ten Times less, viz. 0,57589450, and if the Characteristick was — 2, the Number corresponding would be ten Times less, viz. 0,05758945, &c. in the Room of those negative Characteristicks, — 1, — 2, — 3, &c. the Analysts set down 9, 8, 7, &c. it being understood that those Logarithms are to be diminished by 10.

LVII.

Use of logarithms for performing numeral operations.

The Tables of Logarithms are of great Use for performing with ease, and Expedition the Operations in numeral Arithmetick, because, by the help of those Tables, not only the Logarithm of any given Number, but also the Number corresponding to any given Logarithm may be found.

First example.

A Debt for Example of a 1000 l. 3 Years due, was acquitted Principal and Interest, by paying 1191 l. $\frac{2}{125}$ let the Rate of Interest allowed be required.

Here $a = 1000 l.$ $r = 1191 l.$ $\frac{2}{125}$, $x = 3$, are given to find p .
Consequently we will have $p = x\sqrt{\frac{r}{a}}$, or $\text{Log. } p = \frac{\text{Log. } r - \text{Log. } a}{x}$
$$= \frac{3,0759179 - 3,0000000}{3} = \frac{0,0759179}{3} = 0,0253059$$
 : Since

0,0253059 is the Logarithm of p or of $\frac{d+i}{d}$, adding to it the Log. of d or of 100, the Sum 2,0253059 is the Log. of $d+i$, but to this Logarithm corresponds the Number 106, wherefore $d+i = 106$, wherefore $i = 6$, the rate of Interest required.

The Sum of 1000 l. was borrowed at $7\frac{1}{2}$ per Cent. per Annum, the Sum due at the End of three Years seven Months and 15 Days are required.

Second example.

Here $a = 1000$, $i = 7\frac{1}{2}$, $d = 100$, $p = \frac{107\frac{1}{2}}{100} = \frac{215}{200} = \frac{43}{40}$;
 $x = \frac{1320}{365} = \frac{264}{73}$ Years, consequently $r = p^x$, or $\text{Log. } r = \text{Log. } a + x \text{ Log. } p = 3,0000000 + \frac{264}{73} \times 0,0314085$
 $= 3,0000000 + 0,1135869 = 3,1135869$ to which Logarithm corresponds the Number $1298\frac{29}{30}$ which expresses the Sum due required.

LVIII.

If it be agreed between the Borrower and the Lender, that the Interest tho' not paid when it becomes due, is not to be added to the Principle, so that the Amount shall be converted into a new Principle for the succeeding Term; the Sum r due at the Expiration of any limited Time will be expressed by

$a + \frac{a t i}{d} = a \left(1 + \frac{t i}{d} \right)$ and in this Case the Lender is said to receive simple Interest for the Loan of his Money.

In what cases compound interest is favourable to the borrower, and in what cases disadvantageous.

Now comparing this Expression $a \left(1 + \frac{t i}{d} \right)$ of the Sum due, at the Expiration of a Number of Years t , computed (in the Language of the Analysts) at simple Interest, with the former Expression $a \left(\frac{d+i}{d} \right)^t$

of the Sum due at the Expiration of the same Number of Years, computed at compound Interest, it will appear 1^o that if t is an Integer greater

than Unity, $\left(\frac{d+i}{d} \right)^t > \left(1 + \frac{t i}{d} \right)$ for $\left(\frac{d+i}{d} \right)^t =$

$$\frac{d^t}{d^t} + \frac{t i d^{t-1}}{d^t} + \frac{t \cdot t - 1 \cdot i^2 d^{t-2}}{2 d^t} + \frac{t \cdot t - 1 \cdot t - 2 \cdot i^3 d^{t-3}}{2 \times 3 d^t} + \&c.$$

Now this Quantity is manifestly equal to $1 + \frac{t i}{d} +$ a real and positive Quantity, consequently it is greater than $\left(1 + \frac{t i}{d} \right)$.

2^o if $t = 1$ it will easily appear that those two Quantities will be equal.

3^o if $t = \frac{1}{p}$ we will have $\left(\frac{d+i}{d} \right)^{\frac{1}{p}} < 1 + \frac{t i}{d}$ or $< 1 + \frac{t}{d p}$ for raising the Quantities on both Sides to the Power p there will result on one Side $\frac{d+i}{d}$, and on the other $1 + \frac{i}{d} +$ a real and positive Quantity.

4^o from whence we may conclude in general, that if t is any fractional Number greater than Unity, $a \left(\frac{d+i}{d} \right)^t > a + \frac{t i a}{d}$, and if t is a Fraction less than Unity, $a \left(\frac{d+i}{d} \right)^t > a + \frac{t i a}{d}$.

Whence it appears, that when compound Interest is allowed for the Loan of Money, the Sum due at the Expiration of any Time exceeding a Year, is greater than it would be if simple Interest was allowed, that the Sum due at the Expiration of any Time less, is less than it would be, if simple Interest was allowed, wherefore, if compound Interest is advantageous to the Lender,

in certain Cases it is also to the Borrower in others, the Compensation it is true is not equal, since the Advantage of the Borrower ends with the first Year, and that of the Lender goes on continually increasing as the Number of Years increaseth.

LIX.

Problem.

A Sum a was lent, on Condition that the Principal and Interest at a certain Rate i per Cent per Annum, should be discharged in a Number n of equal Payments, one at the end of every Year; and that at each Payment, the Interest then due should be cleared, and the Remainder by which such Payment exceeds the Interest applied to reduce the Principal, the Value of the Payments is required.

To solve this Question, I observe that the first Payment is composed of two Parts, one of which is the Interest of the Principal at the supposed Rate, due at the End of the first Year; the other is a Portion of the Principal taken to compleat the Payment. That the Principal being diminished by the first Payment, the Interest due at the End of the second Year will be less than the former, and consequently the Portion of the Principal to be taken to compleat the second Payment, will be greater than the Portion taken to compleat the first, and so on for the succeeding Years. Whence there results two Series, the one decreasing, whose Terms express the Interest due at the End of every Year, the other increasing whose Terms represent the different Portions of the Principal taken to compleat the Payments. It is this last Series we shall consider, and to discover the Law of it, let z, y, x &c. express the Portions of the Principal taken to compleat the first, second, third &c. Payments, so that $z + y + x + \&c. = a$.

The first Payment will be $\frac{ai}{d} + z$, the 2^d $\frac{ai - zi}{d} + y$, the Third $\frac{ai - zi - yi}{d} + x$ &c. As those Payments are supposed to be equal, by comparing the first with the second, the second with the third, &c. there will result so many different Equations. Whence is deduced $y = z \times \frac{d+i}{d}$,

$x = z \times \left(\frac{d+i}{d} \right)^2$, &c. from whence it appears, that the Series in

Question is a geometrical Progression, which by putting $\frac{d+i}{d} = p$, will be expressed as follows, z, zp, zp^2, zp^3 &c. $zp^{n-3}, zp^{n-2}, zp^{n-1}$. To find the first Term of this Progression, I observe that the Sum of all the Terms except the first, is equal to the Sum of all the Terms except the last, multiplied by the common Ratio, wherefore $a - z = ap - zp^n$ whence

$ap - a = zp^n$ or $z = a + \frac{p-1}{p^n-1}$, to which adding the Interest of

the Capital due at the End of the first Year, which is $a \times (p - 1)$, there results $r = a \times \left(\frac{p-1}{p^n-1} + p - 1 \right)$, or (reducing the whole to the same Denominator) $r = a \times \left(\frac{p^{n+1} - p^n}{p^n - 1} \right)$, or in Logarithms, $\text{Log. } r = \text{Log. } (p^{n+1} - p^n) - \text{Log. } (p^n - 1)$.

Let the Sum lent, for Example be 120 l. 5 s. Interest 6 per Cent. and to be reimbursed in 7 equal Payments. Here $a = 120, 25$, $p = 1,06$, $n = 7$ whence $\text{Log. } r = \text{Log. } 10,84852 - \text{Log. } 0,50361 = 1,035369 - 9,702113 = 1,333256$, whence $r = 21 \text{ l. } 10 \text{ s. } 9 \frac{1}{2} \text{ d.}$

Application of the foregoing problem to an example.

LX.

When a Debt is discharged by many equal Payments, and the Interest due at the Time of each Payment is cleared, before any Part of the Principal, compound Interest is allowed to the Lender. To make this appear, I argue thus, a Person who borrows a Sum a will owe at the End of the first Year, $a \times \frac{d+i}{d}$, but by Hypothesis he reimburses at the End of this

Year a Sum r , wherefore he will owe at the Beginning of the second Year $a \times \left[\frac{d+i}{d} \right] - r$, and at the End of the second Year he will owe

$a \times \left[\frac{d+i}{d} \right] - r$ (considered as Principal) multiplied by $\frac{d+i}{d}$, that is,

$a \times \left[\frac{d+i}{d} \right]^2 - r \times \frac{d+i}{d}$: and since at the End of the Year he reimburses a Sum r , he will owe at the Beginning of the the third Year

$a \left[\frac{d+i}{d} \right]^2 - r \times \frac{d+i}{d} - r$, and at the beginning of the 4th Year

$a \times \left[\frac{d+i}{d} \right]^3 - r \left[\frac{d+i}{d} \right]^2 - r \times \frac{d+i}{d} - r$, and in general at the End of the n^{th} Year.

$a \times \left[\frac{d+i}{d} \right]^n - r \left[\frac{d+i}{d} \right]^{n-1} - r \left[\frac{d+i}{d} \right]^{n-2} \dots - r$

Wherefore, if the Payment should be made in a Number n Years, the foregoing Quantity must be put equal to nothing. Now I observe, that in this last Quantity all the Terms that are multiplied by r form a geometrical Progression, of which the first Term is $\left[\frac{d+i}{d} \right]^{n-1}$, and the last Term

1. Now, the Sum of this Progression, putting for Brevity's sake, $\frac{d+i}{d} = p$ will be equal to $p^n - 1$ divided by $p - 1$, consequently

$a p^n - r \times \frac{p^n - 1}{p - 1} = 0$, consequently $r = a \times \frac{p^{n+1} - p^n}{p^n - 1}$: the same Value as found before.

When a debt is discharged by several equal payments compound interest is allowed to the lender.

LXI

What is
meant by
annuities.

When a Sum of Money is lent on Condition that the Borrower shall discharge both Principal and Interest in a Number of equal Payments to be made yearly, such Payments is called an Annuity. In Computations relating to Annuities therefore, there are four Things to be considered, the Sum lent a , the Rent received every Year r , the Rate *per Cent* i , the Number of Years n that the Annuity is paid; and any three of those four Things being given, the other may be found, as will appear by the following Examples.

Let it be required to find what a yearly Rent r is worth in ready Money for its Continuance n Years, the Rate i *per Cent per Annum*, compound Interest being allowed to the Purchaser.

Here r , n , p are given to find a , which is equal to $r \times \frac{1 - \frac{1}{p^n}}{p - 1}$, or

$$\text{Log. } \left(r - \frac{r}{p^n} \right) - \text{Log. } (p - 1).$$

Resolution
of the va-
rious ques-
tions relat-
ing to an-
nuities.

To apply this general Solution to an Example, let the Rent be 30*l.* *per Annum*, to be continued 7 Years, and let the present Worth be required, allowing 6 *per Cent* compound Interest to the Purchaser.

Here we have $r = 30$, $n = 7$, and $p = 1.06$ and of Course
 $\text{Log. } a = \text{Log. } 10.048 - \text{Log. } 0.06 = 1.002079 - 8.778151 = 2.223928$
and consequently $a = 167 \text{ l. } 9 \text{ s. } 5 \text{ d.}$

LXII.

Let it be required to find for what Time n a Sum a will purchase an Annuity of r Pounds *per Annum*, at the Rate i *per Cent* compound Interest.

Here a , r , & p are given to find n . Because $p^n = \frac{r}{r + a - ap}$
$$n = \frac{\text{Log. } r - \text{Log. } (r + a - ap)}{\text{Log. } p}$$

Let it be required to find, for Example, for what Time will 167*l.* 9*s.* 5*d.* purchase an Annuity of 30*l.* *per Annum*, at 6 *per Cent* compound Interest. In this Case $a = 167.416$, $r = 30$, $p = 1.06$, hence $n = \frac{\text{Log. } 30 - \text{Log. } 19.9517}{\text{Log. } 1.06} = 7 \text{ Years, the Time required.}$

LXIII.

Annuities are said to be in Arrears when they are payable or due either yearly or half yearly, &c. and are unpaid for any Number of Payments, to compute the Amount m of all those Payments, allowing any Rate of Interest for their Forbearance.

Suppose r the first Years Rent, then $rp + r$ will be the Amount of the first Years Rent more the second Years Rent, and $rp^2 + rp + r$ the Amount of the first and second Years Rent, more the third Years Rent, and in general the Amount of the $n - 1$ Years Rent more the n Years Rent will be expressed by $rp^{n-1} + rp^{n-2} + rp^{n-3} \dots + r$.

Now the Sum of this Progression is $\frac{rp^n - r}{p - 1} = m$ the Amount required, or $\text{Log. } m = \text{Log. } (rp^n - r) - \text{Log. } (p - 1)$.

LXIV.

To apply this Formula to an Example, let a yearly Rent of 30*l.* be unpaid 9 Years, and let its Amount at the Rate of 6 per Cent per Annum, compound Interest be required.

Here $r = 30$, $n = 9$ & $p = 1,06$ & $\text{Log. } m = \text{Log. } 20,68353 - \text{Log. } 0,06 = 1,315626 - 8,778151 = 2,537475$, and consequently $m = 344*l.* 14*s.* 6*d.*$

LXV.

Let it be required to find what Annuity r forborn n Years, will raise a Stock m , at a Rate i per Cent compound Interest.

Here m , p , n are given to find r , which is equal to $\frac{m \times (p - 1)}{p^n - 1}$, or $\text{Log. } r = \text{Log. } (mp - m) - \text{Log. } (p^n - 1)$

Let $a = 344*l.* 5*s.*$ $n = 8$, and $i = 3*l.* 10*s.*$ Consequently $p = 1,035$, wherefore $\text{Log. } r = \text{Log. } 12,04875 - \text{Log. } 0,316803 = 1,080908 - 9,500785 = 1,580123$, and consequently $r = 38,0297*l.* = 38*l.* 0*s.* 7*d.* per Annum.$

LXVI.

Let it be required to find in what Time n , an Annuity of r pounds per Annum, will raise a Stock m , at a Rate i per Cent. Compound Interest.

Here m , p , r are given to find n , which is equal to $\frac{\text{Log. } (mp + r - m) - \text{Log. } r}{\text{Log. } p}$

Let $m = 344,25$, $r = 38,0297$, and $i = 3*l.* 10*s.*$ hence $p = 1,035$ and $n = \frac{\text{Log. } 50,07845 - \text{Log. } 38,0297}{\text{Log. } 1,035} = 8$ Years, the Time required.

LXVII.

As all freehold or real Estates may be considered as Annuities to continue forever, to obtain their true Value a , it suffices to put the Quantity $a p^n - r p^{n-1} - r p^{n-2} - r p^{n-3} - r p^{n-4} \dots - r$, or

$a - \frac{r}{p} - \frac{r}{p^2} - \frac{r}{p^3} \dots - \frac{r}{p^n}$ continued to an infinite Number of Terms equal to nothing, which on Account of all the Terms that are multiply'd by r being a geometrical Progression infinitely decreasing, and consequently equal to $\frac{1}{p-1}$ is reduced to $a - \frac{r}{p-1} = 0$ which gives $a = \frac{r}{p-1}$.

Let it be required, for Example, to find the Value of a freehold Estate of 25*l.* per Annum, allowing 5*l.* 10*s.* per Cent, &c. Compound Interest to the Buyer.

Resolution of the various questions relating to annuities in arrears.

Here are given $r = 25$ l. $i = 5$ l. 10s. and consequently $p = 1,055$ wherefore, $a = 454$ l. 10s. 10 $\frac{1}{2}$ d. the Value required.

LXVIII.

After having solved all the Difficulties that could occur in the Solution of Problems of every Degree, producing Equations consisting of two Terms, Order requires that we should proceed to explain how to solve Problems producing Equations of every Degree consisting of three Terms, but the Analysts have not as yet found a general Method for every Problem of this Kind, the following in its full Extent, produces Equations of every Degree, consisting of three Terms that have been compleatly solved.

LXIX.

Problems
producing
equations
consisting of
three terms
solved by
the method
for equa-
tions of the
second de-
gree.

Let it be required to find two Numbers, whose Product is a , and the Difference of their n^{th} Powers is b .

To solve this Question let the lesser Number be x , then the greater is $\frac{a}{x}$,

Now, by the Question $b = \frac{a^n}{x^n} - x^n$, or $b x^n = a^n - x^{2n}$, consequently

$x^{2n} + b x^n = a^n$, to resolve this Equation, I add the Square of $\frac{b}{2}$ to both Sides, and there results $x^{2n} + b x^n + \frac{1}{4} b^2 = a^n + \frac{1}{4} b^2$ whose Root is $x^n \pm \frac{1}{2} b = \pm \sqrt{a^n + \frac{1}{4} b^2}$, &c $x^n = -\frac{1}{2} b \pm \sqrt{a^n + \frac{1}{4} b^2}$, wherefore $x = \sqrt[n]{-\frac{1}{2} b \pm \sqrt{a^n + \frac{1}{4} b^2}}$.

By means of this Formula, all Equations may be resolved consisting of three Terms, in the first of which, the Index of the unknown Quantity x , is double of the Index of x in the second Term, and the third Term is a given Quantity, and it is easy to perceive, that all Equations in this general Formula $x^{2n} + b x^n = a^n$ cannot have more than four real Roots, and will only have two when n is an odd Number.

LXX.

Example of
the forego-
ing method.

To apply this general Solution to an Example, let it be required to find two Numbers whose Product is 15, and the Difference of their Squares is 16.

Here $a = 15$, $b = 16$, and $n = 2$, consequently
 $x = \pm \sqrt{[-8 \pm \sqrt{289}]} = \sqrt{\pm \sqrt{-8 \pm 17}} = \pm \sqrt{9} = 3$.

LXXI.

Another
example.

Let $x^4 - b b x^2 = b b c c$ be proposed to be solved, adding on both Sides $\frac{1}{4} b^4$ Square of one half of the Coefficient of $x x$, there will result $x^4 - b b x x + \frac{1}{4} b^4 = \frac{1}{4} b^4 + b b c c$, whose Square Root is $x^2 - \frac{1}{2} b b = \pm b \sqrt{(\frac{1}{4} b b + c c)}$, whence is deduced $x x = \frac{1}{2} b b \pm b \sqrt{(\frac{1}{4} b b + c c)}$, and consequently $x = \pm \sqrt{[\frac{1}{2} b b \pm b \sqrt{(\frac{1}{4} b b + c c)}]}$ susceptible, of two real Values, and of two imaginary ones. The two first are $x = \pm \sqrt{[\frac{1}{2} b b + b \sqrt{(\frac{1}{4} b b + c c)}]}$, the two others are

$x = \pm \sqrt{\frac{1}{2}bb - b\sqrt{\frac{1}{2}bb + cc}}$ which are necessarily imaginary, because $b\sqrt{\frac{1}{2}bb + cc}$ is greater than $\frac{1}{2}bb$.

LXXII.

Let the Equation $x^6 - 2a^2bx^3 = a^6$ be proposed, adding to both Sides a^4bb , there results $x^6 - 2aabbx^3 + a^4bb = a^4bb + a^6$, whose Third example.
square Root is $x^3 - aab = a^2\sqrt{a^2 + b^2}$, or $x^3 = a^2b \pm a^2\sqrt{a^2 + b^2}$, and consequently $x = \sqrt[3]{a^2b \pm a^2\sqrt{a^2 + b^2}}$, susceptible of two real Values, one positive and the other negative; the other four Roots are imaginary.

LXXIII.

Let the Equation $x^4 - (aa + bb)xx = -aabb$ be proposed, adding to both Sides $\frac{1}{2}a^4 + \frac{1}{2}aabb + \frac{1}{2}b^4$, there will result $x^4 - (aa + bb)xx + \frac{1}{2}a^4 + \frac{1}{2}aabb + \frac{1}{2}b^4 = \frac{1}{2}a^4 - \frac{1}{2}aabb + \frac{1}{2}b^4$, both Members being perfect Squares, extracting therefore the square Root, there will result Fourth example.
 $xx - \frac{1}{2}aa - \frac{1}{2}bb = \pm \frac{1}{2}aa \mp \frac{1}{2}bb$, which gives $xx = aa$ and $xx = bb$, that is, $x = \pm a$ and $x = \pm b$, which are the four Roots of the Equation $x^4 - (aa + bb)xx = -aabb$.

LXXIV.

Let the Equation $x^4 - (2gb + 4ff)xx = -ggbb$ be proposed to be solved, adding to both Sides the Square of half the Coefficient of the second Term, and extracting the square Root of both Members, there results Fifth example.
 $x^2 - gb - 2ff = \pm 2f\sqrt{gb + ff}$ which gives $x = \pm \sqrt{gb + 2ff \pm 2f\sqrt{gb + ff}}$.

Now on Examination I find that this Quantity is a perfect Square, that of $f \pm \sqrt{ff + gb}$, for the Term $2f\sqrt{gb + ff}$ is double of the Product of f into $\sqrt{gb + ff}$, and the Quantity $gb + 2ff$ contains the Square of f , and the Square $ff + gb$ of the radical Part.

Whence the foregoing Value of x is reduced to $f \pm \sqrt{ff + gb}$ and $-f \pm \sqrt{ff + gb}$, and those are the four Roots of the Equation $x^4 - 2gbxx - 4ffxx = -ggbb$.

LXXV.

In Order to discover when the Roots of Surds may be expressed exactly by other Surds, as in the foregoing Example, and how those Roots may be found, I observe that the Root of a Quantity composed of two Parts, one of which is rational and the other a radical Quantity of the second Degree, should itself consist of two Parts, and that one of them at least should be a radical Quantity.

Let then $A + B$ express in general the proposed Quantity, A denoting the rational Quantity, and B a radical of the second Degree, and let $p + q$ express the Root sought.

I now observe, that whether p denotes the radical Quantity, or whether it be expressed by q , or that p and q are both Radicals, in the Square

$p^2 + 2pq + q^2$, the Term $2pq$ can only be a Radical; comparing therefore this Square with the given Quantity, $2pq$ will represent B and $p^2 + q^2$, A ; that is, we have the two Equations $p^2 + q^2 = A$ and $2pq = B$ for finding p and q . From the second I deduce $p = \frac{B}{2q}$ which substituted

Method of finding the square root of quantities partly rational and partly irrational.

in the first gives $\frac{B^2}{4q^2} + q^2 = A$, or $q^4 - Aq^2 = -\frac{B^2}{4}$ or $q^2 - \frac{1}{2}A = \pm \frac{1}{2}\sqrt{AA - BB}$, consequently $q = \pm \sqrt{[\frac{1}{2}A \pm \frac{1}{2}\sqrt{AA - BB}]}$. Substituting afterwards this Value of q in the Equation $p^2 + q^2 = A$, or $p = \pm \sqrt{A - q^2}$ we have $p = \pm \sqrt{[\frac{1}{2}A \mp \frac{1}{2}\sqrt{AA - BB}]}$; from whence it follows, that the Root required of the Quantity $A + B$ is $\pm \sqrt{[\frac{1}{2}A \pm \frac{1}{2}\sqrt{AA - BB}]} \pm \sqrt{[\frac{1}{2}A \pm \frac{1}{2}\sqrt{AA - BB}]}$, or simply $\pm \sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{AA - BB}]} \pm \sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{AA - BB}]}$.

As to the Signs which should affect the two Parts $\sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{AA - BB}]}$ and $\sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{AA - BB}]}$ of the Root required of $A + B$, they are the same when the radical B is positive, but different when B is negative; for it is easy to perceive that in general $p + q$, or $-p - q$ being the Root of $A + B = pp + qq + 2pq$, $p - q$, or $-p + q$ is the Root of $A - B = pp + qq - 2pq$.

LXXVI.

That the Quantity $\pm \sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{AA - BB}]} \pm \sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{AA - BB}]}$ found for the Root of the Quantity $A + B$ may be expressed by a Binomial Surd, the Quantity $A^2 - B^2$ must be a perfect Square; which will never fail to happen as often as the Root of $A + B$ can be extracted: To make this appear we have only to observe that $p - q$ is expressed by $\sqrt{A - B}$ at the same Time that $p + q$ is expressed by $\sqrt{A + B}$; from whence it is easy to conclude, that $\sqrt{A - B} \times \sqrt{A + B}$, or $\sqrt{A^2 - B^2} = (p - q) \times (p + q)$, or $p^2 - q^2$ a commensurable Quantity.

LXXVII.

To shew the Application of the foregoing Method, let the Quantity $aa + 2c\sqrt{aa - cc}$ be proposed, comparing it with $A + B$ we have $a^2 = A$ and $B = 2c\sqrt{aa - cc}$; and consequently $\sqrt{A^2 - B^2} = aa - 2cc$, whence $\sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{A^2 - B^2}]} = \sqrt{a^2 - c^2}$; in like Manner, $\sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{A^2 - B^2}]} = c$; that is, the Root sought is $c + \sqrt{aa - cc}$, or $-c - \sqrt{aa - cc}$.

Application of the foregoing method to an example.

LXXVIII.

If it was proposed to extract the square Root of $16 + 6\sqrt{7}$, putting $A = 16$ and $B = 6\sqrt{7}$, there results $\sqrt{AA - BB} = 2$, and consequently $\sqrt{[\frac{1}{2}A + \frac{1}{2}\sqrt{AA - BB}]} = 3$, and $\sqrt{[\frac{1}{2}A - \frac{1}{2}\sqrt{AA - BB}]} = \sqrt{7}$, whence $3 + \sqrt{7}$, or $-3 - \sqrt{7}$ will be the Root required.

LXXIX.

Let the Root of $ap - 2a\sqrt{ap - aa}$ be required, putting $A = ap$ and $B = -2a\sqrt{ap - aa}$, there results $\sqrt{AA - BB} = ap - 2aa$ and $\sqrt{\frac{1}{2}A + \frac{1}{2}\sqrt{AA - BB}} = \sqrt{ap - aa}$: In like Manner, $\sqrt{\frac{1}{2}A - \frac{1}{2}\sqrt{AA - BB}} = a$; that is, the Root sought will be $\sqrt{ap - aa} - a$, or $a - \sqrt{ap - aa}$. Another example.

LXXX.

If the Quantity $b^2 - ab + \frac{1}{4}a^2 + 2\sqrt{ab^3 - 2a^2b^2 + \frac{1}{2}a^3b}$ was proposed, then $A = b^2 - ab + \frac{1}{4}aa$, $B = 2\sqrt{ab^3 - 2a^2b^2 + \frac{1}{2}a^3b}$, and $\sqrt{A^2 - B^2} = \sqrt{b^4 - 6ab^3 + \frac{19}{2}a^2b^2 - \frac{3}{2}a^3b + \frac{1}{16}a^4}$ Third example.
 $= bb - 3ab + \frac{1}{4}aa$, which will give $\sqrt{\frac{1}{2}A + \frac{1}{2}\sqrt{A^2 - B^2}} = \sqrt{bb - 2ab + \frac{1}{4}aa}$, and $\sqrt{\frac{1}{2}A - \frac{1}{2}\sqrt{A^2 - B^2}} = \sqrt{ab}$; whence the Root sought is $\sqrt{ab} + \sqrt{bb - 2ab + \frac{1}{4}aa}$, or $-\sqrt{ab} - \sqrt{bb - 2ab + \frac{1}{4}aa}$.

LXXXI.

After the Analysts had found out a Method for discovering among the Quantities partly rational and partly irrational such as were perfect Squares, they, without Doubt, sought also how to distinguish those that were perfect Cubes or higher Powers, as being absolutely necessary to compleat the Resolution of Equations contained in the Formula $x^{2m} + ax^m = b$, or $x = \sqrt[m]{-\frac{1}{2}a \pm \sqrt{b + \frac{1}{4}aa}}$. Let us first examine what must be done when $m = 3$, that is, to find the cube Root of any Quantity $A + B$ in which A is rational, and B a Radical of the second Degree. Method of finding the cube root of quantities, partly rational and partly irrational.

I observe first, that the cube Root of a Quantity of this Nature cannot include more than one Radical of the second Degree, for it is easy to perceive, that the Cube of a Quantity such as $\sqrt{m} + \sqrt{n}$ which contains two of those Radicals, will necessarily be affected by those same Radicals.

I observe next, that the same cube Root sought cannot contain any other kind of Radicals except a cubical Radical, and which is common to the two Parts of the Root. Such would be, for Example, the Quantity of $f^3\sqrt{m} + \sqrt{g} \times \sqrt[3]{m}$, whose Cube $mf^3 + 3mf\sqrt{g} + (3mff + mg)\sqrt{g}$, as the proposed Quantity $A + B$ consists of a commensurable Quantity, and a radical Part of the second Degree.

Now let $p + q$ express the Root sought q being the Part affected by the Radical of the second Degree, whether it be affected as well as p by a cubical Radical or not.

It is manifest that the Cube $p^3 + 3p^2q + 3pq^2 + q^3$ will not contain any other Radical, but the Radical of the second Degree, which is in q , and that only the Terms $3p^2q$ and q^3 will be affected by this Radical; I therefore compare those two Terms with the given Quantity B and the two others with A , which gives the Equations $A = p^3 + 3pq^2$ and $B = 3p^2q + q^3$.

To solve those two Equations we must first exterminate one of the unknown Quantities p or q , which may be effected by any of the Methods explained in the second Chapter, but much easier by observing, that if $p+q$ is the cube Root of $p^3 + 3p^2q + 3pq^2 + q^3$, the first Part of which $p^3 + 3pq^2 = A$, and the second $3p^2q + q^3 = B$, $p-q$ will necessarily be the cube Root of $A-B$; which is in this Case $= p^3 + 3pq^2 - 3p^2q - q^3$, whence $A+B = (p+q)^3$ and $A-B = (p-q)^3$; consequently, $(A+B)(A-B) = (p+q)^3(p-q)^3$, or $AA-BB = (pp-qq)^3$ or $\sqrt[3]{(AA-BB)} = p^2 - q^2$, or $n = p^2 - q^2$, in which n is given, and must be either a commensurable Quantity, or a simple cubical Radical. From the Equation $n = p^2 - q^2$, or $q^2 = p^2 - n$, and from the Equation $A = p^3 + 3pq^2$ is deduced $4p^3 - 3pn - A = 0$, by Means of which Equation the Part p of the cubical Root sought may be obtained, and when found, the second Part of the Root sought will be obtained by Means of the Equation $q = \sqrt{p^2 - n}$.

As to the radical Sign it will be positive, if the Radical of the proposed Quantity has the Sign $+$, and negative, if the Radical of the proposed Quantity has the Sign $-$; for it is easy to perceive, that the radical Part of the Cube of $p+q$ which is $(3pp+qq) \times q$ will always have the same Sign as q .

If the Root of the proposed Quantity $A+B$ does not contain a cubical Radical affecting all its Terms, n or $\sqrt[3]{(AA-BB)}$ will be a commensurable Quantity, and consequently, the Equation $4p^3 - 3pn - A = 0$ will not be affected by any radical Quantity, and as p in this Case will be rational, we cannot fail of finding it by investigating all the Divisors of this Equation.

If the Root sought should have its two Parts affected by a cubical Radical, which will happen as often as $AA-BB$ is not a perfect Cube, then $AA-BB$ must be multiplied by a Quantity Q , such as that the Product $A^2Q - B^2Q$ may be a perfect Cube; and now instead of $A+B$ we must extract the cube Root of $(A+B) \times \sqrt[3]{Q}$, which being found and divided by the cube Root of $\sqrt[3]{Q}$ will give the Root required.

LXXXII.

Application
of the fore-
going me-
thod to an
example.

To shew the Application of this Method, let it be required to find the cube Root of the Quantity $7a^3 - 3a^2b + (5aa-ab) \times \sqrt{(2aa-ab)}$.

Comparing this Quantity with $A+B$ I find $7a^3 - 3a^2b = A$, $[5aa - ab \sqrt{(2aa-ab)}] = B$; and consequently, $A^2 - B^2 = -a^6 + 3a^5b - 3a^4b^2 + a^3b^3$, and n or $\sqrt[3]{(A^2 - B^2)} = -aa + ab$, substituting this Value of n , as also that of A in the Equation $4p^3 - 3pn - A = 0$, there results $4p^3 + 3paa - 3pab - 7a^3 + 3a^2b$, which is divisible by $p-a$, that is, the Value of p is a . Substituting this Value of p in the Equation $q = \sqrt{p^2 - n}$, there results $q = \sqrt{(2aa-ab)}$, whence the Root sought is $a + \sqrt{(2aa-ab)}$.

LXXXIII.

Let it be required to find the cube Root of $2aac - abc - bbc - [2a - b \sqrt{aacc - bbcc}]$, putting $2aac - abc - bbc = A$ and $B = -[2a - b \sqrt{aacc - bbcc}]$, there results $A^2 - B^2 = 2b^4cc - 2ab^3cc$, which is not a perfect Cube. To discover what will make it a perfect Cube, I resolve it into its Factors, and it becomes $2 \times cc \times (b - a) \times b^3$; whence it appears that by multiplying by

Another example.

$\frac{4 \times (b - a)^2}{cc}$ it will become a perfect Cube, that of $2 \times (b - a) \times b$, or of $2bb - 2ab$; and consequently if the proposed Quantity be multiplied by $\frac{2b - 2a}{c}$ squareRoot of $\frac{4 \times (b - a)^2}{cc}$, there will result a new Quantity $[(2aa - ab - bb) \times (2b - 2a) - (2a - b) \times (2b - 2a) \times \sqrt{aacc - bbcc}]$, the first Part of which representing A , and the second B , will give for $\sqrt[3]{(AA - BB)}$, that is, for n , $2bb - 2ab$. Whence the Equation $4p^3 - 3pn - A = 0$ will be transformed into $4p^3 - 3p \times (2bb - 2ab) + (bb + ab - 2aa) \times (2b - 2a) = 0$, which is divisible by $p + a - b$, that is, $p = b - a$, which Value being substituted in $q = \sqrt{pp - n}$ there results $q = \sqrt{aa - bb}$; wherefore $b - a - \sqrt{aa - bb}$ will be the cube Root of the Product of the proposed Quantity into $\frac{2b - 2a}{c}$; wherefore $\frac{[b - a - \sqrt{aa - bb}] \times \sqrt[3]{c}}{\sqrt[3]{(2b - 2a)}}$ is the cube Root required of the proposed Quantity.

LXXXIV.

If the Terms of the Quantity whose cube Root is to be extracted have Divisors, they are first to be reduced to the same Denominator, and then the cube Root of the Numerator is to be divided by that of the Denominator.

LXXXV.

If it be proposed to extract the Root of a numerical Quantity partly rational and partly irrational, it may be performed much easier than by the foregoing Method.

For supposing first, that the Root sought should not be affected by a cubical Radical, but should consist of an Integer, and a radical Part, also an Integer Number, because $p - q = \sqrt[3]{(A - B)}$ when $p + q = \sqrt[3]{A + B}$ we will have $p = \frac{\sqrt[3]{(A - B)} + \sqrt[3]{(A + B)}}{2}$; whence the rational Part of the Root will be obtained by computing in the nearest integer Numbers, $\sqrt[3]{(A - B)}$ and $\sqrt[3]{(A + B)}$, and taking the one half of these two Numbers. For assuming for $\sqrt[3]{(A - B)}$ and for $\sqrt[3]{(A + B)}$ the nearest integer Numbers, neglecting the fractional Parts, the Error that can arise in each of these Quantities will be less than $\frac{1}{2}$, and consequently the Integer Number that results for the Value of $\frac{\sqrt[3]{(A - B)} + \sqrt[3]{A + B}}{2}$, that is, for p , will not

Method of finding the cube roots of numeral quantities, partly rational and partly irrational.

differ from its true Value by Unity, and as this Value of p should be an Integer, it will be exactly determined by this Means.

Having thus obtained the Value of p , and that of n or of $\sqrt[3]{(AA-BB)}$ being given, substituting the Values of p and n in $q = \sqrt{(pp-n)}$, the second Part of the cube Root required will be obtained.

If the Root sought should have its two Terms affected by a cubical Radical, which happens as often as $A^2 - B^2$ is not a perfect Cube, then as in specious Quantities we must extract the cube Root of $A\sqrt{2} + B\sqrt{2}$, 2 being a Number such as that $A^2 2 - B^2 2$ may be a perfect Cube, and having found this Root, divide it by the cube Root of $\sqrt{2}$.

LXXXVI.

To shew the Application of this Method, let the Cube Root of $7 + 5\sqrt{2}$ be required, putting $A=7$, $B=5\sqrt{2}$, I find n or $\sqrt[3]{(A^2 - B^2)} = -1$, I afterwards observe that the Value of $\sqrt[3]{(A+B)}$, or of $\sqrt[3]{(7 + 5\sqrt{2})}$ is nearer 2 than 3. I therefore assume 2 for its Value, in like Manner I observe that $\sqrt[3]{(A-B)}$, or $\sqrt[3]{(7 - 5\sqrt{2})}$ is nearer 0 than 1, I assume 0 for this

Application of the foregoing method to an example.

Quantity, whence p , or $\frac{\sqrt[3]{(A+B)} + \sqrt[3]{(A-B)}}{2} = 1$. I substitute this Value of p in $q = \sqrt{(pp-n)}$, and I find $q = \sqrt{2}$, whence I conclude, that if the cube Root of the proposed Quantity $7 + 5\sqrt{2}$ can be extracted, it will be $1 + \sqrt{2}$, and in Effect, $1 + \sqrt{2}$ raised to the Cube, gives $7 + 5\sqrt{2}$.

LXXXVII.

Let the cube Root of $5 + 3\sqrt{3}$ be required. Here $AA-BB = -2$, but 2 not being a perfect Cube, I examine what Number $\sqrt{2}$, by which $5 + 3\sqrt{3}$ being multiplied, will render $AA-BB$ a perfect Cube, and finding 2 for this Number, I investigate the cube Root of $10 + 6\sqrt{3}$, and I find $n = -2$, and the nearest Value of $\frac{\sqrt[3]{(A+B)} + \sqrt[3]{(A-B)}}{2}$ or of $\frac{\sqrt[3]{(10 + 6\sqrt{3})} + \sqrt[3]{(10 - 6\sqrt{3})}}{2}$ in Integers to be 1, substituting those Values of p and n in $q = \sqrt{(pp-n)}$ I find $q = \sqrt{3}$; and finding upon Examination that $p + q$ or $1 + \sqrt{3}$ is the cube Root of $10 + 6\sqrt{3}$, I conclude that $\frac{1 + \sqrt{3}}{\sqrt[3]{2}}$ is the cube Root sought of $5 + 3\sqrt{3}$.

Another example.

LXXXVIII.

The foregoing method rendered more simple.

The Computation for determining p may be rendered more simple by observing that instead of $\frac{\sqrt[3]{(A+B)} + \sqrt[3]{(A-B)}}{2}$ we may put $\frac{\sqrt[3]{(A+B)} + \frac{n}{\sqrt[3]{(A+B)}}}{2}$, Since n or $\sqrt[3]{(A^2 - B^2)} =$

$\sqrt[3]{(A+B)} \times \sqrt[3]{(A-B)}$. Which is simpler than $\frac{\sqrt[3]{(A+B)} + \sqrt[3]{(A-B)}}{2}$ because it is easier to divide the Number n already found by $\sqrt[3]{(A+B)}$ than to compute separately $\sqrt[3]{(A-B)}$.

LXXXIX.

To shew the Application of this new Formula, let it be applied to the Example of Art. LXXXV. where A was $= 7$, and $B = 5\sqrt{2}$. After having found, as in the same Article, that $n = -1$, and that $\sqrt[3]{(A+B)}$ in the nearest integer Number, to be 2, instead of searching, as in the same Article, the cube Root of $7 - 5\sqrt{2}$ in the nearest integer Number, I divide n or -1 by the Value 2 of $\sqrt[3]{(A+B)}$ which gives $-\frac{1}{2}$ which being substituted in the foregoing Formula, which expresses the Value of p ,

$\sqrt[3]{A+B} + \frac{n}{\sqrt[3]{(A+B)}}$ there results $\frac{2 - \frac{1}{2}}{2}$, or 1 (assuming the nearest integer Number) for the Value of p , as found in the above-mentioned Article by the Formula $\frac{\sqrt[3]{(A+B)} + \sqrt[3]{(A-B)}}{2}$, the Remainder of the Operation is performed as in the same Article.

XC.

It is to be observed however, that the new Formula may lead into Error, if A and B have not the same Sign, for when these Quantities have different Signs, the real Value of $\sqrt[3]{(A+B)}$ may be so small with respect to n , that the nearest integer Number assumed for this Value, will give for

$$\sqrt[3]{(+B)} + \frac{n}{\sqrt[3]{(A+B)}}$$

a Number that will differ from its true Value by several Units. For Example, if it was required to extract the cube Root of $45 - 29\sqrt{2}$, putting $A = 45$ and $B = -29\sqrt{2}$, the Value of $\sqrt[3]{(A+B)}$ or of $\sqrt[3]{(45 - 29\sqrt{2})}$ in the nearest integer Number will be 1, and as $\sqrt[3]{(A^2 - B^2)}$ is here equal to 7, the Value of p , or of

$$\sqrt[3]{(A-B)} + \frac{n}{\sqrt[3]{(A+B)}}$$

will be found in this Case to be 4, which in Reality is only equal to 3, as may be proved by the Expression

$$\frac{\sqrt[3]{(A+B)} + \sqrt[3]{(A-B)}}{2}$$
 of the foregoing Method.

But provided that this new Method of finding the Value of p is exact, when A and B have the same Sign, it is of little Consequence whether it be applicable or not, when those Quantities have different Signs. For it is easy to perceive that in this Case, we have only to suppose A and B to be

This new method defective when A and B have different signs.

ELEMENTS OF

both positive, and extract the Root $p + q$, then give p the same Sign that A has, and q the same Sign that B has.

It remains then to examine, whether when A and B have the same Sign, or what amounts to the same Thing, if A and B be both positive, we

What is to
be done in
this case.

may substitute in $\frac{3\sqrt[3]{(A+B)} + \frac{n}{3\sqrt[3]{(A+B)}}}{2}$ in the Room of

$3\sqrt[3]{(A+B)}$ its Value in the nearest integer Number, so that the Value which will result will not differ from its true Value by Unity, to be assured of it, let the Value of $3\sqrt[3]{(A+B)}$ in the nearest integer Number be supposed to differ from its true Value by $\frac{1}{2}$, which is more than can ever hap-

pen. In this Case, the Value of p will be $\frac{3\sqrt[3]{(A+B)} \pm \frac{1}{2} + \frac{n}{3\sqrt[3]{(A+B)} \pm \frac{1}{2}}}{2}$

to shew that this Expression of p can not differ from its true Value by Unity, let $p + q$ be substituted in the Room of $3\sqrt[3]{(A+B)}$ and $pp + qq$

in the Room of n , and there will result $\frac{p + q \pm \frac{1}{2} + \frac{pp + qq}{p + q \pm \frac{1}{2}}}{2}$ from

which deducting p , the Remainder $\frac{\pm q + \frac{1}{2}}{2p + 2q \pm 1}$ will express the difference between the Value of p in the nearest Integer, and its true Value, now, it is manifest, that this Quantity can never equal $\frac{1}{2}$, for in the first

Expression $\frac{q + \frac{1}{2}}{2p + 2q + 1}$ that it includes, the Numerator $q + \frac{1}{2}$ being less than the one half of $2q + 1$, will be much less than the one half of

$2p + 2q + 1$: and in the second Expression $\frac{-q + \frac{1}{2}}{2p + 2q - 1}$ which it likewise includes, the Numerator $-q + \frac{1}{2}$, being less than the one Half of $2q$, will be also less than the one half of $2q + 2p - 1$.

Whence the Value of p determined by the foregoing Method, cannot differ from its true Value by Unity, and consequently as often as a Quantity $A + B$, partly rational and partly irrational, (in which only Integers enter either under the radical Sign, or before this Sign) will have a cube Root $p + q$, expressed in Integers, this Root may be found by the foregoing Method.

XCI.

But if the Quantity $A + B$, though clear of Fractions, shall have a fractional Root, such as the Quantity $2 + \sqrt{5}$, whose cube Root is $\frac{1}{2} + \frac{1}{2}\sqrt{5}$, it cannot be found by either of the two foregoing Methods.

To remedy this Inconveniency, the Analysts have sought in a direct Manner all the fractional Roots, which when cubed, become integer Numbers.

Case in
which the
two fore-
going me-
thods fail.

Let all those Roots be expressed by $\frac{p}{m} + \frac{q}{n}$; p and m expressing Integers that have no common Divisor, and q the Root of an integer Number, which will admit of no Reduction with the Number n . Raising this Quantity to the Cube, there will result $\frac{p^3}{m^3} + \frac{3pq^2}{mn^2}$ for the rational Part, and $\left[\frac{3p^2}{nm^2} + \frac{q^2}{n^3} \right] \times q$ for the irrational Part. Now, by the Conditions of the Problem, the first Part $\frac{p^3}{m^3} + \frac{3pq^2}{mn^2}$, and the Coefficient $\frac{3p^2}{nm^2} + \frac{qq}{n^3}$ of the second Part should be integer Numbers.

How to remedy this deficiency.

Let $\frac{3p^2}{m^2n} + \frac{q^2}{n^3}$ be put equal to b , which will express an integer Number. Then $q^2 = bn^3 - \frac{3p^2n^2}{m^2}$; but this Quantity by Hypothesis should be an integer Number; consequently $\frac{3p^2n^2}{m^2}$ should be also an Integer; wherefore $\frac{3n^2}{m^2}$ should be likewise an Integer, and consequently n should be a Multiple of m .

Now let $n = ml$, consequently $q^2 = bm^3l^3 - 3p^2l^2$, substituting those Values of n and q^2 in $\frac{p^3}{m^3} + \frac{3pq^2}{mn^2}$, this Quantity will become after the Reductions $\frac{8p^3}{m^3} + 3pbl$, which should be an Integer, but p and m having no common Divisor, this Quantity cannot be an integer Number, unless m be 1 or 2. As to n or ml , it is easy to perceive that it should be equal to m , because the Equation $q^2 = bm^3l^3 - 3p^2l^2$ will give $\frac{q}{l} = \sqrt{(bm^3l - 3p^2)}$ and consequently $\frac{q}{ml}$, that is,

$\frac{q}{n} = \frac{1}{m} \sqrt{(bm^3l - 3p^2)}$ whence it appears that the second Part of the Root can have no other Denominator than what the first has, and consequently must be either 2 or 1.

When therefore the Root of a Quantity $A + B$, the rational Part of which A , and the irrational Part B is expressed in Integers, cannot be found by the foregoing Method, we have no more to do than to multiply this Quantity by 8, and investigate by the foregoing Method the Cube Root of the Quantity resulting from this Multiplication, and if this does not succeed, no farther Trial is to be made: if it succeeds, the one Half of the cube Root found, will be the Root required.

XCII.

When the Number A and the Radical B are Fractions, they are to be reduced to the same Denominator, and the Root of the Numerator and Denominator are to be extracted separately. Thus, to extract the cube Root of $\sqrt{242 - 12\frac{1}{2}}$, this reduced to a common Denominator is $\frac{\sqrt{968 - 25}}{2}$, and the Roots of the Numerator and Denominator found

separately give the Root $\frac{2\sqrt[3]{2} - 1}{3\sqrt[3]{2}}$. And if you are to extract the cube

Root out of $3\sqrt[3]{3993} + 6\sqrt[3]{17578125}$, divide its Parts by the common Divisor $3\sqrt[3]{3}$, and the Quotient being $11 + \sqrt[3]{125}$, the cube Root of the proposed Quantity will be found by taking the cube Roots of $3\sqrt[3]{3}$, and of $11 + \sqrt[3]{125}$, and multiplying them into each other.

XCIII.

What is to be done when the cube root should be the Sum of two radicals.

If it be proposed to extract the cube Root of a Quantity composed of two Radicals of the second Degree, whether it be a specious or numeral Quantity, we have no more to do than to multiply it by the Cube of one of the Radicals that it contains, the Product that results being a Quantity partly rational and partly irrational, Its Root may be extracted by the foregoing Method, which when found is to be divided by the Radical, by the Cube of which the proposed Quantity was multiplied.

XCIV.

How the root denominated by an even number is extracted.

If the fourth Root of a Quantity as $A + B$ was required; first, the square Root is to be extracted, which, if it cannot be found, much less can the Root of the fourth Power be found. In like Manner, all even Roots are to be sought, continually depressing them by extracting the square Root.

XCV.

In general, if the m^{th} Root of the Quantity $A + B$ was required, then $(p + q)^m = A + B$, and raising $p + q$ to the Power m , it will appear

that A is the Sum of all the odd Terms $p^m, \frac{m \cdot (m-1)}{2} q^2 p^{m-2},$

Investigate of the rule for extracting the root of any power of quantities partly rational and partly irrational.

$\frac{m(m-1)(m-2)(m-3)}{2 \times 4} q^4 p^{m-4}, \&c.$ and that B is the Sum of

all the even Terms $m q p^{m-1}, \frac{m(m-1)(m-2)}{2 \times 3} q^3 p^{m-3}.$

$\frac{m(m-1)(m-2)(m-3)(m-4)}{2 \times 3 \times 4 \times 5} q^5 p^{m-5}, \&c.$ whence, when

$A + B$ is the m^{th} Power of $p + q$, $A - B$ will be the m^{th} Power of $p - q$; consequently $\sqrt[m]{A - B} \times \sqrt[m]{A + B} = (p + q) \times (p - q),$

or $pp - qq = n$, or $q = \sqrt{p^2 - n}$, also $\frac{\sqrt[m]{A+B} + \sqrt[m]{A-B}}{2}$

$$= \frac{p+q+p-q}{2}, \text{ or } p = \frac{\sqrt[m]{(A+B)} + \frac{n}{\sqrt[m]{(A+B)}}}{2}, \text{ because}$$

$\sqrt[m]{(AA-BB)} = \sqrt[m]{(A+B)} \times \sqrt[m]{(A-B)}$, and thus from the Composition of the Binomial $A+B$ we are lead to its Resolution, when A is rational, and A^2-B^2 is a perfect m Power.

XCVI.

If the m^{th} Root of A^2-B^2 cannot be taken, multiply A^2-B^2 by a Number $\mathcal{Q}$, such as that the Product may be the least perfect m Power $n^m = A^2 \mathcal{Q} - B^2 \mathcal{Q}$; and now instead of $A+B$ extract the m Root of $(A+B) \times \sqrt[m]{\mathcal{Q}}$, as above, and when found, divide it by $\sqrt[m]{\mathcal{Q}}$, and the Quotient will be the Root required.

To find this least perfect Power m (n^m) that shall be a Multiple of A^2-B^2 by a whole Number $\mathcal{Q}$, let this given Number A^2-B^2 be represented by the Product $a^p b^q d^r f^s$, whose single Divisors let be $a a a \dots b b b \dots d, f$, and the Product of these Divisors raised to the Power m , which is $a^m b^m d^m f^m$ divided by $a^p b^q d^r f^s$, will give the Quotient $a^{m-p} b^{m-q} d^{m-r} f^{m-s} = \mathcal{Q}$ a whole Number, provided some Index as n or p be not greater than m ; if it is, take, instead of the single Divisor, a or b , a^2 or b^2 , a^3 or b^3 , &c. till there be no negative Index in the Quotient, that is, till $\mathcal{Q}$ be a whole Number.

XCVII.

Having shewn how Equations are solved by compleating them into perfect Squares, the Square on the right Hand Side being a Quantity entirely known, we shall now proceed to explain how Equations are compleated into perfect Squares, the Squares on both Sides being affected by the unknown Quantity. Let the Equation $x^4 - 9x^3 + 15x^2 - 27x + 9 = 0$; be proposed to be solved according to the Rules of the Extraction of the square Root, I find the two first Terms of the Root of this Quantity to be $x^2 - \frac{9}{2}x$. Let the whole Root be expressed by

Resolution of equations of even dimensions by compleating them in to perfect squares explained by an example.

$x^2 - \frac{9}{2}x + A$; now observing that the last Term 9 of the proposed Equation is a perfect Square, as likewise the last Term of the assumed Equation $(x^2 - \frac{9}{2}x + A)^2 = x^4 - 9x^3 + (2A + \frac{81}{4})x^2 - 9Ax + A^2$. I put $A = 3$ which will transform this Equation into

$x^4 - 9x^3 + \frac{105}{4}x^2 - 27x + 9$, whose Terms agree with all the Terms of the proposed Equation, except the third; the Difference of those Terms being $\frac{45}{2}x^2$; to compleat therefore the proposed Equation into a

perfect Square, add to both Sides of it $\frac{45}{2} x^2$, and extracting the Square Root of it, I find $x^2 - \frac{9}{2} x + 3 = \frac{3}{2} x \sqrt{5}$, and consequently,
 $x = \frac{9 \pm 3}{4} \sqrt{5} \pm \sqrt{\left[\frac{39}{4} \pm \frac{27}{8} \sqrt{5} \right]}$.

XCVIII.

Let the Equation $x^4 - 2ax^3 + (2a^2 - cc)x^2 - 2a^3x + a^4 = 0$ be proposed to be solved, according to the Rules of the Extraction of Roots I find the two first Terms of the Root of this Quantity to be $x^2 - ax$: Let the whole Root be expressed by $x^2 - ax + A$; now observing that the last Term a^4 of the proposed Equation to be a perfect Square, as likewise the last Term of the assumed Equation $(x^2 - ax + A)^2 = x^4 - 2ax^3 + (2A + a^2)x^2 - 2aAx + A^2$: I put $A = a^2$, which will transform this Equation into $x^4 - 2ax^3 + 3a^2x^2 - 2a^3x + a^4$, whose Terms agree with the Terms of the proposed Equation, except the third, the Difference of those Terms being $(3a^2 - 2a^2 + c^2)x^2 = (a^2 + c^2)x^2$; to compleat therefore the proposed Equation into a perfect Square, I add to both Sides of it $(a^2 + c^2)x^2$, and extracting the square Root of it, I find $x^2 - ax + a^2 = x\sqrt{(a^2 + c^2)}$, and consequently $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{(aa+cc)} \pm \sqrt{\left[\left(\frac{1}{4}cc - \frac{1}{2}aa \pm \frac{1}{2}a \right) \sqrt{(aa+cc)} \right]}$.

XCIX.

Let in general the Equation $x^4 + px^3 + qx^2 + rx + s = 0$, be proposed to be solved, in which p, q, r, s , are the given Coefficients, and the Equation is clear of Fractions and Surds;

Investigation of the method of completing equations of four dimensions into perfect squares.

$$\begin{array}{rcl}
 x^4 + px^3 + qx^2 + rx + s & \left\{ \begin{array}{l} 2x^2 + px + \mathcal{Q} \\ x^2 + \frac{1}{2}px + \mathcal{Q} \\ 2x^2 + \frac{1}{2}px \end{array} \right. \\
 - x^4 & & \\
 \hline
 + px^3 + qx^2 + rx + s & & \\
 - px^3 - \frac{1}{4}p^2x^2 & & \\
 \hline
 + (q - \frac{1}{4}p^2)x^2 + rx + s & & \\
 + A^2x^2 + 2ABx + B^2 & & \\
 - 2\mathcal{Q}x^2 - \mathcal{Q}px - \mathcal{Q}^2 & & \\
 \hline
 \end{array}$$

o o o

According to the Rules of the Extraction of Roots I find the two first Terms of the Root of this Quantity to be $x^2 - \frac{1}{2}px$. Let the Square $AAx^2 + 2ABx + B^2$ compleat it into a perfect Square; so that the whole Root shall be expressed by $x^2 + \frac{1}{2}px + \mathcal{Q}$.

I observe that $-\frac{1}{4}p^2 + A^2$, if they are Fractions they should have the same Denominator, since they destroy each other, or their Result is an Integer; consequently $\mathcal{Q}$ is an Integer, or half of an Integer.

Now from the Equations $q - \frac{1}{2} p^2 + A^2 - 2Q = 0$, $r + 2AB - pQ = 0$, and $s + B^2 - Q^2 = 0$, putting $q - \frac{1}{2} p^2 = \alpha$ we have $2Q = A^2 + \alpha$, $pQ = 2AB + r$, $Q^2 = B^2 + s$; substituting therefore the Value of Q , as given by the first Equation, in the other two, we shall get $\frac{1}{2} p A^2 - 2AB = \beta$, and $\frac{1}{2} A^4 + \frac{1}{2} \alpha A^2 - B^2 = \zeta$: Supposing $\beta = r - \frac{1}{2} \alpha p$, and $\zeta = s - \frac{1}{2} \alpha \alpha$. In which Equations the unknown Quantities appertaining to the latter of the two assumed Squares are only concerned, and from which their Values might be found; but as the resulting Equation, when one of the Quantities is exterminated, rises to the sixth Dimension, and would require more Trouble to reduce it than even the original one propounded, little Advantage would be reaped there from: The Analysts therefore, instead of proceeding in a direct Manner, have endeavoured to discover such Properties or Relations of those Quantities as might enable them to guess at their Values, which may be afterwards tried by Means of the Equations here exhibited.

The reduction of equations by completing them into perfect squares cannot otherwise be effected than by trial.

C.

To explain what they have imagined in this Respect, it is to be observed, that A and B may be either rational Quantities or Radicals, and in this last Case, as $A^2 x^2 + 2ABx + B^2$ should be a rational Quantity, A and B must be affected by the same Radical. Let then $A = k\sqrt{n}$, and $B = l\sqrt{n}$, by this Means the two Equations derived above will be changed to $\frac{1}{2} p k^2 n - 2kl n = \beta$, and $\frac{1}{2} k^4 n^2 + \frac{1}{2} \alpha k^2 n - l^2 n = \zeta$, or to $\frac{1}{2} p k^2 - 2kl = \frac{\beta}{n}$, and $\frac{1}{2} k^4 n + \alpha k^2 - 2l^2 = \frac{2\zeta}{n}$ respectively.

Now since k and l are Integers, or Halves of Integers, consequently n an Integer, it is plain, that $\frac{\beta}{n}$ and $\frac{2\zeta}{n}$ must be Integers likewise, or at least the Halves of Integers, and consequently that n (whose Value we are seeking) ought to be some common integral Divisor of β and 2ζ .

How equations of four dimensions may be reduced in this manner

Moreover, with Regard to k and l , it is evident from the first of those Equations $\frac{1}{2} p k - 2l = \frac{\beta}{n}$ that the former k ought to be some Divisor

of $\frac{\beta}{n}$, and that if the Quotient $\frac{\beta}{n k}$ be taken from $\frac{1}{2} p k$ the Remainder will be the Double of l .

It further appears, from the Equations $Q = \frac{A^2 + \alpha}{2}$, and $Q^2 = B^2 + s$, by substituting for A^2 and B^2 their Equals $n k^2$ and $n l^2$, that Q will be

$\frac{3}{2} Z$

perfect Square, add to both Sides of it $\frac{45}{2} x^2$, and extracting the Square Root of it, I find $x^2 - \frac{9}{2} x + 3 = \frac{3}{2} x \sqrt{5}$, and consequently,
 $x = \frac{9 \pm 3}{4} \sqrt{5} \pm \sqrt{\left[\frac{39}{4} \pm \frac{27}{8} \sqrt{5}\right]}$.

XCXVIII.

Another example.

Let the Equation $x^4 - 2ax^3 + (2a^2 - cc)x^2 - 2a^3x + a^4 = 0$ be proposed to be solved, according to the Rules of the Extraction of Roots I find the two first Terms of the Root of this Quantity to be $x^2 - ax$: Let the whole Root be expressed by $x^2 - ax + A$; now observing that the last Term a^4 of the proposed Equation to be a perfect Square, as likewise the last Term of the assumed Equation $(x^2 - ax + A)^2 = x^4 - 2ax^3 + (2A + a^2)x^2 - 2aAx + A^2$: I put $A = a^2$, which will transform this Equation into $x^4 - 2ax^3 + 3a^2x^2 - 2a^3x + a^4$, whose Terms agree with the Terms of the proposed Equation, except the third, the Difference of those Terms being $(3a^2 - 2a^2 + c^2)x^2 = (a^2 + c^2)x^2$; to compleat therefore the proposed Equation into a perfect Square, I add to both Sides of it $(a^2 + c^2)x^2$, and extracting the Square Root of it, I find $x^2 - ax + a^2 = x\sqrt{(a^2 + c^2)}$, and consequently $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{(aa + cc)} \pm \sqrt{\left[\left(\frac{1}{4}cc - \frac{1}{2}aa \pm \frac{1}{2}a\right)\sqrt{(aa + cc)}\right]}$.

XCXIX.

Investigation of the method of completing equations of four dimensions into perfect squares.

Let in general the Equation $x^4 + px^3 + qx^2 + rx + s = 0$, be proposed to be solved, in which p, q, r, s , are the given Coefficients, and the Equation is clear of Fractions and Surds;

$$\begin{array}{rcl}
 x^4 + px^3 + qx^2 + rx + s & \left\{ \begin{array}{l} 2x^2 + \frac{p}{2}x + \frac{p^2}{4} \\ x^2 + \frac{1}{2}p x + \frac{p^2}{4} \\ 2x^2 + \frac{1}{2}p x \end{array} \right. & \\
 - x^4 & & \\
 \hline
 + px^3 + qx^2 + rx + s & & \\
 - px^3 - \frac{1}{2}p^2x^2 & & \\
 \hline
 + (q - \frac{1}{2}p^2)x^2 + rx + s & & \\
 + A^2x^2 + 2ABx + B^2 & & \\
 - 2Qx^2 - Qpx - Q^2 & & \\
 \hline
 \end{array}$$

o o o

According to the Rules of the Extraction of Roots I find the two first Terms of the Root of this Quantity to be $x^2 - \frac{1}{2}px$. Let the Square $AAx^2 + 2ABx + B^2$ compleat it into a perfect Square; so that the whole Root shall be expressed by $x^2 + \frac{1}{2}px + Q$.

I observe that $-\frac{1}{2}p^2 + A^2$, if they are Fractions they should have the same Denominator, since they destroy each other, or their Result is an Integer; consequently Q is an Integer, or half of an Integer.

Now from the Equations $q - \frac{1}{2}p^2 + A^2 - 2Q = 0$, $r + 2AB - pQ = 0$, and $s + B^2 - Q^2 = 0$, putting $q - \frac{1}{2}p^2 = a$ we have $2Q = A^2 + a$, $pQ = 2AB + r$, $Q^2 = B^2 + s$; substituting therefore the Value of Q , as given by the first Equation, in the other two, we shall get $\frac{1}{2}pA^2 - 2AB = \beta$, and $\frac{1}{4}A^4 + \frac{1}{2}aA^2 - B^2 = \zeta$: Supposing $\beta = r - \frac{1}{2}ap$, and $\zeta = s - \frac{1}{4}aa$. In which Equations the unknown Quantities appertaining to the latter of the two assumed Squares are only concerned, and from which their Values might be found; but as the resulting Equation, when one of the Quantities is exterminated, rises to the sixth Dimension, and would require more Trouble to reduce it than even the original one propounded, little Advantage would be reaped there from: The Analysts therefore, instead of proceeding in a direct Manner, have endeavoured to discover such Properties or Relations of those Quantities as might enable them to guess at their Values, which may be afterwards tried by Means of the Equations here exhibited.

The reduction of equations by compleating them into perfect Squares cannot otherwise be effected than by trial.

C.

To explain what they have imagined in this Respect, it is to be observed, that A and B may be either rational Quantities or Radicals, and in this last Case, as $A^2 x^2 + 2ABx + B^2$ should be a rational Quantity, A and B must be affected by the same Radical. Let then $A = k\sqrt{n}$, and $B = l\sqrt{n}$, by this Means the two Equations derived above will be changed to $\frac{1}{2}pk^2n - 2kl n = \beta$, and $\frac{1}{4}k^4n^2 + \frac{1}{2}ak^2n - l^2n = \zeta$, or to $\frac{1}{2}pk^2 - 2kl = \frac{\beta}{n}$, and $\frac{1}{4}k^4n + ak^2 - 2l^2 = \frac{2\zeta}{n}$ respectively.

Now since k and l are Integers, or Halves of Integers, consequently n an Integer, it is plain, that $\frac{\beta}{n}$ and $\frac{2\zeta}{n}$ must be Integers likewise, or at least the Halves of Integers, and consequently that n (whose Value we are seeking) ought to be some common integral Divisor of β and 2ζ .

How equations of four dimensions may be reduced in this manner

Moreover, with Regard to k and l , it is evident from the first of those Equations $\frac{1}{2}pk - 2l = \frac{\beta}{n}$ that the former k ought to be some Divisor

of $\frac{\beta}{n}$, and that if the Quotient $\frac{\beta}{n}$ be taken from $\frac{1}{2}pk$ the Remainder will be the Double of l .

It further appears, from the Equations $Q = \frac{A^2 + a}{2}$, and $Q^2 = B^2 + s$, by substituting for A^2 and B^2 their Equals $n k^2$ and $n l^2$, that Q will be

$\frac{3}{2}Z$

$= \frac{\alpha + nk^2}{2}$, and $l^2 = \frac{2\mathcal{Q} - s}{n}$, from the former of which $\mathcal{Q}$ will be known, when n and k are known; by Means whereof and the other Equation, l may be a second Time found, and the Agreement or Coincidence of this Value, with that before determined for l , will be a Proof that n and k have been rightly assumed, and that adding to the given Equation the Quantity $n \times (kx + l)^2$ it will be completed into the Square $(x^2 + \frac{1}{2}p x + \mathcal{Q})^2$.

CI.

Since the foregoing Method of Solution depends upon the assuming proper Divisors of β , 2ζ and $\frac{\beta}{n}$ for the Values of n and k , it will be expedient, in Order to diminish the Number of Trials, to discover such of the Divisors as are not to the Purpose, which may be effected from the Consideration of the Properties of even and odd Numbers.

The number of trials diminished from the consideration of even and odd numbers.

In Order to which $\mathcal{Q} = \frac{\alpha + nk^2}{2}$ being previously transformed to $n k^2 = 2\mathcal{Q} - \alpha = 2\mathcal{Q} - q + \frac{1}{2}p]^2 = \frac{1}{2}p]^2 - f$ (by putting $q - 2\mathcal{Q} = f$). It is evident from thence, that if p be an odd Number $p^2 - 4f$, and consequently its Equal $4nk^2$, will likewise be an odd Number, because an even Number $4f$, subtracted from the Square of an odd one, always leaves odd; therefore seeing $4k^2 \times n$ is here an odd Number, both n and $4k^2$ must be odd, (for the Product of two even Numbers, or of an odd one and an even one is even and not odd); whence it follows, because $2k]^2$ is odd, that $2k$ must be odd too; and consequently k the Half of an odd Number.

Now seeing p , n , and $2k$ are all of them odd Numbers (when p is such) they may therefore be expressed by $2a + 1$, $2b + 1$ and $2c + 1$, respectively; a , b , and c being Integers: In Consequence of which Assumption, the Equation $4nk^2 = p^2 - 4f$ will by Substitution be changed to $8bc^2 + 8bc + 2b + 4c^2 + 4c + 1 = 4a^2 + 4a + 1 - 4f$, or $2bc^2 + 2bc + \frac{1}{2}b + c^2 + c = a^2 + a - f$; from whence it is manifest, as all the Terms but $\frac{1}{2}b$ are known to be Integers, that $\frac{1}{2}b$ must be an Integer likewise: And so b being an even Number, it follows that n , or $2b + 1$, must be double of an even Number, (or a Multiple of 4) increased by Unity; therefore all the Divisors of β and 2ζ that have not this Property, may be safely rejected, as not for the Purpose.

In like Manner, if p be even the same Limitations will take Place, provided that r is odd, which will be the Case when $\mathcal{Q}$ is the Half of an odd Number; for when $\mathcal{Q}$ is an Integer, $A^2 = \frac{1}{2}p^2 - f$, and $B^2 = \mathcal{Q}^2 - s$, being Integers, their Product $A^2 B^2$ will be an Integer, and consequently, the square Root thereof AB being rational, will likewise be an Integer, and so $p\mathcal{Q}$ and $2AB$ being both even Numbers, their Difference r , as

given by the Equation $p \mathcal{Q} = 2AB + r$, would be even, and not odd.

Let then $\mathcal{Q} = \frac{2x+1}{2}$, then $\mathcal{Q}^2 - s = \frac{4x^2+4x+1-4s}{4}$

but B^2 , or its Equal $n l^2$, being here equal to the Square of Half of an odd Number ($\mathcal{Q}$) joined to an Integer ($-s$), l will be also Half of an odd Number.

Let then $l = \frac{2y+1}{2}$, then $n l^2 = \mathcal{Q}^2 - s = \frac{4n y^2 + 4n y + n}{4}$

consequently $\frac{4x^2+4x+1-4s}{4} = n \times \left(\frac{4y^2+4y+1}{4} \right)$, and

$n = \frac{1+4x+4x^2-4s}{1+4y+4y^2} = 1 + 4 \times \left(\frac{x+x^2-s-y-y^2}{1+4y+4y^2} \right)$;

whence n is positive, and the Double of an even Number, or a Multiple of 4 increased by Unity, when the Fraction $\frac{x+x^2-s-y-y^2}{1+4y+4y^2}$ is positive, but negative and a Multiple of 4 less Unity, when this Fraction is negative.

From those several Conclusions is derived the following Rule for the Reduction of Equations of four Dimensions, contained in the Formula

$$x^4 + p x^3 + q x^2 + r x + s = 0.$$

CII.

Make $\alpha = q - \frac{1}{4} p^2$, $\beta = r - \frac{1}{2} \alpha p$, and $\zeta = s - \frac{1}{4} \alpha \alpha$, then put for n some common integral Divisor of β and 2ζ , that is neither a Square, nor divisible by a Square, and which being divided by 4 shall leave Unity, if n be positive, but a Multiple of 4 less Unity if n is negative, if either p or r be odd. Put also for k some Divisor of $\frac{\beta}{n}$ if p be even, or Half of the odd Divisor if p be odd: Take the Quotient from $\frac{1}{2} p k$, and call Half the Remainder l . Make $\mathcal{Q} = \frac{\alpha + n k^2}{2}$, and try if n divides $\mathcal{Q} \mathcal{Q} - s$, and the Root of the Quotient be equal to l ; if so it happen, then the proposed Equation, by Means of the Values thus determined, will be reduced to $x x + \frac{1}{2} p x + \mathcal{Q} = \pm \sqrt{n} \times (k x + l)$.

General rule derived from the foregoing conclusions for reducing an equation of four dimensions.

That the Divisor n ought not here to be a Square, is evident from what has been already remarked, since both A and B would then be rational Quantities; and that it ought not to be divisible by a Square, will also appear, if it be considered that k and l in the Equation, $k \sqrt{n} = A$ and $l \sqrt{n} = B$ are to be taken the greatest, and n the least, that the Case will admit of.

CIII.

Let the Equation $x^4 + 12x - 17 = 0$ be propose to be reduced.

ELEMENTS OF

Application
of the fore-
going rule
to an ex-
ample.

Here $p = 0$, $q = 0$, $r = 12$, $s = -17$; consequently, $a = 0$, $\beta = 12$, $\xi = -17$, and β and 2ξ ; that is, 12 and -34 , having only 2 for a common Divisor, it must be $n = 2$. Again $\frac{\beta}{n} = 6$, whose Divisors 1, 2, 3, 6, are to be successively put for k , and -3 , $-\frac{3}{2}$, -1 , $-\frac{1}{2}$, for l respectively, but $\frac{a + nk^2}{2}$; that is, k^2 is

equal to $\mathcal{Q}$, and $\sqrt{\frac{\mathcal{Q}^2 - s}{n}} = l$; and when the even Divisors 2 and 6 are substituted for k , $\mathcal{Q}$ becomes 4 and 36, and $\mathcal{Q}^2 - s$ being an odd Number, is not divisible by $n = 2$; wherefore 2 and 6 are to be set aside: but when 1 and 3 are written for k , $\mathcal{Q}$ is 1 or 9, and $\mathcal{Q}^2 - s$ is 18 or 98 respectively; which Numbers can be divided by 2, and the Roots of the Quotients extracted being ± 3 and ± 7 ; but only one of them, Viz. -3 coincides with l . I put therefore $k = 1$, $l = -3$, $\mathcal{Q} = 1$, and adding to both Sides of the Equation $nk^2x^2 + 2nklx + nl^2$, that is, $2x^2 - 12x + 18$, there results $x^4 + 2x^2 + 1 = 2x^2 - 12x + 18$; and extracting the Root of each, $x^2 + 1 = \pm \sqrt{2} \times (x - 3)$, and again, extracting the Root of this last, the four Values of x according to the Varieties in the Signs are $-\frac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \frac{1}{2}}$, $-\frac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \frac{1}{2}}$; $\frac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \frac{1}{2}}$, $\frac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \frac{1}{2}}$.

CIV.

Let the Equation $x^4 - 6x^3 - 58x^2 - 114x - 11 = 0$ be proposed to be reduced.

The forego-
ing rule ap-
plied to an-
other exam-
ple.

Here $p = -6$, $q = -58$, $r = -114$, $s = -11$, consequently $-67 = a$, $-315 = \beta$, and $-1135\frac{1}{4} = \xi$. The Numbers β and 2ξ , that is, -315 and $-\frac{4533}{2}$ have but one common Divisor 3, that is

$n = 3$, and the Divisors of $-105 = \frac{\beta}{n}$ are 3, 5, 7, 15, 21, 35, and 105.

Wherefore I first make Trial with $3 = k$ and dividing $\frac{\beta}{n}$ or -105 by it, get the Quotient -35 , and this subtracted from $\frac{1}{2}pk = -3 \times 3$, leaves 26, whose half, 13, ought to be equal to l , but $\frac{a + nk^2}{2}$ or $\frac{-67 + 27}{2}$

that is, -20 is equal to $\mathcal{Q}$, and $\mathcal{Q}^2 - s = 411$, which is indeed divisible by $n = 3$; but the Root of the Quotient 137 cannot be extracted, therefore I reject the Divisor 3, and try with $5 = k$, by which, dividing

$\frac{\beta}{n} = -105$ the Quotient is -21 , and this taken from $\frac{1}{2}pk = -3 \times 5$,

leaves $6=2l$. At the same Time $\mathcal{Q} = \frac{a + nk^2}{2} = \frac{-67+75}{2} = 4$

and $\mathcal{Q}^2 - s$, or $16 + 11$ is divisible by n , and the Root of the Quotient 9, that is 3 coincides with l . Whence I conclude, that putting $l=3, k=5, \mathcal{Q}=4, n=3$, adding to both Sides of the Equation, the Quantity $nk^2x^2 + 2nklx + nl^2$, that is, $75x^2 + 90x + 27$, and extracting the Roots, it will be

$x^2 + \frac{1}{2}px + \mathcal{Q} = \sqrt{n \times (kx + l)}$, or $x^2 - 3x + 4 = \pm \sqrt{3 \times (5x + 3)}$

and again extracting the Root, $x = \frac{3 \pm 5\sqrt{3}}{2} \pm \sqrt{17 \pm \frac{21 \times \sqrt{3}}{2}}$.

cv.

No Regard in the foregoing Rule is had to that Circumstance in which β happens to be nothing. In this Case, the Equation $\frac{1}{2}pk^2 - 2kl$

$= \frac{\beta}{n}$ becomes $\frac{1}{2}pk^2 - 2kl = 0$; where one Root, or Value of k

must necessarily be nothing. Therefore $\mathcal{Q}$ being $= \frac{1}{2}a$, we have $l\sqrt{n} = \sqrt{(\mathcal{Q}^2 - s)} = \sqrt{(\frac{1}{4}a^2 - s)}$; so that by a direct Process our given Equation is reduced to $x^2 + \frac{1}{2}px + \frac{1}{2}a = \sqrt{(\frac{1}{4}aa - s)}$ wherein a is given $= q - \frac{1}{4}pp$.

Thus the Equation $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$, where $a = -38, \beta = 0$, is reduced to $x^2 + x - 19 = \pm 6\sqrt{10}$.

Besides the Value of $k=0$ when $\beta=0$, there are two other Values which equally fulfil the several Conditions required, and bring out the very same Conclusion.

For subtracting the Square of half the Second of the original Equations, $2\mathcal{Q} - a = AA, p\mathcal{Q} - r = 2AB, \mathcal{Q}^2 - s = B^2$: from the Product of the other two, there will be obtained the Equation $\mathcal{Q}^3 + \frac{1}{2}q\mathcal{Q}^2 + (\frac{1}{4}pr - s) \times \mathcal{Q} + \frac{1}{2} \times (as - \frac{1}{4}rr) = 0$, wherein the unknown Quantity $\mathcal{Q}$ is alone concerned, which Equation being of three Dimensions, the Root $\mathcal{Q}$ and consequently

$k = \frac{2\mathcal{Q} - a}{n}$ will admit of three different Values.

Thus the Value of k in the proposed Equation $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$ may be 0, 3, or 4, or which comes to the same, the Equation itself may be

reduced to $x^2 + x - 19 = \pm 6\sqrt{10}$, $x^2 + x + \frac{7}{2} = \pm \sqrt{5 \times (3x + \frac{3}{2})}$,

or to $x^2 + x - 3 = \pm \sqrt{2 \times (4x + 2)}$, all which are, in Effect, but one and the same Equation, as will appear by squaring both Sides of each,

Case in which the foregoing rule cannot be applied.

Rule to be observed in this case.

Application of this rule to an example.

ELEMENTS OF

and properly transposing; from whence, in every Case, the given Equation $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$ will emerge.

CVI.

Let the Equation $x^4 - 3x^3 + 44x^2 - 123x + 97 = 0$ be proposed to be reduced.

Third example.

$$\text{Here } p = -3, q = 44, r = -123, s = 97, q - \frac{1}{4}p^2 = 44 - \frac{9}{4} = \frac{167}{4}, \frac{p\alpha}{2} = -\frac{501}{8}, r - \frac{1}{2}\alpha p = -123 + \frac{501}{8} = \frac{-984 + 501}{8} = -\frac{483}{8} = \beta, \frac{\alpha\alpha}{4} = +\frac{2789}{16};$$

$$s - \frac{1}{4}\alpha\alpha = 97 - \frac{2789}{16} = \frac{1552 - 2789}{16} = -\frac{2637}{16} = \zeta,$$

the Divisors of β are $-3, -7, -23$, and those of ζ are -3 , and -8779 ; whence β and 2ζ , having only -3 for a common Divisor,

it must be $n = -3$; wherefore $\frac{\beta}{n} = +\frac{483}{8 \times 3} = +\frac{161}{8} : I$

therefore try whether $k = +\frac{7}{2}$ (because p is an odd Number) will

$$\text{succeed. } \frac{p k}{2} - \frac{\beta}{n k} = -\frac{21 - 23}{4} = -\frac{44}{4} = -11;$$

$$\text{and } l = -\frac{11}{2}. \text{ Whence is deduced } \frac{\alpha + n k^2}{2} = \frac{147}{8} - \frac{3}{2} \times \frac{49}{4}$$

$$= \frac{167 - 147}{8} = +\frac{5}{2} = \mathcal{Q}, \text{ and } \mathcal{Q}^2 = \frac{25}{5};$$

$$\mathcal{Q}^2 - s = \frac{25 - 388}{8} = -\frac{363}{4}. \text{ This Number divided by}$$

$$n = -3 \text{ gives } \frac{121}{4}, \text{ whose Root is } \pm \frac{11}{2} = l; \text{ wherefore the}$$

proposed Equation can be reduced, and there results

$$x^2 - \frac{3}{2}x + \frac{5}{2} = \left(\frac{7}{2}x - \frac{11}{2} \right) \sqrt{} - 3.$$

CVII.

Application of the foregoing rule for reducing equations having fracted terms.

If the highest Term of the Equation to be reduced has a Coefficient different from Unity, the Equation may be transformed into one that shall have the Coefficient of the highest Term Unity, and the foregoing Rules may be applied to the new Equation.

Thus, if the Equation $m x^4 + p x^3 + q x^2 + r x + s = 0$ be multiplied by m^3 , it will be transformed into

$$m^4 x^4 + m^3 p x^3 + m^3 q x^2 + m^3 r x + m^3 s = 0.$$

Let $m x = y$, and $y^4 + p y^3 + m q y y + m m r y + m^3 s = 0$ will result; to which the foregoing Rules may be applied.

For Example, if the Equation $2 x^4 - 6 x^3 + 137 x^2 - 400 x + 315 = 0$ was proposed, here $2 x = y$, and consequently $m = 2$, $m^2 = 4$; $m^3 = 8$. Substituting y for x , and multiplying the Coefficient of the third Term by 2, the fourth by 4, and the last by 8, there will result,

$$y^4 - 6 y^3 + 274 y^2 - 1600 y + 2520 = 0.$$

Here $p = -6$, $q = 274$, $r = 1600$, $s = 2520$, $q - \frac{1}{2} p^2 = 274 - 9 = 265 = \alpha$, $\frac{p \alpha}{2} = -3 \times 265 = -795$, and $r - \frac{1}{2} p \alpha =$ Example.
 $-1600 + 795 = -805 = \beta$, $s - \frac{1}{4} \alpha \alpha = 2520 - \frac{70225}{4} =$
 $= \frac{60145}{4} = \zeta$.

The Divisors of β are -5 , -7 , -23 , and those of ζ are -5 , -23 , -523 ; wherefore $n = -5$, or $n = -23$. I put $n = -5$, whence $\frac{\beta}{n} = \frac{-805}{-5} = +161$, and $k = 7$, or $k = 23$.

Let $k = 7$, $\frac{p k}{2} - \frac{\beta}{n k} = -21 - 23 = -44 = 2 l$, and
 $l = -22$; whence is deduced $\frac{\alpha + n k^2}{2} = \frac{265 - 5 \times 49}{2} = 10 = \mathcal{Q}$
and $\mathcal{Q} \mathcal{Q} = 100$, $\mathcal{Q} \mathcal{Q} - s = 100 - 2520 = -2420$, and $\frac{\mathcal{Q} \mathcal{Q} - s}{n} =$
 $\frac{-2420}{-5} = 484 = (-22)^2 = l^2$; the Equation therefore reduced is
 $y^2 - 3 y + 10 = (7 y - 22) \sqrt{-5}$, and restoring $4 x^2$ for y^2 and
 $2 x$ for y , and dividing all the Terms by 4, $x^2 - \frac{3}{2} x + \frac{5}{2}$
 $= \left(\frac{7}{2} x - \frac{11}{2} \right) \sqrt{-5}$.

CVIII.

Hitherto we have applied the Rule to the Extraction of surd Roots, to apply the same to the Extraction of rational Roots, it suffices to make use of Unity for the Quantity n , and by this Means we may find whether an Equation that has no fracted or surd Terms can admit of any Divisor, either rational or surd, of two Dimensions.

Thus, if the Equation $x^4 - x^3 - 5 x^2 + 12 x - 6 = 0$ was proposed, by substituting -1 , -5 , $+12$, and -6 , for p , q , r and s re-

Application of the foregoing rule for extracting the rational roots of equations of four dimensions.

ELEMENTS OF

ſpectively, I find $\alpha = -5 \frac{1}{4}$, $\beta = 9 \frac{3}{8}$, and putting $n = 1$, the Diviſors of the Quantity $\frac{\beta}{n}$, or $\frac{75}{8}$, are 1, 3, 5, 15, 25, 75, the Halves whereof (if p be odd) are to be tried for k , and for k trying $\frac{5}{2}$, I find $\frac{1}{2} p k - \frac{\beta}{n k} = -5$, and its Half $-\frac{5}{8} = l$. Alſo $\frac{\alpha + n k k}{2} = \frac{1}{2} = \mathcal{Q}$, and $\frac{\mathcal{Q} \mathcal{Q} - s}{n} = 6 \frac{1}{4}$, the Root whereof agrees with l . I therefore conclude, that the Quantities $n, k, l, \mathcal{Q}$, are rightly found, and having added to each Part of the Equation the Terms $n k^2 x^2 + 2 n k l x + n l l$, that is, $6 \frac{1}{4} x x - 12 \frac{1}{2} x + 6 \frac{1}{4}$, the Root may be extracted on both Sides, and by that Extraction there will come out $x x + \frac{1}{2} p x + \mathcal{Q} = \pm \sqrt{n \times (k x + l)}$, that is, $x x - \frac{1}{2} x + \frac{1}{2} = \pm 1 \times (2 \frac{1}{2} x - 2 \frac{1}{2})$, or $x x - 3 x + 3 = 0$, and $x x + 2 x - 2 = 0$, and ſo by theſe two quadratick Equations the biquadratick One propoſed may be divided.

CIX.

Observation
which ſerve
to diminifh
the number
of trials.

If at any Time there are many Diviſors of the Quantity $\frac{\beta}{n}$, ſo that it may be too difficult to try all of them for k , their Number may be diminifhed by ſeeking all the Diviſors of the Quantity $\alpha s - \frac{1}{4} r r$. For the Quantity $\mathcal{Q}$ ought to be equal to ſome of thoſe, or to the Half of ſome odd one; for $\alpha s = \alpha \mathcal{Q}^2 - \alpha l n^2$, and taking from both $\frac{1}{4} r^2 = \frac{1}{4} p^2 \mathcal{Q}^2 - p \mathcal{Q} n k l + n^2 k^2 l^2$, ſeeing the Remainder $\alpha \mathcal{Q}^2 - (\alpha + n k^2) \times n l^2 - \frac{1}{4} p^2 \mathcal{Q}^2 + p \mathcal{Q} n k l = \alpha \mathcal{Q}^2 - 2 \mathcal{Q} \times n l^2 - p^2 \mathcal{Q}^2 + p \mathcal{Q} n k l$ has $\mathcal{Q}$ in every Term, the Thing is manifeſt.

Thus in the laſt Example $\alpha s - \frac{1}{4} r r$ is $-\frac{9}{2}$, ſome one of theſe Diviſors 1, 3, 9, or of them halved $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, ought to be $\mathcal{Q}$; Wherefore by trying ſingly the halved Diviſors of the Quantity $\frac{\beta}{n}$ Viz. $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, $\frac{25}{2}$ and $\frac{75}{2}$ for k . I reject all that do not make $\frac{1}{2} \alpha + \frac{1}{2} n k^2$, or $\frac{21}{8} + \frac{1}{2} k k$; that is, $\mathcal{Q}$ to be one of

the Numbers 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$; but by writing $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, &c. for k , there come out respectively $-\frac{5}{2}$, $-\frac{3}{2} + \frac{1}{2} + \frac{51}{2}$, &c. for $\mathcal{Q}$, out of which only $-\frac{3}{2}$ and $\frac{1}{2}$ are found among the aforesaid Numbers 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, and consequently the rest being rejected, either k will be $= \frac{3}{2}$, and $\mathcal{Q} = -\frac{3}{2}$, or $k = \frac{5}{2}$ and $\mathcal{Q} = \frac{1}{2}$; which two Cases let be examined.

CX.

If the Equation to be reduced is of six Dimensions, let it be $x^6 + p x^5 + q x^4 + r x^3 + s x^2 + t x + v = 0$, and let there be assumed $(x^3 + \frac{1}{2} p x^2 + \mathcal{Q} x + R)^2 - (A x^2 + B x + C)^2 = x^6 + p x^5 + q x^4 + r x^3 + s x^2 + t x + v = 0$, which, by Invo-

lution and Transposition, will give

$$(2 \mathcal{Q} + \frac{1}{4} p^2 - q) x^4 + (2 R + p \mathcal{Q} - r) x^3 + (p R + \mathcal{Q}^2 - s) x^2 + (2 \mathcal{Q} R - t) x + R^2 - v = A^2 x^4 + 2 A B x^3 + (2 A C + B^2) x^2 + 2 B C x + C^2$$

Investigation of the method of reducing equations of six dimensions by trial.

From whence by equating the Coefficients of the homologous Powers, and writing $a = q - \frac{1}{4} p^2$,

$$\begin{aligned} \text{We have } 1^\circ. 2 \mathcal{Q} - a &= A^2; \quad 2^\circ. 2 R + p \mathcal{Q} - r = 2 A B; \\ 3^\circ. p R + \mathcal{Q}^2 - s &= 2 A C + B^2; \quad 4^\circ. 2 \mathcal{Q} R - t = 2 B C; \\ 5^\circ. R^2 - v &= C^2. \end{aligned}$$

If now the Value of $\mathcal{Q} = \frac{1}{2} A^2 + \frac{1}{2} a$, as given by the first of these Equations, be substituted for $\mathcal{Q}$, in the second, we shall get $2 R + \frac{1}{2} p A^2 + \frac{1}{2} p a - r = 2 A B$, and consequently, $R = A B - \frac{1}{4} p A^2 + \frac{1}{2} \beta$ (by putting $\beta = r - \frac{1}{2} p a$), which Value, together with that of $\mathcal{Q}$, being substituted in the three remaining Equations, we shall have,

$$\begin{aligned} 1^\circ. p A B - \frac{1}{4} p^2 A^2 + \frac{1}{2} p \beta + \frac{1}{4} A^4 + \frac{1}{2} a A^2 + \frac{1}{4} a^2 - s &= 2 A C + B^2 \\ 2^\circ. A^3 B - \frac{1}{2} p A^4 + \frac{1}{2} \beta A^2 + a A B - \frac{1}{4} p a A^2 + \frac{1}{2} a \beta - t &= 2 B C \\ &4 A \end{aligned}$$

3°. $A^2 B^2 - \frac{1}{2} p A^3 B + \beta A B + \frac{1}{16} p^2 A^4 - \frac{1}{4} p \beta A^2 + \frac{1}{4} \beta^2 - v = C^2$
 which by putting $\gamma = r - \frac{1}{2} p \beta$, $\zeta = \gamma - \frac{1}{4} a a$, $\eta = r - \frac{1}{2} a \beta$, and
 $\theta = v - \frac{1}{4} \beta \beta$, will be reduced to

$$p A B - \frac{1}{4} p^2 A^2 + \frac{1}{4} A^4 + \frac{1}{2} a A^2 - \zeta = 2 A C + B^2$$

$$A^3 B - \frac{1}{4} p A^4 + \frac{1}{2} \beta A^2 + a A B - \frac{1}{4} p a A^2 - \eta = 2 B C$$

and $A^2 B^2 - \frac{1}{2} p A^3 B + \beta A B + \frac{1}{16} p^2 A^4 - \frac{1}{4} p \beta A^2 - \theta = C^2$
 respectively.

Now as the Values of A , B , and C , are either rational Quantities, or if irrational, affected by the same radical Quantity; let them be expressed by $k \sqrt{n}$, $l \sqrt{n}$, and $m \sqrt{n}$, respectively; then Substitution being made, and every Equation divided by n , we shall have

$$1^\circ. p k l - \frac{1}{4} p^2 k^2 + \frac{1}{4} n k^4 + \frac{1}{2} a k^2 - \frac{\zeta}{n} = 2 k m + l^2$$

$$2^\circ. n k^3 l - \frac{1}{2} p n k^4 + \frac{1}{2} \beta k^2 + a k l - \frac{1}{4} p a k^2 - \frac{\eta}{n} = 2 l m$$

$$3^\circ. n k^2 l^2 - \frac{1}{2} p n k^3 l + \beta k l + \frac{1}{16} p^2 n k^4 - \frac{1}{4} p \beta k^2 - \frac{\theta}{n} = m^2$$

From whence it appears (since k , l , and m , are here considered as Integers, or as the Halves of such) that ζ , η , and θ , ought to be all of them divisible by n , or which is the same, that n ought to be some common integral Divisor of the Quantities ζ , η , and θ .

With Regard to k , let the several Terms in the former Part of the first of our three Equations, in which k is found, (in order to abreviate the Work) be denoted by $F k$, then will the Equation itself be changed to $F k - \frac{\zeta}{n} = 2 k m + l^2$, and in the very same Manner our other

two Equations will be changed to $G k - \frac{\eta}{n} = 2 l m$, and

$H k - \frac{\theta}{n} = m^2$, respectively.

Let now the Square of Half the Second of these Equations be subtracted from the Product of the first and third, then will

$$F H k^2 - \frac{F k \theta}{n} - \frac{H k \zeta}{n} + \frac{\zeta \theta}{n n} - \frac{1}{4} G^2 k^2 + \frac{G k \eta}{2 n} - \frac{\frac{1}{2} \eta^2}{n n} = 2 k m^3,$$

which, by dividing the whole by $2 k$, and putting $\lambda = \zeta \theta - \frac{1}{4} \eta^2$, will at Length become,

$$\frac{1}{2} F H k - \frac{1}{8} G^2 k - \frac{1}{2} H \times \frac{\zeta}{n} + \frac{1}{4} G \times \frac{\eta}{n} - \frac{1}{2} F \times \frac{\theta}{n} + \frac{\lambda}{2 n n k} = m^3.$$

where ξ , n , and θ , being all divisible by their common Divisor n , (as is shewn above); it is manifest, that $\frac{\lambda}{2nn}$ (in order that m may be an Integer, or the Half of an Integer), ought also to be divisible by its Divisor k ; that is, k ought to be some Divisor of the Quantity $\frac{\lambda}{2nn}$.

Again, with Regard to l , let the Value of $R = AB - \frac{1}{2}pQ + \frac{1}{2}r$, as given by the second of the five original Equations, be substituted in the fourth, by which Means we have $2QAB = pQ^2 + rQ - t = 2BC$, or $2Qnk l - pQ^2 + rQ - t = 2nlm$ (because $A = k\sqrt{n}$, $B = l\sqrt{n}$, $C = m\sqrt{n}$) and consequently $\frac{rQ - pQ^2 - t}{nl} = 2m - 2kQ$;

where $2m - 2kQ$, being an Integer, it is evident that $\frac{rQ - pQ^2 - t}{nl}$ must be an Integer also; and consequently, l some Divisor of the Quantity $\frac{rQ - pQ^2 - t}{n}$, from whence l being found in Numbers, the Value

of $R = AB - \frac{1}{2}pQ + \frac{1}{2}r = nk l - \frac{1}{2}pQ + \frac{1}{2}r$ will be had likewise; and then by Means of the three last of the five original Equations, the Value of m may be also found three different Ways, and the Truth of the Solution thereby confirmed: For those Equations, by substituting for A , B , and C , their Equals $k\sqrt{n}$, $l\sqrt{n}$, and $m\sqrt{n}$, do become $R^2 - v = nm^2$, $2QR - t = 2nlm$, and $pR + Q^2 - s = 2nkm + nll$;

from the first of which $m = \sqrt{\frac{R^2 - v}{n}}$; from the second

$m = \frac{2QR - \frac{1}{2}t}{nl}$; and from the third $m = \frac{2Q + pR - nll - s}{2nk}$,

which Values therefore, when Q , R , &c. are rightly assumed, will be all found equal among themselves.

CXI.

As the foregoing Method of Solution depends upon the assuming proper Divisors of ξ , n , θ , &c. for n , &c. To bring the Work into less Compass, it will be expedient to reject such Divisors of those Quantities as are not to the Purpose, and this may be effected as in the precedent Case, from the Consideration of the Properties of even and odd Numbers, the Reasoning thereon being the very same in this Case as in the former, which therefore it will be unnecessary to repeat. And from the Conclusions above derived is deduced the following Rule for reducing an Equation of six Dimensions

$$x^6 + px^5 + qx^4 + rx^3 + sx^2 + tx + o = 3.$$

The number of trials diminished from the consideration of the properties of even and odd numbers.

CXII.

General rule derived from the foregoing method for reducing an equation of six dimensions.

Make $q - \frac{1}{2} p p = \alpha$, $r - \frac{1}{2} p \alpha = \epsilon$, $s - \frac{1}{2} p \beta = \gamma$, $\gamma - \frac{1}{2} \alpha \alpha = \zeta$, $t - \frac{1}{2} \alpha \beta = \eta$, $v - \frac{1}{2} \beta \beta = \theta$, $\zeta \theta - \frac{1}{2} \eta \eta = \lambda$.

Then for n take of the Terms 2ζ , η , 2θ , some common Integer Divisor, that is not a Square, and that likewise is not divisible by a Square, and which also divided by the Number 4, shall leave Unity, provided any one of the Terms p , r , t , be odd. For k take some Integer Divisor of the Quantity

$\frac{\lambda}{2 n n}$ if p be even, or the Half of an odd Divisor, if p be odd, for Q take the Quantity, $\frac{1}{2} \alpha + \frac{1}{2} n k k$, take for l some One of the Divisors of the Quantity $\frac{2 r - 2 Q p - t}{n}$, if Q be an Integer, or the Half of an odd Divisor, if Q be a Fraction that has for its Denominator the Num-

ber 2; or 0, if that Dividual $\frac{2 r - 2 Q p - t}{n}$ be nothing. And for R the Quantity $\frac{1}{2} r - \frac{1}{2} Q p + n k l$. Then try if $R R - v$ can be divided by n , and the Root of the Quotient extracted; and besides, if that

Root be equal as well to the Quantity $\frac{2 R - \frac{1}{2} t}{n l}$ as to the Quantity

$\frac{2 Q + p R - n l l - s}{2 n k}$; if all these happen, call that Root m , and in

Room of the Equation proposed, write this $x^3 + \frac{1}{2} p x x + Q x + R = \pm \sqrt{n} \times (k x^2 + l x + m)$; for this Equation, by squaring its Parts, and taking from both Sides the Terms on the right Hand, will produce the Equation proposed.

CXIII.

Application of the foregoing rule to an example.

Let $x^6 - 2 a x^5 + 2 b b x^4 + 2 a b b x^3 - (2 a a b b - 2 a^3 b + 4 a b^3) x^2 + 3 a a b^4 - a^4 b b = 0$ be proposed to be reduced.

Here writing $-2 a$, $+2 b b$, $+2 a b b$, $-2 a a b b + 2 a^3 b - 4 a b^3$, 0 , and $3 a a b^4 - a^4 b b$ for p , q , r , s , t , and v , respectively, there will come out $2 b b - a a = \alpha$, $4 a b b - a^3 = \epsilon$, $2 a^3 b + 2 a a b b - 4 a b^3 - a^4 = \gamma$, $-b^4 + 2 a^3 b + 3 a a b b - 4 a b^3 - \frac{5}{4} a^4 = \zeta$, $-\frac{1}{2} a^5 - 3 a^3 b b - 4 a b^4 = \eta$, and $-a a b^4 + a^4 b b - \frac{1}{2} a^5 = \theta$; and the common Divisor of the Terms 2ζ , η and 2θ , is $a a - 2 b b$, or $2 b b - a a$, according as $a a$, or $2 b b$, is the greater. But let $a a$ be greater than $2 b b$, and $a a - 2 b b$ will be n ; for n must always be affirmative. Moreover we have, $\frac{\epsilon}{n} = -\frac{5}{4} a a + 2 a b + \frac{1}{2} b b$,

$\frac{\eta}{n} = -\frac{1}{2} a^3 + 2 a b b$, and $\frac{\theta}{n} = -\frac{1}{4} a^4 + \frac{1}{2} a a b b$, and

$$\frac{\xi}{2n} \times \frac{\theta}{n} = \frac{\theta\xi}{8nn}, \text{ or } \frac{\lambda}{2nn} = \frac{1}{8} a^6 - \frac{1}{4} a^5 b - \frac{1}{8} a^4 b b$$

$$+ \frac{1}{2} a^3 b^3 - \frac{3}{8} a^2 b^4, \text{ the Divisors whereof are } 1, a, a a; \text{ but be-}$$

cause $\sqrt{n} \times k$ cannot be of more than one Dimension, and that $\sqrt{n}$ is of one, therefore k will be of none; and, consequently, can only be a Number. Wherefore, rejecting a and $a a$, there remains only 1. for k , be-

$$\text{sides } \frac{1}{2} a + \frac{1}{2} n k k \text{ gives } 0 \text{ for } Q, \text{ and } \frac{2r - 2Qp - t}{n} \text{ is also}$$

Nothing; and consequently l , which ought to be its Divisor, will be Nothing. Lastly, $\frac{1}{2} r - \frac{1}{2} p Q + n k l$ gives $a b b$ for R , and $R R - v$ is $-2 a a b^4 + a^4 b b$, which may be divided by n , or $a a - 2 b b$, and the Root of the Quotient $a a b b$ be extracted; and that Root taken negative-

ly, Viz. $-a b$ is not unequal to the indefinite Quantity $\frac{2R - \frac{1}{2} p t}{n l}$,

$$\text{or } \frac{0}{0}, \text{ but equal to the definite Quantity } \frac{2Q + p R - n l l - s}{2 n k},$$

wherefore that Root $-a b$ will be m ; and in the Room of the Equation proposed, there may be writ $x^3 - \frac{1}{2} p x x + Q x + R = \sqrt{n} \times (k x x + l x + m)$, that is, $x^3 - a x x + a b b = \sqrt{(a a - 2 b b) \times (x x - a b)}$; the Truth of which Conclusion you may prove by squaring the Parts of the Equation found, and taking away the Terms on the right Hand from both Sides; for from that Operation will be produced the Equation $x^6 - 2 a x^5 + 2 b b x^4 + 2 a b b x^3 - (2 a a b b - 2 a^3 b + 4 a b^3) x^2 + 3 a a b^4 - a^4 b b = 0$, which was proposed to be reduced.

CXIV.

When $\lambda = 0$; in which Case k (or one Value of k at least) will also be $= 0$, the Reduction will be performed by a direct Process. For k being Nothing, the three Equations wherein k , l , and m are first introduced

$$\text{will become } -\frac{\xi}{n} = l^2, -\frac{\eta}{n} = 2 l m, \text{ and } -\frac{\theta}{n} = m^2; \text{ whence}$$

$l \sqrt{n} = \sqrt{-\xi}$, $m \sqrt{n} = \sqrt{-\theta}$; and consequently $\lambda = \xi \theta - \frac{1}{4} \eta^2 = 0$, as it ought to be; therefore, by substituting those Values, and writing also, instead of Q and R , their Equals $\frac{1}{2} a$, and $\frac{1}{2} \beta$, the Equation given is here reduced to $x^3 + \frac{1}{2} p x^2 + \frac{1}{2} a x + \frac{1}{2} \beta = \pm x \sqrt{-\xi} \pm \sqrt{-\theta}$.

CXV.

If the Equation proposed is of eight Dimensions, let it be $x^8 + p x^7 + q x^6 + r x^5 + s x^4 + t x^3 + v x^2 + w x + z = 0$; then by assuming $(x^4 + \frac{1}{2} p x^3 + Q x^2 + R x + S)^2 - (A x^3 + B x^2 + C x + D)^2 = x^8 + p x^7 + q x^6 + r x^5 + s x^4 + t x^3 + v x^2 + w x + z = 0$;

Case in which the foregoing rule cannot be applied. Rule to be observed in this case.

and proceeding as in the former Cases we shall have here, 1^o. $2Q - a = A^2$,
 2^o. $2R + pQ - r = 2AB$, 3^o. $2S + pR + QQ - s = 2AC + BB$,
 4^o. $pS + 2QR - t = 2AD + 2BC$, 5^o. $2QS + RR - v = 2BD + CC$, 6^o. $2RS - w = 2CD$. 7^o. $SS - z = DD$.

Method for
discovering
whether an
equation of
eight dimen-
sions can be
reduced.

Putting now (as before) $A = k\sqrt{n}$, $B = l\sqrt{n}$, $C = m\sqrt{n}$,
 $D = b\sqrt{n}$; putting also, to shorten the Work, $Q = Q'n + \frac{1}{2}a$
 $R = R'n + \frac{1}{2}\beta$, $S = S'n + \frac{1}{2}\gamma$; that is, let the Quotients of Q , R ,
 and S , when divided by the common Divisor n , be Q' , R' , S' , and the Re-
 mainders $\frac{1}{2}a$, $\frac{1}{2}\beta$, and $\frac{1}{2}\gamma$, respectively: Then to determine these last,
 which must be first known, before n can be known, let Substitution be
 made in the second and third Equations, every where disregarding such
 Terms wherein n and its Powers are involved. Thus Substitution being
 made in the second Equation, we have $2R'n + \beta + pQ'n + \frac{1}{2}pa - r = 2kl n$;
 where the homologous Terms in which n enters not are β ,
 $\frac{1}{2}pa$, and $-r$: The others therefore being here disregarded, we have
 $\beta + \frac{1}{2}pa - r = 0$, or $\beta = r - \frac{1}{2}pa$; in the very same Manner from
 the third Equation $\gamma + \frac{1}{2}p\beta + \frac{1}{4}aa - s = 0$; and consequently,
 $\gamma = s - \frac{1}{2}p\beta - \frac{1}{4}aa$.

Let Substitution be now made in the fourth, fifth, sixth, and seventh,
 Equations (still disregarding such Terms as would involve the Powers of n)
 and there will come out, 1^o. $\frac{1}{2}p\gamma + \frac{1}{2}a\beta - t$, 2^o. $\frac{1}{2}a\gamma + \frac{1}{4}\beta\beta - v$,
 3^o. $\frac{1}{2}\beta\gamma - w$, 4^o. $\frac{1}{4}\gamma\gamma - z$. Now, as all the other Terms that would
 arise in these Equations (besides those put down) are affected with n , and
 are therefore divisible thereby; it is manifest, that the four Quantities
 $t - \frac{1}{2}p\gamma - \frac{1}{2}a\beta$, $v - \frac{1}{2}a\gamma - \frac{1}{4}\beta\beta$, $w - \frac{1}{2}\beta\gamma$, and $z - \frac{1}{4}\gamma\gamma$;
 here brought out, which, for Brevity sake, I represent severally by δ , ϵ , ζ , η ,
 must likewise be all of them divisible, by the same common Divisor n ,
 when the Equation given is capable of being reduced. If therefore no such
 common Divisor, (under the Restrictions specified in the preceding Cases,
 depending on any Coefficient p , r , t , or w , in a Place denominated from an
 even Number, being an odd Number,) can be discovered, the Work will
 then be at an End.

CXVI.

From the same Method of Operation, which may be looked upon as a
 sort of Examination, whether the Equation be reducible or not, we may
 find all the Quantities, to which n ought to be a common Divisor, when
 the Equation given is of 10, 12, or a greater Number of Dimensions.

General me-
thod for dis-
covering
whether an
equation of

Let $x^{2m} + px^{2m-1} + qx^{2m-2} + rx^{2m-3} + sx^{2m-4} + tx^{2m-5} + \&c. = 0$
 be given, and let there be assumed,

SPECIOUS ARITHMETICK.

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$[x^m + \frac{1}{2} p x^{m-1} + (Q'n + \frac{1}{2} \alpha) x^{m-2} + (R'n + \frac{1}{2} \beta) x^{m-3} + (S'n + \frac{1}{2} \gamma) x^{m-4} \&c.]^2$ any numb-
 $- n (k x^{m-1} + l x^{m-2} + m x^{m-3} \&c.)^2 = x^{2m} + p x^{2m-1} + q x^{2m-2} \&c.$ of even di-
 then by squaring $x^m + \frac{1}{2} p x^{m-1} + (Q'n + \frac{1}{2} \alpha) x^{m-2}$, &c. and men-
 transposing $x^{2m} + p x^{2m-1} + q x^{2m-2}$, &c. it will appear that sions
 the Terms of this Equation, in which n enters not, will be, can be re-
 duced.

$$\left. \begin{array}{l} \frac{1}{4} p^2 \\ -q \end{array} \right\} \times x^{2m-2} + \left. \begin{array}{l} \frac{1}{2} p \alpha \\ -r \end{array} \right\} \times x^{2m-3} + \left. \begin{array}{l} \frac{1}{4} p \beta \\ -s \end{array} \right\} \times x^{2m-4} + \left. \begin{array}{l} \frac{1}{2} p \gamma \\ -t \end{array} \right\} \times x^{2m-5} + \left. \begin{array}{l} \frac{1}{4} p \delta \\ -v \end{array} \right\} \times x^{2m-6}$$

From the former Half of which Terms all the Quantities α, β, γ , &c. will be determined by assuming the Coefficients equal to Nothing: Thus we have $\alpha = q - \frac{1}{4} p p$, $\beta = r - \frac{1}{2} p \alpha$, $\gamma = s - \frac{1}{2} p \beta - \frac{1}{4} \alpha \alpha$, $\delta = t - \frac{1}{2} p \gamma - \frac{1}{2} \alpha \beta$, &c. and those Quantities being known, the Coefficients of the remaining Terms will likewise be known, which ought all of them to be divisible by n , in Order that the Reduction may succeed; that is, they ought to be such, as to admit of a common Divisor (n) under the Restrictions before specified.

CXVII.

For Example, if the Equation given is of twelve Dimensions, as
 $x^{12} + p x^{11} + q x^{10} + r x^9 + s x^8 + t x^7 + v x^6 + a x^5 + b x^4$ Application
 $+ c x^3 + d x^2 + e x + f = 0$, we should have $\alpha = q - \frac{1}{4} p p$ of the fore-
 $\beta = r - \frac{1}{2} p \alpha$, $\gamma = s - \frac{1}{2} p \beta - \frac{1}{4} \alpha \alpha$, $\delta = t - \frac{1}{2} p \gamma - \frac{1}{2} \alpha \beta$, and going me-
 $\epsilon = v - \frac{1}{2} p \delta - \frac{1}{2} \alpha \gamma - \frac{1}{4} \beta \beta$, and the Coefficients of the other six thod to an
 Terms (whereof n ought to be a common Divisor) would be example.
 $\frac{1}{2} p \epsilon + \frac{1}{2} \alpha \delta + \frac{1}{2} \beta \gamma - a$, $\frac{1}{2} \alpha \epsilon + \frac{1}{2} \beta \delta + \frac{1}{4} \gamma \gamma - b$, $\frac{1}{2} \beta \epsilon + \frac{1}{2} \gamma \delta - c$,
 $\frac{1}{2} \gamma \epsilon + \frac{1}{4} \delta \delta - d$, $\frac{1}{2} \delta \epsilon - e$, and $\frac{1}{4} \epsilon \epsilon - f$.

These Operations for finding of n will most commonly end the Work. If such a Value however should be found for n as to answer all the Conditions above specified, and consequently there are Hopes of a future Reduction. It may be tried by the following Steps, which though applied only for reducing an Equation of eight Dimensions, yet it is easy to perceive, are general for all Equations.

CXVIII.

Because by the 7th Equation $SS - z = n b^2$, if n be even, seek a square Number b^2 , to which, after it is multiplied into n , the last Term of the Equation z , being added with its proper Sign $z + n b^2$, shall make

a square Number, which may be expeditiously performed, by adding to z when n is an even Number, or to $4z$ when it is odd, those Quantities successively $n, 3n, 5n, 7n, 9n, 11n$, and so on till the Sum becomes equal to some Number in the Table of square Numbers; and if no such Number occurs, before the square Root of that Sum, augmented by the square Root of the Excess of that Sum, above the last Term of the Equation, is four Times greater than the greatest of the Terms of the proposed Equation p, q, r, s, t, v , &c. There will be no Occasion to try any farther; for then the Equation cannot be reduced: But if such a square Number does occur, let its Root be S , if n be even, or $2S$, if n be odd;

and call $\sqrt{\frac{SS-z}{n}} = b$; but if n is odd, then S and b may be Fractions

that have 2 for their Denominator: And if one is a Fraction, the other ought to be so too, which is also to be observed of the Numbers R and m , Q and l , p and k , hereafter to be found. And all the Numbers S and b within this

General
rule for re-
ducing equa-
tions that
have been
found by the
foregoing
method to
the reducible

Limit must be collected in a Catalogue.

Having thus found n, b , and S ; k is next to be found by successively trying all Numbers which do not make $nk \pm \frac{1}{2}p$ four Times greater than the greatest Term of the Equation; when k is had, Q is to be found by

the Equation $Q = \frac{nk^2 + z}{2}$, Q being found, all Numbers are to be

tried for l which do not make $nl \pm Q$ greater than quadruple the greatest Term of the Equation, and l being found we have

$R = \frac{-npk^2 + 2\beta}{4} + nkl$. Lastly to find m , all Numbers are

successively to be tried, which do not make $nm \pm R$ greater than quadruple the greatest Term of the Equation. Of the Values of the Letters registered in the Catalogue, those only are to be assumed which agree when found several Ways, thus, $S = \sqrt{z + nb^2}$ by the 7th Equation; and also

$= \frac{s - pR - QQ + nll}{2} + nkm$ by the third; and its Correspondent

$b = \frac{\sqrt{SS-z}}{n}$ by the 7th Equation, and $= \frac{pS + 2QR - t - 2nlm}{2nk}$

by the fourth; also $= \frac{2QS + R^2 - v - nm^2}{2nl}$ by the 5th, and

$= \frac{2RS - w}{2nm}$ by the 6th Equation; which Values therefore, when

Q, R , &c. are rightly assumed, will be all found equal among themselves, and the Equation proposed will be then reduced to

$x^4 + \frac{1}{2}px^3 + Qx^2 + Rx + S = \sqrt{n} \times (kx^3 + lx^2 + mx + b)$

CXIX.

Let $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x - 5 = 0$,
be proposed to be reduced.

Here $q = \frac{1}{4}p p = -1 - 4 = -5 = u$, $r = \frac{1}{2}p u = -10 + 10$
 $= 0 = \beta$, $s = \frac{1}{2}p \beta - \frac{1}{4}u u = 5 - \frac{25}{4} = -\frac{5}{4} = \gamma$, $t = \frac{1}{2}p \gamma - \frac{1}{2}u \beta$
 $= -5 + \frac{5}{2} = -\frac{5}{2} = \delta$, $v = \frac{1}{2}u \gamma - \frac{1}{4}\beta \beta = -10 - \frac{25}{8}$
 $= -\frac{105}{8}$, $w = \frac{1}{4}\beta \gamma = -10 = \zeta$, $z = \frac{1}{4}\gamma \gamma = -5 - \frac{25}{64}$
 $= -\frac{345}{64} = \eta$, therefore, 2δ , 2ϵ , 2ζ , 8η , respectively, are

-5 , $-\frac{105}{4}$, -20 , and $-\frac{345}{8}$, and their common Divisor 5, which
 divided by 4, leaves 1, as it ought, because the Term r is odd. Since
 therefore, the common Divisor n , or 5, is found, which gives Hope to a fu-
 ture Reduction, and because it is odd, to 42, or -20 , I successively add
 n , $3n$, $5n$, $7n$, $9n$, &c. or 5, 15, 25, 35, 45, &c. and there arises
 -15 , 0, 25, 60, 105, 160, 225, 300, 385, 480, 585, 700, 825,
 960, 1105, 1260, 1425, 1600. of which only 0, 25, 225, and 1600,
 are Squares.

Application
 of the fore-
 going rule
 to an exam-
 ple.

Wherefore, the Halves of those Roots $0, \frac{5}{2}, \frac{15}{2}, 20$ are to be collected in
 a Table for the Values of S , & the Values of $\sqrt{\frac{SS - x}{n}}$, that is, $1, \frac{3}{2}, \frac{7}{2}, 9$
 respectively for b . But because $S + nb$, if 20 be taken for S , and 9 for b ,
 becomes 65, a Number greater than four Times the greatest Term of the
 Equation. Therefore I reject 20 and 9, and write only the rest in the
 Table as follows, $b \left\{ 1, \frac{3}{2}, \frac{7}{2} \right\}$ $S \left\{ 0, \frac{5}{2}, \frac{15}{2} \right\}$.

Then I try for k , all the Numbers which do not make $2 \pm 5k$ greater
 than 40, (four Times the greatest Term of the Equation), that is, the Num-
 bers. $-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7$,
 putting $\frac{nk + a}{2}$ or $\frac{5k + 5}{2}$, that is, the Numbers $\frac{315}{2}, 120, \frac{175}{2},$
 $60, \frac{75}{2}, 20, \frac{15}{2}, 0, -\frac{5}{2}, 0, \frac{15}{2}, 20, \frac{75}{2}, 60, \frac{175}{2}, 120$ respec-
 tively for Q : But since $Q \pm n$, and much more Q ought not to be greater
 than 40, I perceive I am to reject $\frac{315}{2}, 120, \frac{175}{2}$ and 60, and their
 Correspondents $-8, -7, -6, -5, 5, 6, 7$ and consequently that only
 $-4, -3, -2, -1, 0, 1, 2, 3, 4$ must respectively be tried for k ,

and $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, respectively for $\mathcal{Q}$.

Let us therefore try -1 for k , and 0 for $\mathcal{Q}$, and in this Case for l there will be successively to be tried all the Numbers which do not make $\mathcal{Q} \pm nl$ greater than 40, that is, all the Numbers between 10 and -10 ; and for

R you are respectively to try the Numbers $\frac{2\beta - npkk}{4} + nkl$,

or $-5-5l$, that is, $-55, -50, -45, -40, -35, -30, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, 30, 35, 40, 45$, the three former of which, and the last, because they are greater than 40, may be neglected. Let us try therefore, -2 for l , and 5 for R , and in this Case, for m there will be besides to be tried all the Numbers which do not make $R \pm nm$, or $5 \pm 5m$ greater than 40, that is, all the Numbers between 7 and -9 , and see whether

or not, by putting $\frac{s - \mathcal{Q}\mathcal{Q} - pR + nll}{2}$ that is, $5 - 20 + 20$ or $5 = 2H$

$H + nkm$, or $\frac{5}{2} - 5m = S$, that is, if any of these Numbers $-\frac{65}{2}$,

$-\frac{55}{2}$, $-\frac{45}{2}$, $-\frac{35}{2}$, $-\frac{25}{2}$, $-\frac{15}{2}$, $-\frac{5}{2}$, $\frac{5}{2}$, $\frac{15}{2}$,

$\frac{25}{2}$, $\frac{35}{2}$, $\frac{45}{2}$, $\frac{55}{2}$, $\frac{65}{2}$, $\frac{75}{2}$, $\frac{85}{2}$, is equal to any of the

Numbers 0, $\pm \frac{5}{2}$, $\pm \frac{15}{2}$, which were brought into the Catalogue

for S , and we meet with four of these $-\frac{15}{2}$, $-\frac{5}{2}$, $\frac{5}{2}$, $\frac{15}{2}$ to which

Answer $\pm \frac{7}{2}$, $\pm \frac{3}{2}$, $\pm \frac{3}{2}$, $\pm \frac{7}{2}$ written for b in the same Table, &

2, 1, 0, -1 substituted for m , but let us try $-\frac{5}{2}$ or S , 1 for m , and

$\pm \frac{3}{2}$ for b , and you will have $\frac{2RS - w}{2nm} = \frac{-25 + 10}{10} = -\frac{3}{2}$,

and $\frac{2\mathcal{Q}S + RR - v - nmm}{2nl} = \frac{35 + 10 - 5}{-20} = -\frac{1}{2}$, and

$\frac{pS + 2\mathcal{Q}R - t - 2nlm}{2nh} = \frac{-10 + 5 + 20}{-10} = -\frac{3}{2}$.

Wherefore, since there comes out in all Cases $-\frac{3}{2}$, or b , I conclude all the

Numbers to be rightly found, and consequently that in the Room of the Equation proposed, we may write $x^4 + 2x^3 + 5x - 2\frac{3}{2} = \sqrt{5} \times (-x^3 - 2xx + x - 1\frac{1}{2})$. For by squaring the Parts of

this, there will be produced that Equation of eight Dimensions which was at first proposed.

CXX.

Whatever Efforts the Analysts have made, to find a general Method for solving Problems producing Equations of every Degree, besides those expressed by $ax^m = b$, and $ax^{2m} + bx^m = c$, they have not hitherto succeeded, except for such as produce Equations of the third and fourth Degree, which we shall proceed to explain, beginning by the Resolution of Problems, producing Equations of the third Degree.

Resolution
of Equations of third
and fourth
degree..

CXXI.

The Compound Interest of a certain Sum of Money, put out for 4 Years, amounted to a Sum a, but the simple Interest thereof for the same Time, and at the same Rate, would have been a Sum. b. What was the Sum put out? And what the Rate of Interest?

Let y denote the Interest of 1 l. for one Year, therefore since the simple Interest of 1 l. for 4 Years, is $4y$, and the compound Interest $(1+y)^4 - 1$ or $4y + 6y^2 + 4y^3 + y^4$, we have, as $4y + 6y^2 + 4y^3 + y^4 : 4y = a : b$ by the Nature of the Question, and consequently $6y + 4y^2 + y^3 = \frac{4a - 4b}{b}$

from the Resolution of which Equation, y will be found, and consequently the Rate of Interest required.

CXXII.

Let in General the Equation to be solved, be $y^3 + dy^2 + ey + f = 0$, if we could reduce this Equation to another, including only the first and last Term, it is manifest that the Equation would be solved. Now the most natural Method of transforming an Equation into another, in which we are left at Liberty to make some Variation, is to substitute in this Equation in the Room of the unknown Quantity some other Quantity, in which a Letter is left undetermined in order to make use of it at pleasure.

Equation
the most
complicated
of the third
degree..

CXXIII.

Let then $x + r$ be substituted in this Equation, in the Room of y , and there will result $x^3 + (3r + d)xx + (3rr + 2dr + e)x + r^3 + dr^2 + er + f = 0$. As r in this Equation, is at our Disposal, it is easy to perceive, that by its Means, any Term may be taken away out of the Equation.

Transformation by
means of
which any
term is
taken away
from an
Equation..

Let, for Example, $3r + d$ be put $= 0$, or $r = -\frac{1}{3}d$, and the

Equation is reduced to $x^3 + (e - \frac{d^2}{3})x + \frac{2}{27}d^3 - \frac{de}{3} + f = 0$, the second Term vanishing. If $3rr + 2dr + e$ be put $= 0$, or

$r = -\frac{1}{3}d \pm \sqrt{\left[-\frac{e}{3} + \frac{1}{9}d^2\right]}$, the third Term will vanish, but the two others will remain.

It is easy to conceive, that by substituting $x + r$ in the Room of y , no more than one Term at a Time can be made to vanish, but if by this Transformation, we have not entirely effected the Reduction of the proposed Equation to two Terms, at least the Question has been rendered more simple, since an Equation consisting only of three Terms remains to be solved.

Of the two transformed Equations that result by taking away either the 2d or 3d Term, the first, $x^3 + \left(e - \frac{d^2}{3}\right)x + \frac{2}{27}d^3 - \frac{de}{3} + f = 0$ is the most simple, which we shall therefore endeavour to resolve, by still trying to diminish the Number of its Terms: but before we proceed to this Research, we shall explain how the Method employed for transforming an Equation of the third Degree, and exterminating its intermediate Terms, may be applied to Equations of every Degree.

CXXIV.

The foregoing transformation applied to an Equation of the fourth degree.

Let, for Example, the Equation proposed be $y^4 + ay^3 + by^2 + cy + d = 0$, the most general of the fourth Degree, substituting $x + r$ in the Room of y , there results.

$$x^4 + (4r + a)x^3 + (6r^2 + 3ar + b)x^2 + (4r^3 + 3ar^2 + 2br + c)x + r^4 + ar^3 + br^2 + cr + d = 0.$$

In which putting $4r + a = 0$, or $r = -\frac{1}{4}a$, there will result an Equation of the fourth Degree wanting the second Term.

The second term most commonly exterminated and why.

In like Manner, if r be determined by Means of the Equation $6r^2 + 3ar + b = 0$, there will result an Equation of the fourth Degree, wanting the third Term, the fourth Term in like Manner may be made to vanish by solving an Equation of the 3d Degree, &c. But the Analysts seldom or never take away any Term but the Second, because the other Terms cannot be taken away without employing Calculations complicated with Radicals.

CXXV.

The second term exterminated in an Equation of the fifth degree.

By substituting in the Equation $y^5 + ay^4 + by^3 + cy^2 + dy + e = 0$, $x + r$ for y , there will result $x^5 + (5r + a)x^4 + \&c.$ in which the second Term will be taken away by solving the Equation $5r + a = 0$, or putting

$r = -\frac{1}{5}a$: and in General, in an Equation of the m^{th} Degree represented by

In an Equation of the m^{th} degree.

$y^m + ay^{m-1} + by^{m-2} + \&c. = 0$, if y be put $= x + r$ it will be transformed into $x^m + (mr + a)x^{m-1} + \&c.$ whereby it appears that the second Term may be taken away from the proposed Equation by putting $y = x - \frac{a}{m}$.

CXXVI.

After this little Digression, let us proceed to resolve the Equation of the third Degree, $x^3 + (e - \frac{d^2}{3})x + (\frac{2}{27}d^3 - \frac{de}{3} + f) = 0$, to which the general Equation $y^3 + dy^2 + ey + f = 0$, was reduced by putting $y = x - \frac{1}{3}d$, and to shorten the Calculation, let it be expressed thus, $x^3 + px + q = 0$.

Pursuing the same Method already employed, let the foregoing Equation be again transformed, by substituting, for Example, $u + z$ in the Room of x , not with a View to take away a Term of this Equation, as before, for it would soon appear that the Term taken away would return, but to resolve this Equation into others more simple. Without perceiving that such a Method would succeed; yet it is manifest, that the Transformation of an Equation into another, in which a Letter is left undetermined, cannot but be of Use.

$u + z$ being substituted for x , there results $u^3 + 3uuz + 3uzz + z^3 + pu + pz + q = 0$. Let one of the unknown Quantities u or z be supposed such, that $u^3 + z^3 + q = 0$. In this Case we will have $3u^2z + 3uz^2 + pu + pz = 0$, which being divided by $u + z$, gives $3uz + p = 0$, or $u = -\frac{p}{3z}$. By which Equation one of the unknown Quantities introduced may be exterminated; to find the Equation that will determine the other unknown Quantity, let this Value of u be substituted in the first Equation $u^3 + z^3 + q = 0$, and there will result

$$-\frac{p^3}{27z^3} + z^3 + q = 0, \text{ or } z^6 + qz^3 = \frac{p^3}{27}, \text{ whence is deduced}$$

$$z = \sqrt[3]{-\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}}. \text{ Wherefore } u \text{ or } -\frac{p}{3z}$$

$$= -\frac{\frac{1}{3}p}{\sqrt[3]{-\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}}} \text{ which is reduced to}$$

$-\sqrt[3]{\left[-\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right]}$; for it is easy to perceive that the Product of $-\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}$ into $+\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}$ is $\frac{1}{27}p^3$ consequently $\frac{1}{3}p$ is the Product of the cube Roots of those Quantities. Now, adding together those Values of u and z , we will have $u + z$ or $x = \sqrt[3]{-\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}} - \sqrt[3]{+\frac{1}{2}q \pm \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}}$, or $x = \sqrt[3]{-\frac{1}{2}q + \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}} - \sqrt[3]{+\frac{1}{2}q + \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}}$ for the Value of x will be the same, whether the Radical $\sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}$ be taken positively or negatively.

CXXVII.

The foregoing Formula expresses only one of the Values of the three Roots of the Equation, to find the two others we must divide the Equation $x^3 + px + q = 0$ by the Root found, and the Equation of the second Degree that results for Quotient, will when resolved, give the two other Roots required.

To find the general Expression of these two Roots, I put, in order to abbreviate the Work $\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} = m$ and $\sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} = n$, then observing that $mn = \frac{1}{3}p$ and $m^3 - n^3 = q$, I divide the Equation $x^3 + px + q = 0$ by $x + m - n$, and there results the Equation $x^2 + nx - mx + mm + nn + mn = 0$ whose two Roots, are $x = \frac{m-n}{2} \pm \frac{1}{2} \times (m+n) \sqrt{-3}$, or $x = \frac{1}{2} \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} - \frac{1}{2} \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} \pm \frac{1}{2} \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} + \frac{1}{2} \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} \times \sqrt{-3}$ which are necessarily imaginary when $\sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}$ is a real Quantity.

CXXVIII.

Case in which the value of x cannot be discovered by means of the foregoing formula on account of the imaginary quantities it includes.

The foregoing Solution of Equations of the third Degree is liable to this Inconveniency, that it gives no Value for x when $\frac{1}{27}p^3$ is negative, and greater than $\frac{1}{4}qq$, in this Case, the Value of the Quantity $\sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}$ is imaginary, whence the two Quantities $\sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}}$ and $-\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}}$ which compose the Value of x are imaginary, we are not however to conclude, that the Value of x in this Case is imaginary, it being real, as may be proved after the following Manner.

Let the Value $\sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}} - \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}}$ of x be resumed, and let a be put in the Room of $\frac{1}{2}q$ and $b\sqrt{-1}$ in the Room of $\sqrt{\frac{1}{4}q^2 + \frac{1}{27}p^3}$ supposed imaginary, and there will result $x = \sqrt[3]{-a + b\sqrt{-1}} - \sqrt[3]{a + b\sqrt{-1}}$ Now $\sqrt[3]{-a + b\sqrt{-1}}$ and $-\sqrt[3]{a + b\sqrt{-1}}$ being reduced into Series by Means of the binomial Theorem.

The value of x however in this case is real.

The first will become $-a^{\frac{1}{3}} + \frac{1}{3}a^{-\frac{2}{3}}b\sqrt{-1} - \frac{1}{9}a^{-\frac{5}{3}}b^2 - \frac{5}{81}a^{-\frac{8}{3}}b^3\sqrt{-1} + \frac{10}{243}a^{-\frac{11}{3}}b^4 + \&c.$

And the Second will become $-a^{\frac{1}{3}} - \frac{1}{3}a^{-\frac{2}{3}}b\sqrt{-1} - \frac{1}{9}a^{-\frac{5}{3}}b^2 + \frac{5}{21}a^{-\frac{8}{3}}b^3\sqrt{-1} + \frac{10}{243}a^{-\frac{11}{3}}b^4 - \&c.$ and x will be =

$$-2a^{\frac{1}{3}} - \frac{2}{3}a^{-\frac{5}{3}}b^2 + \frac{10}{243}a^{-\frac{11}{3}}b^4 - \frac{308}{6561}a^{-\frac{17}{3}}b^6 + \&c. \text{ or}$$

$$-2a^{\frac{1}{3}} \times \left(1 + \frac{b^2}{9a^2} - \frac{10b^4}{243a^4} + \frac{14b^6}{6561a^6} - \&c.\right) \text{ Expression entirely real.}$$

Wherefore when $\frac{1}{27}p^3$ is negative and greater than $\frac{1}{4}qq$, the Value $\sqrt[3]{-\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}q^2)}}$ — $\sqrt[3]{\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}q^2)}}$ of x , though exhibited under an imaginary Form, is however real. But the Analysts have not been hitherto able to give it a real Form, without admitting an infinite Number of Terms in its Expression.

CXXIX.

Though the Value of x cannot be determined exactly, we may however approximate to it by Means of the foregoing Series; for supposing a greater than b , the Terms of this Series will decrease very swiftly, and a few of the first Terms will give a near Value of the Root required. But if a is less than b , in finding the Value of $\sqrt[3]{(-a + b\sqrt{-1})}$ & $\sqrt[3]{(a + b\sqrt{-1})}$ by Means of the binomial Theorem, $b\sqrt{-1}$ must be put for the first Term of the Binomial, and a for the Second, and there will result for the Value of x , or of $\sqrt[3]{(-a + b\sqrt{-1})} - \sqrt[3]{(a + b\sqrt{-1})}$, the Quantity

$$+ \frac{2}{3}b^{-\frac{2}{3}}a - \frac{10}{81}b^{-\frac{8}{3}}a^3 + \frac{44}{729}b^{-\frac{14}{3}}a^5 - \frac{748}{19693}b^{-\frac{20}{3}}a^7 + \&c.$$

$$\text{or } -\frac{2a}{3b^{\frac{2}{3}}} \times -1 + \frac{5a^2}{27b^4} - \frac{22a^4}{243b^4} + \frac{374a^6}{6561b^6} - \&c.$$

Expression entirely Real, and whose Terms decrease very swiftly, when a is less than b .

CXXX.

Of the three Roots of the Equation $x^3 + px + q = 0$, those $x = \frac{1}{2}\sqrt[3]{\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}}$ — $\frac{1}{2}\sqrt[3]{\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}}$ $\pm [\frac{1}{2}\sqrt{(\frac{1}{2}q + \sqrt{(\frac{1}{4}q^2 + \frac{1}{27}p^3)})} + \frac{1}{2}\sqrt{(-\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)})}] \times \sqrt{-3}$ are both imaginary, when $\sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}$ is a real Quantity. We shall now shew that these two Roots are real when $\sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}$ is imaginary, that is, when p is negative, and $\frac{1}{27}p^3$ is greater than $\frac{1}{4}qq$.

For changing by Means of the Denominations already employed, the Expression of those two Values of y into $\frac{1}{2}\sqrt[3]{(-a + b\sqrt{-1})} - \frac{1}{2}\sqrt[3]{(a + b\sqrt{-1})} \pm [\frac{1}{2}\sqrt[3]{(a + b\sqrt{-1})} + \frac{1}{2}\sqrt[3]{(-a + b\sqrt{-1})}] \times \sqrt{-3}$ and reducing it into Series, there will result for the the two Values of x .

Method of approximating to the value of x in this case.

The two other values of x , are real in the same case.

$$a^{\frac{1}{3}} \times 1 + \frac{bb}{9aa} - \frac{10b^4}{243a^4} + \frac{154b^6}{6561a^5} \&c. \pm \frac{b}{a^{\frac{1}{3}} \sqrt[3]{3}} \times 1 - \frac{5bb}{27aa} \\ + \frac{22b^4}{243a^4} - \frac{374b^6}{6561a^6} + \&c. \text{ Expression entirely real.}$$

CXXXI.

How from
the roots of
the Equa-
tion $x^3 + px + q = 0$ are
deduced the
roots of the
equation
 $y^3 + dy^2 + ey + f = 0$.

As the most general Equation of the third Degree represented by $y^3 + dy^2 + ey + f = 0$ is reduced to $x^3 + \left(e - \frac{d^2}{3}\right)x + \frac{1}{27}d^3 - \frac{de}{3} + f = 0$, or to $x^3 + px + q = 0$, by supposing $y = x - \frac{1}{3}d$ it follows that the Values of y in the general Equation $y^3 + dy^2 + ey + f = 0$ will be found by resolving this last Equation $x^3 + px + q = 0$, and deducting $\frac{1}{3}d$ from its three Roots.

CXXXII.

An Equa-
tion of the
third degree
has three
real roots, or
one real root
and two im-
possible
roots.

Whence it appears, that every Equation of third Degree has at least one real Root, and that the two others are either both real or imaginary.

How to di-
stinguish
those cases.

To discover which of those two Cases happens in any proposed Equation of the third Degree, the second Term must be exterminated, in order to compare it with the Equation $x^3 + px + q = 0$; which being done, if p be positive, or if negative, $\frac{1}{27}p^3$ is not greater than $\frac{1}{4}qq$, the Equation will have only one real Root expressed by the Formula of Art. cxxvi. But if p is negative, and $\frac{1}{27}p^3$ is greater than $\frac{1}{4}qq$, the three Roots will be real, but cannot otherwise be determined than by Approximation.

Nature of
the roots
when $\frac{1}{27}p^3$
is negative
and equal
 $\frac{1}{4}qq$.

If $\frac{1}{27}p^3$ is negative and equal to $\frac{1}{4}qq$ the first Value of x expressed by $3\sqrt{[-\frac{1}{2}q + \sqrt{(\frac{1}{4}q^2 + \frac{1}{27}p^3)}]} - 3\sqrt{[\frac{1}{2}q + \sqrt{(\frac{1}{4}q^2 + \frac{1}{27}p^3)}]}$ is reduced to $-2\sqrt[3]{\frac{1}{2}q}$ and the two other Values expressed in general by $\frac{1}{2}\sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]} - \frac{1}{2}\sqrt[3]{[-\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]}$ $\pm [\frac{1}{2}\sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]} + \frac{1}{2}\sqrt[3]{[-\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}]}] \times \sqrt{-3}$ is reduced to $\pm \sqrt[3]{\frac{1}{2}q}$. Consequently in this Case the three Roots of the Equation are real.

CXXXIII.

To solve the Equation $y^3 + 4y^2 + 6y = \frac{4a - ab}{b}$ produced by the

Application
of the fore-
going Me-
thods to the
resolution of
the proposed
problem.

foregoing Problem: Let $a = 344,811$; $b = 3201$; whereby this Equation is transformed into $y^3 + 4y^2 + 6y = 0,310125r$. I first exterminate the second Term by putting $y = x - \frac{4}{3}$, and by this Substitution there results the Equation $x^3 + 0,666x - 3,569384 = 0$; which compared with $x^3 + px + q = 0$, gives $p = 0,666$ &c. and $q = -3,569384$: those Values being substituted in $3\sqrt{(\frac{1}{4}qq + \frac{1}{27}p^3)}$, this Quantity becomes

$\sqrt[3]{3,19538}$ which is a real Quantity, whence the Value required of x will be obtained by Means of the Formula of Art. cxxvi. Substituting therefore in $\sqrt[3]{[-\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{7}{27}p^3)]} - \sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{4}qq + \frac{7}{27}p^3)]}$, $-1,784692$ for $\frac{1}{2}q$, and $\sqrt[3]{3,19538}$ for $\sqrt{(\frac{1}{4}qq + \frac{7}{27}p^3)}$; this Value of x becomes $\sqrt[3]{1,784692 + \sqrt[3]{3,19538}} - \sqrt[3]{-1,784692 + \sqrt[3]{3,19538}}$ and substituting this Value of x in $y = x - 1,333$, &c. there results $y = \sqrt[3]{[1,784692 + \sqrt[3]{3,19538}]} - \sqrt[3]{[-1,784692 + \sqrt[3]{3,19538}]} - 1,333 = .05$, which is the only real Root of the Equation, therefore the Rate of Interest was 5 per Cent. and the Sum put out 1600*l*.

CXXXIV.

Every Equation of the sixth Degree, in which the Dimensions of the unknown Quantity are all even, may be resolved by the foregoing Method. Let for Example, the Equation $z^6 + 9z^4 + 39zz + 55 = 0$, be proposed; putting zz equal to a new unknown Quantity x less the third Part of the Coefficient of the second Term, that is, putting $zz = x - 3$, the Equation will be transformed into $x^3 + 12x - 8 = 0$, which is only of the third Degree, and wants the second Term.

Comparing now this Equation with $x^3 + px + q = 0$, we have $p = 12$, $q = -8$, and consequently $\sqrt{(\frac{1}{27}p^3 + \frac{1}{4}qq)} = \sqrt{80}$; whence x or $\sqrt[3]{[-\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}qq)}]} - \sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}qq)}]}$ $= \sqrt[3]{(4 + \sqrt{80})} - \sqrt[3]{(\sqrt{80} - 4)}$, and since by Supposition $zz = x - 3$, or $z = \sqrt{(x - 3)}$, we will have the unknown Quantity sought $x = \pm \sqrt{[\sqrt[3]{(4 + \sqrt{80})} - \sqrt[3]{(\sqrt{80} - 4)} - 3]}$ and the two Values given by this Expression are the only real Ones of the Six that the proposed Equation admits of.

In general, all Equations as $z^{3m} + az^{2m} + bz^m + c = 0$, is reduced to one of the third Degree wanting the second Term, by putting $z^m = x - \frac{1}{3}a$.

CXXXV.

Let it be proposed to divide the Number 24 into two such Parts, that the Difference of their Cubes may be 3584.

Let $12 + x$ express the greater Part, and $12 - x$ the Lesser, then will $(12 + x)^3 - (12 - x)^3 = 3584$. That is, $x^3 + 432x = 1792$, which being compared with $x^3 + px + q = 0$, I find $p = 432$, $q = -1792$, and as p is positive, I conclude Art. cxxxii. that the Formula of Art. cxxvi will succeed in this Case, substituting in Effect the Values of p and of q in the general Formula $x = \sqrt[3]{[-\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}qq)}]} - \sqrt[3]{[\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}qq)}]}$ we will have $x = \sqrt[3]{896 + \sqrt{3788800}} - \sqrt[3]{-896 + \sqrt{3788800}}$, which by the Rule of Art. lxxxviii. will be found to be reducible, since $896 + \sqrt{3788800}$ is the Cube of $2 + \sqrt{148}$ and $-896 + \sqrt{3788800}$ is the Cube of $-2 + \sqrt{148}$, whence the Value of x is reduced to 4, whence 8 and 16 are the two required Numbers.

4 C

What two Numbers are those, whose Product is 98, and if from the Square of the greater, the lesser be taken, the Remainder will be 90.

Suppose x the Greatest, and y the least of those Numbers, then

Problem
producing
an Equation
of the third
degree that
cannot be
solved by
the formula
of Art.
cxxxvi.

$xy = 98$, or $y = \frac{98}{x}$, & $xx - y = 90$, or $y = xx - 90 = \frac{98}{x}$
or $x^3 - 90x = 98$. Comparing this Equation with the general Equation
 $x^3 + px + q = 0$, we have $p = -90$, $q = -98$. Now p being negative,
and $\frac{1}{27}p^3$ being greater than $\frac{1}{4}qq$, the Equation cannot be resolved by the
foregoing Formula. I therefore employ the Method of Art. cxxxix. to find
an approximate Value of x .

To this End, I substitute for p and q their Values -90 , and -98 ,
in $x = \sqrt[3]{-\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}qq)}} - \sqrt[3]{\frac{1}{2}q + \sqrt{(\frac{1}{27}p^3 + \frac{1}{4}qq)}}$
and there Results $x = \sqrt[3]{49 + \sqrt{-24599}} - \sqrt[3]{-49 + \sqrt{-24599}}$
which compared with $\sqrt[3]{(-a + b\sqrt{-1})} - \sqrt[3]{(a + b\sqrt{-1})}$ or
 $-\frac{2a}{3b^{\frac{2}{3}}} \times -1 + \frac{5a^2}{27b^2} - \frac{22a^4}{243b^4} + \frac{274a^6}{6561b^6}$ &c. gives $a = -49$

Application
of the me-
thod of Art.
cxxxix. for
approximat-
ing to the
roots of the
Equation
produced by
the forego-
ing problem.

$bb = +24599$, which are to be substituted in this infinite Series.

To perform this Substitution, I first extract the cube Root of 24599 to
obtain $b^{\frac{2}{3}}$, which I find to be nearly 29,08, and consequently $\frac{2a}{3b^{\frac{2}{3}}}$ or

$$\frac{98}{\sqrt[3]{24599}} \text{ is nearly } \frac{10000}{8902}.$$

Squaring afterwards a , and dividing it by bb , I find $\frac{aa}{bb}$ to be nearly
0,0976, whose Square 0,00952 is the Value of $\frac{a^4}{b^4}$; as to the Value of
 $\frac{a^6}{b^6}$ and of the higher Powers, they are to be disregarded in this Series, which

converges very swiftly. Substituting those Values of $\frac{a^2}{b^2}$, $\frac{a^4}{b^4}$ in the infinite
Series, I find $-1,104$ for the Value of x given by the Formula of Art. cxxxvi.
to obtain the other two Values of x , which are also real, Art. cxxxii. I divide
the Equation $x^3 - 90x - 98 = 0$, by $x + 1,104$ and there results for
Quotient $x^2 - 1,104x - 88,781$ with a Remainder 0,0142, which may
be rejected as exceeding small. So that $xx - 1,104x - 88,781 = 0$,
may be considered as the exact Quotient arising from the Division of
 $x^3 - 90x - 98 = 0$ by $x + 1,104$, and as the Product of
the two Roots sought; resolving therefore this Equation, I find
 $x = 0,552 \pm \sqrt{89,0857}$, that is, $+9,990$ and $-8,886$, consequently,
the three Values of x in the proposed Equation $x^3 - 90x - 98 = 0$,

are $-1, 104$; $+ 9,990$; $- 8,886$, whereof the second only is for our Purpose.

The foregoing Method of resolving by Approximation, Equations of the third Degree, whose three Roots are real, is liable to this Inconveniency, that when a differs but little from b , the Terms of the Series expressing the Value of x , decrease so slowly, that a great Number will be required to give a near Value of the Root, whereby the Computation is rendered extremely laborious. To remedy which, the Analysts have sought a Method in general more commodious in Practice, which we shall proceed to explain.

Inconveniency to which the foregoing Method of approximating to the roots of Equations of the third degree is liable.

CXXXVII.

With this View, let the Equation $x^3 + px + q = 0$, or rather $x^3 - px + q = 0$, (the Case in which the Method of Art. cxxvi. fails, being comprehended among those in which p is negative) be resumed, which for greater Simplicity I propose to reduce to this Form, $z^3 - z = r$, to effect which, I put $x = mz$, which transforms the Equation $x^3 - px + q = 0$ into $z^3 - \frac{p}{m} z = -\frac{q}{m^3}$, whence it appears, that by putting $m = -\sqrt{p}$, if q is positive, and $m = \sqrt{p}$, if q is negative, the Equation $x^3 - px + q = 0$ will be reduced to the Form $z^3 - z = r$.

Another general method of approximation more commodious in practice.

I now observe, that if this Equation be of the Number of those that cannot be solved by the Formula of Art. cxxvi. r must be less than $\frac{1}{3}\sqrt{\frac{4}{3}}$, or $\sqrt{\frac{4}{27}}$, and at the same Time one of the Roots of the Equation must be positive, and greater than Unity, but less than $\sqrt{\frac{4}{3}}$, for if z exceeded $\sqrt{\frac{4}{3}}$, there would result for r , that is, for $z^3 - z$ a Number greater than $\sqrt{\frac{4}{27}}$, and consequently the Equation $z^3 - z = r$, would be of the Number of those that can be solved by Means of the Formula of Art. cxxvi.

This being premised, I put $z = 1 + \delta$, and substituting this Value in the Equation $z^3 - z = r$, there results $-r + 2\delta + 3\delta^2 + \delta^3 = 0$, which multiplied by four Terms of the Series $1 + A\delta + B\delta^2 + C\delta^3 + \&c.$ the Product will be

$$-r + (2 - rA)\delta + (3 + 2A - rB)\delta^2 + (1 + 3A + 2B - rC)\delta^3 + (A + 3B + 2C)\delta^4 + (B + 3C)\delta^5 + C\delta^6,$$

in which the third, fourth and fifth Terms may be made to vanish, by determining A , B , and C by Means of the Equations $3 + 2A - rB = 0$, $1 + 3A + 2B - rC = 0$, $A + 3B + 2C = 0$, whence is deduced

$$B = \frac{2A}{r} + \frac{3}{r}, C = \frac{2B}{r} + \frac{3A}{r} + \frac{1}{r}, 0 = 2C + 3B + A,$$

whence exterminating C , $\frac{2B}{r} + \frac{3A}{r} + \frac{1}{r} + \frac{3B}{2} + \frac{A}{2} = 0$,

$$\text{or } \left[\frac{2}{r} + \frac{3}{2} \right] \times B + \left[\frac{3}{r} + \frac{1}{2} \right] \times A + \frac{1}{r} = 0,$$

which by substituting the Value of B becomes

$$\left[\frac{2}{r} + \frac{3}{2}\right] \times \left[\frac{2A}{r} + \frac{3}{r}\right] + \left[\frac{3}{r} + \frac{1}{2}\right] \times A + \frac{1}{r} = 0, \text{ or,}$$

$$\left[\frac{4}{r^2} + \frac{6}{r} + \frac{1}{2}\right] \times A + \frac{6}{r^2} + \frac{9}{2r} + \frac{1}{r} = 0, \text{ multiplying by } 2r^2,$$

$$(8 + 12r + r^2) \times A + 12 + 9r + 2r = 0, \text{ whence } A \text{ is found}$$

$$= -\frac{12 + 11r}{8 + 12r + r^2}, B = \frac{14 + 3r}{8 + 12r + r^2}, C = -\frac{15 + r}{8 + 12r + r^2},$$

substituting those Values in the foregoing Equation there results

$$-r + \left[\frac{16 + 36r + 13r^2}{8 + 12r + r^2}\right] \delta - \left[\frac{31 - 6r}{8 + 12r + r^2}\right] \delta^5 - \frac{15 - r}{8 + 12r + r^2} \delta^6 = 0$$

which by rejecting the two last Terms as exceeding small, is reduced to,

$$+ 8r + 12r^2 + r^3 = (16 + 36r + 13r^2) \times \delta; \text{ hence}$$

$$\delta = \frac{8r + 12r^2 + r^3}{16 + 36r + 13r^2}, \text{ Consequently } 1 + \delta \text{ or } z = \frac{16 + 44r + 25r^2 + r^3}{16 + 36r + 13r^2}.$$

CXXXVIII.

To discover the Error produced in the Value of z by rejecting the

Terms — $\frac{31 - 6r}{8 + 12r + r^2} \delta^5 - \frac{15 - r}{8 + 12r + r^2} \delta^6$, it suffices to try

that Case in which those Terms are greatest, let then z or $1 + \delta$ be supposed to be equal to $\sqrt{\frac{4}{3}}$, which is greater than can ever happen, when the proposed Equation is of the Number of those that cannot be solved by the Method of Art. cxxvi. in this Case $r = \frac{1}{3} \sqrt{\frac{4}{3}}$, and the foregoing Method

will give for the Value of z , $\frac{1596 + 1192\sqrt{\frac{4}{3}}}{1452 + 972\sqrt{\frac{4}{3}}}$ instead of the true Value

$\sqrt{\frac{4}{3}}$, now it is easy to perceive that these two Quantities differ only by the one ten thousandth Part of an Unit, and this is consequently the greatest

Error that can be committed by taking $\frac{16 + 44r + 25r^2 + r^3}{16 + 36r + 13r^2}$ for the posi-

tive Root of the Equation $z^3 - z = r$, if this Equation be of the Number of those to which the Formula of Art. cxxvi. cannot be applied,

After the Value of z has been computed by the Expression

$$\frac{16 + 44r + 25r^2 + r^3}{16 + 36r + 13r^2}, \text{ it must be multiplied by } m, \text{ to obtain an}$$

approximate Value of x .

The foregoing method of approximation give the value of x true to the fifth decimal place.

CXXXIX.

If still a greater Degree of Exactness is required, more Terms of the Series $1 + A\delta + B\delta^2 + C\delta^3$ &c. must be taken in. Now, it is evident that in all Cases, $B = \frac{2A}{r} + \frac{3}{r}$, $C = \frac{2B}{r} + \frac{3A}{r} + \frac{1}{r}$; $D = \frac{2C}{r} + \frac{3B}{r} + \frac{A}{r}$; $E = \frac{2D}{r} + \frac{3C}{r} + \frac{B}{r}$, &c. where the last Value is always equal to nothing.

Substituting in the first of these Equations $B = \frac{2A}{r} + \frac{3}{r}$, $\mathcal{Q}$ and q for $\frac{2}{r}$ and $\frac{3}{r}$ respectively, it becomes $B = \mathcal{Q}A + q$, this Value of B wrote in the second Equation $C = \frac{2B}{r} + \frac{3A}{r} + \frac{1}{r}$, gives $C = \frac{2\mathcal{Q}A}{r} + \frac{2q}{r} + \frac{3A}{r} + \frac{1}{r}$, which by substituting R and r' for $\frac{2\mathcal{Q}+3}{r}$ and $\frac{2q+1}{r}$ respectively, becomes $RA + r'$.

Method of approximating to the value of x to any assigned degree of exactness.

Those Values of B and C substituted in the third Equation, gives $D = \frac{2RA}{r} + \frac{2r'}{r} + \frac{3\mathcal{Q}A}{r} + \frac{3q}{r} + \frac{A}{r}$, which by substituting S and s for $\frac{2R+3\mathcal{Q}+1}{r}$ and $\frac{2r'+3q}{r}$ respectively, becomes $SA + s$.

After the same Manner the fourth Equation. will be reduced to $E = TA + t$, by substituting T and t for $\frac{2S+3R+\mathcal{Q}}{r}$, $\frac{2s+3r'+q}{r}$ respectively, &c.

Whence is derived the following Rule for obtaining an Approximation to take in as many Terms of the proposed Series as you please. Put

$$\mathcal{Q} = \frac{2}{r}; R = \frac{2\mathcal{Q}+3}{r}; S = \frac{2R+3\mathcal{Q}+1}{r}; T = \frac{2S+3R+\mathcal{Q}}{r};$$

$$V = \frac{2T+3S+R}{r}, \text{ \&c.}$$

$$q = \frac{3}{r}, r' = \frac{2r+1}{r}, s = \frac{2r'+3q}{r}, t = \frac{2s+3r'+q}{r}, v = \frac{2t+3s+r'}{r}$$

Divide the last of these Quantities q, r, s, t , &c. by the corresponding

Quantity of the upper Series $\mathcal{Q}$, R , S , &c. then $\frac{r}{2+r \times \frac{q}{\mathcal{Q}}}$, $\frac{r}{2+r \times \frac{r}{R}}$,

$\frac{r}{2+r \times \frac{s}{S}}$ &c. or $\frac{1}{\frac{2}{r} + \frac{q}{\mathcal{Q}}}$, $\frac{1}{\frac{2}{r} + \frac{r}{R}}$, $\frac{1}{\frac{2}{r} + \frac{s}{S}}$, $\frac{1}{\frac{2}{r} + \frac{t}{T}}$, $\frac{1}{\frac{2}{r} + \frac{v}{V}}$, &c.

will be so many different Values of δ , whereof each is more exact than the preceding One.

CXL.

To shew the Application of the foregoing Approximations by an Example, let the Equation $z^3 - z = \frac{1}{3}$ be proposed.

Application
of this Me-
thod to an
Example.

Here $\frac{2}{r} = 6$, $\frac{3}{r} = 9$, $\frac{1}{r} = 3$, Hence $\mathcal{Q} = \frac{2}{r} = 6$, $R = \frac{2\mathcal{Q}}{r} + \frac{3}{r} = 45$,

$S = \frac{2R}{r} + \frac{3\mathcal{Q}}{r} + \frac{1}{r} = 327$, $T = \frac{2S + 3R + \mathcal{Q}}{r} = 2385$,

$V = \frac{2T + 3S + R}{r} = 17388$, Also $q = \frac{3}{r} = 9$, $r' = \frac{2q + 1}{r} = 57$,

$s = \frac{2r' + 3q}{r} = 423$, $t = \frac{2s + 3r' + q}{r} = 3078$, $v = \frac{2t + 3s + r'}{r}$

$= 22446$, Therefore $\frac{q}{\mathcal{Q}} = \frac{9}{6}$, $\frac{r'}{R} = \frac{57}{45}$, $\frac{s}{S} = \frac{423}{327}$, $\frac{t}{T} = \frac{3078}{2385}$,

$\frac{v}{V} = \frac{22446}{17388}$. From whence $\delta = 0$, 137158, and consequently

$z = 1, 137158$.

CXL.

There are three Numbers in geometrical Progression whose Sum is a , and the Sum of their Cubes b ; what are those Numbers.

Problem
producing
an Equation
of the third
degree
solved by
this method.

If x , y and z denote the Numbers required, then by the Question $x + y + z = a$, $x^3 + y^3 + z^3 = b$, or by Transposition $x + z = a - y$ and $x^3 + z^3 = b - y^3$ & $(x + z)^3 = (a - y)^3$. Now the Difference of those two last Equations is $3x^2z + 3xz^2 = a^3 - 3a^2y + 3ay^2 - b$,

but $x + z = a - y$ and $xz = y^2$, whence $y^3 - a^2y + \frac{a^3 - b}{3} = 0$.

Let a and b be such that $a^2 = 13$, & $\frac{a^3 - b}{3} = 5$. Then will the Equation be transformed into $y^3 - 13y + 5 = 0$, and substituting $-x \sqrt{13}$

for y , it will become $z^3 - z = \frac{5}{13\sqrt{13}}$, whence z is deduced, and consequently $y = -3,784341$, &c.

CXLII.

Having shewn how to solve Problems producing Equations of the third Degree, we shall proceed to explain the Methods employed by the Analysts for solving Problems producing Equations of the fourth Degree, of which the following is one of the most general of this Order.

The Sum a, the Sum of the Squares b, the Sum of the Cubes c, and the Sum of the Biquadrates d, of any four Numbers being given; to determine the Numbers.

Let the four Numbers be denoted by x, y, z and u , and put $A = a - u$, $B = b - u^2$, $C = c - u^3$, and $D = d - u^4$. From whence by the Conditions of the Problem.

$$x + y + z = A, \quad x^2 + y^2 + z^2 = B, \quad x^3 + y^3 + z^3 = C, \quad x^4 + y^4 + z^4 = D$$

Now, if the second of these Equations be subtracted from the Square of the first, we shall have $2xy + 2xz + 2yz = A^2 - B$.

And if, in the like Manner, the fourth Equation be subtracted from the Square of the 2d, we shall have, $2x^2y^2 + 2x^2z^2 + 2y^2z^2 = B^2 - D$.

Moreover, if from the Square of the former of these last Equations, the Double of the Latter be deducted, there will come out

$$8x^2yz + 8y^2xz + 8z^2xy = A^4 - 2A^2B - B^2 + 2D;$$

or, $8xyz \times (x + y + z) = A^4 - 2A^2B - B^2 + 2D;$

whence, $xyz = \frac{A^4 - 2A^2B - B^2 + 2D}{8A}$ (because $x + y + z = A$)

Again, by multiplying the first and fifth Equations into each other, we get $2x^2y + 2x^2z + 2y^2x + 2y^2z + 2z^2x + 2z^2y + 6xyz = A^3 - AB$, and by multiplying the first and third, Equations together, there arises $x^3 + y^3 + z^3 + x^2y + x^2z + y^2x + y^2z + z^2x + z^2y = AB$.

The Double of which last taken from the Precedent, leaves $6xyz - 2x^3 - 2y^3 - 2z^3 = A^3 - 3AB$, and this added to $2x^3 + 2y^3 + 2z^3 = 2C$, gives $6xyz = A^3 - 3AB + 2C$.

$$\text{Hence } \frac{A^3 - 3AB + 2C}{6} = xyz = \frac{A^4 - 2A^2B - B^2 - 2D}{8A}$$

and consequently by Reduction $A^4 - 6A^2B + 8AC + 3B^2 - 6D = 0$.

In which Equation let the several Values of A, B, C and D be

Problem
producing
an Equation
the most
complicated
of the fourth
degree.

now substituted, and dividing the Whole by 24, we at Length, have

$$u^4 - a u^3 + \frac{a^2 - b}{2} \times u^2 - \frac{a^3 - 3ab + 2c}{6} \times u + \frac{a^4 - 6a^2b + 8ac + 3b^2 - 6d}{24} = 0. \quad \text{Whose Roots Answer}$$

all the Conditions of the Problem.

CXLIII.

Resolution
of the gene-
ral Equation
of the fourth
degree.

Let in general $u^4 + a u^3 + b u^2 + c u + d = 0$ represent all Equations of the fourth Degree, the Difficulty is soon reduced to solve an Equation represented by $z^4 + p z^2 + q z + r = 0$ by putting $u = z - \frac{1}{4}a$.

Now, the Manner the most natural of attempting the Resolution of this Equation, is to consider it as the Product of two Equations of the second Degree, and so contrive that the Determination of the Coefficients affecting the Terms of those Equations of the second Degree, depend on Equations easier to be solved than the proposed one.

Let first $zz + xz + t = 0$ be taken for one of those Equations; it is manifest that the other should have for second Term $-xz$, since the Product of those two Equations should give an Equation wanting the second Term. Let then $zz - xz + s = 0$, be taken for this second Equation, multiplying those two Equations into one another there will result.

$z^4 + (s - x^2 + t)zz + (sx - tx)z + ts = 0$, which being compared with the proposed Equation, gives for determining s, t, x the three Equations, $s - x^2 + t = p$; $sx - tx = q$; $ts = r$.

To make Use of those three Equations, I multiply the first by x , and then add it to the second Equation, and there results $2sx - x^3 = px + q$, whence is deduced $s = \frac{q + px + x^3}{2x}$, which substituted in the Equation

The Resolution of an Equation of the third degree depends on one of the third degree which is called by the Analysts the reduced Equation.

$ts = r$ gives $t = \frac{2rx}{x^3 + px + q}$, and substituting those two Values of

s and t in the Equation $sx - tx = q$, there results at Length

$$\frac{x^3 + px + q}{2} - \frac{2rx^2}{x^3 + px + q} = q, \text{ or, } x^6 + 2px^4 + (pp - 4r)x^2 - q^2 = 0,$$

Equation of the sixth Degree, which by the Method of Art. cxxxiv. is transformed into one of the Third, whence the Difficulty of Equations of the fourth Degree is reduced to that of the Third, for this last Equation (which is called the reduced) being solved, we have no more to do but to substitute the Value it gives for x , in the Equations $zz - xz + s = 0$, $zz + xz + t = 0$,

or rather in $xx - xx + \frac{1}{2}xx + \frac{1}{2}p + \frac{q}{2x} = 0$ and in

$$xx + xx + \frac{2r}{xx + p + \frac{q}{x}} = 0, \text{ and afterwards solve those Equations,}$$

or what amounts to the same, substitute the Value of x in the Roots

$$x = \frac{1}{2}x \pm \sqrt{\left[-\frac{1}{2}xx - \frac{1}{2}p - \frac{q}{2x}\right]}, \text{ \& } x = -\frac{1}{2}x \pm \sqrt{\left[-\frac{2r}{x^2 + p + \frac{q}{x}} + \frac{1}{2}xx\right]}$$

of those two Equations, and the four Roots required of the Equation $x^4 + px^2 + qx + r = 0$, and consequently of the proposed Equation, $u^4 + au^3 + bu^2 + cu + d = 0$ will be obtained.

CKLIV.

It appears at first View, from the Expression of those Values, that in the fourth Degree, as in the Third, one Expression cannot be found for all the Roots of the Equation. However, when it is observed that the Quantity x included in the two foregoing Expressions is necessarily a square Radical, since it resulted from the Resolution of an Equation in which x is always of an even Number of Dimensions, it will easily appear that each of those Expressions may denote four Roots, the first being then expressed thus, The four Roots of an Equation of the fourth degree may be expressed by one formula.

$$x = \pm \frac{1}{2}x \pm \sqrt{\left(-\frac{1}{2}xx - \frac{1}{2}p + \frac{q}{2x}\right)} \text{ and the Second.}$$

$$\text{thus, } x = \mp \frac{1}{2}x + \sqrt{\left(\frac{1}{2}xx - \frac{2r}{xx + p + \frac{q}{x}}\right)}.$$

Those two Equations in Appearance differing; may render the foregoing Reasoning suspect, since it seems to lead to this Absurdity, that an Equation of the fourth Degree may have eight Roots, but this Difference is only apparent, for the Identity of those two Expressions is reduced to that of

$$\sqrt{\left(\frac{1}{2}xx - \frac{2r}{xx + p + \frac{q}{x}}\right)} \text{ and of } \sqrt{\left(-\frac{1}{2}xx - \frac{1}{2}p + \frac{q}{2x}\right)}, \text{ that}$$

$$\text{is, of } -\frac{1}{2}xx - \frac{1}{2}p + \frac{q}{2x}, \text{ and of } -\frac{2r}{xx + p + \frac{q}{x}}. \text{ But the Identity of those two last Quantities cannot fail to take Place, when } x \text{ has been}$$

determined by Means of the Equation $x^6 + 2px^4 + (p^2 - 4r)x^2 - q^2 = 0$.

4. D

Since this Equation has been manifestly deduced from the Equation

$$-\frac{1}{2}xx - \frac{1}{2}p \mp \frac{q}{2x} = -\frac{2r}{x^2 + p \mp \frac{q}{x}}.$$

CXLV.

The first, $z = \pm \frac{1}{2}x \pm \sqrt{-\frac{1}{4}xx - \frac{1}{2}p - \frac{q}{2x}}$ of the two foregoing Expressions is to be chosen, as being the most Commodious.

CXLVI.

Since the Equation from whence is deduced the Value of x , gives necessarily three Values of x preceded by the Sign $\pm$, and there is no Reason of preferring one of those Values to another, besides, it is known, that an Equation of the fourth Degree cannot have more than four Roots, it seems natural to conclude, that any one of those three Values of x preceded by the Sign $\pm$, may be employed indifferently, and still the same Expression for the four Values of z be deduced.

Let which
ever of the
Roots of the
reduced
Equation be
employed
the same
Roots will
always re-
sult for the
proposed
Equation.

To render this manifest, the most natural Method would be to find the three Values of x preceded by $\pm$ given by the Equation $x^3 + 2px^2 + (pp - 4r)x - q^2 = 0$, and substitute them one

after the other in the Expression $z = \pm \frac{1}{2}x \pm \sqrt{-\frac{1}{4}xx - \frac{1}{2}p \mp \frac{q}{2x}}$. to be assured of the Identity of the three different Expressions arising from those Substitutions. But the Calculation that this Method would require is so long, that one could scarce have Patience to continue it to the End, we shall therefore proceed to explain how the Analysts have remedied this Inconveniency.

I observe first, that whatever Value of x is substituted, in the general Expression $z = \pm \frac{1}{2}x \pm \sqrt{-\frac{1}{4}xx - \frac{1}{2}p \mp \frac{q}{2x}}$, the four Values of z expressed at once by this Quantity, may be represented by $i + k$, $i - k$, $-i + l$, $-i - l$; or what amounts to the same Thing, that the four Roots of the Equation $z^4 + pzz + qz + r = 0$, may be represented by $z - i - k$, $z - i + k$, $z + i - l$, $z + i + l$; i denoting the Part $\frac{1}{2}x$ of the Value of z , k and l the two Quantities $\sqrt{-\frac{1}{4}xx - \frac{1}{2}p - \frac{q}{2x}}$, and $\sqrt{-\frac{1}{4}xx - \frac{1}{2}p + \frac{q}{2x}}$, multiplying therefore those four Roots into one another, there results the Equation

$$x^4 - (2ii + kk + ll)z^2 - (2ikk - 2ill)z + i^4 - i^2k^2 - i^2l^2 + k^2l^2 = 0.$$

which compared with $z^4 + pz^2 + qz + r = 0$, gives the Equations
 $p = -2i^2 - k^2 - l^2$, $q = -2ikk + 2ill$, $r = i^4 - i^2k^2 - i^2l^2 + k^2l^2$,
 by Means of which, the Equation $x^6 + 2px^4 + (p^2 - 4r)x^2 - q = 0$,
 is transformed into.

$$x^6 - (4i^2 + 2k^2 + 2l^2)x^4 + (8i^2k^2 + 8i^2l^2 + k^4 - 2k^2l^2 + l^4)x^2 - 4i^2k^4 + 8i^2k^2l^2 - 4i^2l^4 = 0$$

whose Roots are, $x = \pm 2i$; $x = \pm k \pm l$; $x = \pm k \mp l$; now, if we
 substitute any one of those Values of x in the four Expressions contained in

$$z = \pm \frac{1}{2}x \pm \sqrt{\left(-\frac{1}{4}xx - \frac{1}{2}p \pm \frac{q}{2x}\right)}, \text{ or rather in}$$

$$z = \pm \frac{1}{2}x \pm \sqrt{\left(-\frac{1}{4}xx + ii + \frac{1}{2}k^2 + \frac{1}{2}l^2 \mp \frac{i^2l^2 - i^2k^2}{x}\right)}, \text{ \& there}$$

will still result the four Values of z , $i + k$, $i - k$, $-i + l$, $-i - l$.

CXLVII.

It follows from hence, that the Roots of an Equation of the fourth
 Degree are all four real, or all four imaginary; or of the four Roots two
 are real and the other two imaginary, for it is not possible to make any
 other Suppositions with Respect to the four Roots of the given Equation,
 since it has been proved that those four Roots are all expressed at once by the

An Equation of the fourth degree is exactly resolvable when of its four roots two are real and two imaginary.

$$\text{Formula } z = \pm \frac{1}{2}x \pm \sqrt{\left(-\frac{1}{4}xx - \frac{1}{2}p \pm \frac{q}{2x}\right)}.$$

It follows also from hence, that when of the four Roots of an Equation
 of the fourth Degree two are imaginary, and two are real, the reduced
 $x^6 + 2px^4 + (p^2 - 4r)x^2 - qq = 0$, will be of the Number of

Equations solvable by the Formula of Art. cxxvi, and consequently in this
 Case the Equation $z^4 + pz^2 + qz + r = 0$, can be compleatly solved.
 but the contrary happens when the four Roots are all real or all imaginary.
 That this may appear, we are to consider that the general reduced Equation

$$x^6 - (4i^2 + 2l^2 + 2k^2)x^4 + (8i^2k^2 + 8i^2l^2 + k^4 - 2k^2l^2 + l^4)x^2 - 4i^2k^4 + 8i^2k^2l^2 - 4i^2l^4 = 0$$

is the Product of three Roots $xx - 4ii$, $xx - k^2 - 2kl - ll$, $x^2 - k^2 + 2kl - ll$:
 Now, if one of the two Quantities k or l only are imaginary, the
 two Roots, $x^2 - kk - 2kl - ll$, $xx - kk + 2kl - ll$ are imaginary,
 and consequently the reduced Equation is in this Case solvable by the For-
 mula of Art. cxxvi.

But if k and l are both real, or both imaginary, the three Roots
 $xx - 4ii$, $xx - k^2 - 2kl - ll$, $xx - k^2 + 2kl - ll$ of the re-
 duced Equation will be all three real, and consequently cannot be obtained
 by the Formula of Art. cxxvi. whence in the fourth Degree as in the Third
 the Formulas of the Resolution can be applied only to Equations that have
 two impossible Roots.

The contrary happens when the four roots are all real or all imaginary.

How to distinguish the case of the four real roots from that of the four imaginary ones.

Conditions of the four real roots.

Conditions of the four imaginary roots.

Every Equation of the fourth degree wanting the second term and whose third term is positive has imaginary roots.

How to obtain approximate values

When an Equation of the fourth Degree is proposed to be solved, and by its Means the reduced Equation $x^4 - 2p x^2 + (pp - 4r) x^2 - qq = 0$, has been formed, if it be found to be of the Number of those that are not resolvable by the Formula of Art. cxxvi. and it is proposed to find whether in this Case the four Roots are real, or whether they are all four imaginary, it may be effected from the two following Observations.

1^o. When the four Roots are real, the general reduced Equation $x^4 - 2 \times (2i^2 + k^2 + l^2) x^2 + [8i^2(k^2 + l^2) + (k^2 - l^2)^2] x^2 - 4i^2(k^2 - l^2)^2 = 0$, has necessarily the second Term negative, and the third Term positive, since $-2 \times (2i^2 + k^2 + l^2)$ can neither vanish, nor be positive, when i, k, l , are real Quantities, nor can $8i^2 \times (k^2 + l^2) + (k^2 - l^2)^2$ likewise vanish, or be negative, under the same Restrictions.

2^o. But if the four Roots are imaginary, or what amounts to the same Thing, if k, k and l are negative, the reduced which is then expressed thus, $x^4 + 2(k^2 + l^2 - 2ii) x^2 + [(k^2 + l^2)(k^2 + l^2 - 8ii) - 4k^2 l^2] x^2 - 4i^2(k^2 - l^2)^2 = 0$, cannot have at the same Time the second Term negative, and the third positive, for if $k^2 + l^2$ is less than $2i^2$ which would render the second Term negative, the third Term whose Coefficient is $(k^2 + l^2) \times (k^2 + l^2 - 8i^2) - 4k^2 l^2$ will be necessarily negative.

CLXX.

The Inspection of the Equation $z^4 - (2i^2 + k^2 + l^2) z^2 - (2ik^2 - 2il^2) z - &c.$ given in the Art. cxlvi. furnishes an Observation whereby sometimes may be discovered, if an Equation that should have its four Roots either all real or all imaginary, is in the first of those two Cases or in the second. The Observation is, that every Equation of the fourth Degree, wanting the second Term, having the third Term positive, necessarily contains impossible Roots. Since the third Term of all those Equations represented by $(-2ii + kk + ll) z z$ can never be positive, when ii, kk, ll , are positive; that is, when the Roots are real. Knowing therefore that an Equation of the fourth Degree has impossible Roots, and being assured on the other Hand, that it should have its four Roots all real, or all impossible, it is easy to perceive which of those two Cases should take Place.

CL.

When it has been found, that the four Roots of an Equation of the fourth Degree are all real, one of the Roots of its reduced Equation is to be sought by the Method of Art. cxxix. and being substituted in the general Formula

$z = \pm \frac{1}{2} x \pm \sqrt{\left[-\frac{1}{2} x x - \frac{1}{2} p \mp \frac{q}{2x}\right]}$, there will result the Values of the four Roots required.

of the four roots when they are real.

To apply the foregoing Rules to the Solution of the Equation,

$$u^4 - 4u^3 + \frac{a^2 - b}{2} \times u^2 + \frac{a^3 - 3ab + 2c}{6} \times u + \frac{a^4 - 6a^2b + 8ac + 3b^2 - 6d}{24} = 0,$$

produced by the foregoing Problem, let $a = -16$, $b = 58$, $c = -28$,

$d = -2222$, whereby the Equation will be reduced to $u^4 + 16u^3 + 99u^2$

$+ 228u + 144 = 0$, and substituting in this Equation $z = 4$ for u , to

make the 2d Term vanish, it will be transformed into $z^4 + 3z^2 - 52z + 48 = 0$,

which being compared with the general Equation $z^4 + pz^2 + qz + r = 0$,

will give $p = 3$, $q = -52$, $r = 48$; and consequently the reduced

Equation will be $x^6 + 6x^4 - 183x^2 - 2704 = 0$, in which exter-

minating the second Term, by substituting $y = 2$ in the Room of x^2 ,

there will result $y^3 - 195y - 2322 = 0$, which is resolvable by the

Formula of Art. cxxvi. and shews consequently, that the proposed Equa-

tion is of the Number of those that can be exactly solved, that is, of those

that have two real and two impossible Roots. In order to find them, I

employ the Formula of Art. cxxvi. to solve $y^3 - 195y - 2322 = 0$,

and the Value it gives is $y = \sqrt[3]{(1161 + \sqrt{1073296})} - \sqrt[3]{(-1161 + \sqrt{1073296})}$

which is reduced to $\sqrt[3]{(1161 + 1036)} - \sqrt[3]{(-1161 + 1036)}$, or

to $\sqrt[3]{2197} + \sqrt[3]{135}$, or at Length to 18; substituting this Value of

y in $x = \sqrt{(y - 2)}$, there results $x = \sqrt{16} = 4$, which substituted in

the general Value of $z = \pm \frac{1}{2} x \pm \sqrt{\left(-\frac{1}{2} x x - \frac{p}{2} \mp \frac{q}{2x}\right)}$,

there will result for the two real Roots of the Equation

$$z^4 + 3z^2 - 52z + 48 = 0; z = 2 \pm \sqrt{\left(-4 - \frac{3}{2} + \frac{13}{2}\right)},$$

or $z = 2 \pm 1$, that is, either 3, or 1, and for the two impossible Roots

$$z = -2 \pm \sqrt{\left(-4 - \frac{3}{2} - \frac{13}{2}\right)}, \text{ or } z = -2 \pm \sqrt{-12}.$$

Then substituting these four Values in $u = z - 4$, there will result

for the four Roots of the proposed Equation

$$u^4 + 16u^3 + 99u^2 + 228u + 144 = 0, u = -1, u = -3, u = -6 \pm \sqrt{-12}.$$

CLII.

The Sum (a), and the Sum of the Cubes (b), of five Numbers in continued geometrical Proportion being given; to find the Numbers.

Let x , z and y denote the three middle Numbers taken in Order. Then

$\frac{x^2}{z}$ will be the first Number, and $\frac{y^2}{z}$ the last; and we shall have

$$\frac{x^2}{z} + x + z + y + \frac{y^2}{z} = a, \quad \frac{x^6}{z^3} + x^3 + z^3 + y^3 + \frac{y^6}{z^3} = b$$

Another Problem producing an Equation of the fourth degree solved by the foregoing method.

by the Question. Put $u = x + y$, then from the first Equation

$$\frac{x^2}{z} + \frac{y^2}{z} = a - u - z, \text{ and cubing the two Members}$$

$$\frac{x^6}{z^3} + \left[\frac{3x^2}{z} + \frac{3y^2}{z} \right] \times \frac{x^2 y^2}{z^2} + \frac{y^6}{z^3} = (a - u - z)^3, \text{ whence}$$

$$\frac{x^6}{z^3} + \frac{y^6}{z^3} = (a - u - z)^3 - (3a - 3u - 3z) \times z^2, \text{ and}$$

$$x^3 + y^3 = (x + y)^3 - (x + y) \times 3xy = u^3 - 3uz^2, \text{ \& substituting those}$$

$$\text{Values above, we have } (a - u - z)^3 - (3a - 3u - 3z) \times z^2 + u^3 - 3uz^2 + z^3 = b$$

$$\text{and } az - u^2 - uz + z^2 = 0, \text{ the first of which Equations by Reduction,}$$

$$\text{becomes } a^3 - 3a^2 \times (u + z) + 3a \times (u^2 + 2uz) - 3u^2 z - 3uz^2 + 3z^3 = b,$$

$$\text{from whence the other Equation multiplied by } 3z \text{ being subtracted, there}$$

$$\text{remains } a^3 - 3a^2 \times (u + z) + 3a \times (u^2 + 2uz - z^2) = b, \text{ therefore}$$

$$u^2 + 2uz - z^2 - a \times (u + z) = \frac{b}{3a} - \frac{a^2}{3}, \text{ but, by the second}$$

$$\text{Equation, } u^2 + 2uz - z^2 = az + uz, \text{ whence, by Substitution}$$

$$az + uz - a \times (u + z) = \frac{b}{3a} - \frac{a^2}{3}; \text{ that is, } uz - az = \frac{b}{3a} - \frac{a^2}{3}$$

$$\text{or } az - uz = d \text{ (by putting } \frac{a^2}{3} - \frac{b}{3a} = d \text{) from which the second}$$

$$\text{Equation being subtracted, there results } u^2 - z^2 = d, \text{ wherein let } \frac{az - d}{z}$$

$$\text{the Value of } u \text{ (found from the former Equation) be now substituted, and we}$$

$$\text{shall have } \frac{(az - d)^2}{z^2} - z^2 = d, \text{ and consequently, there will result,}$$

$$z^4 - (a^2 - d) \times z^2 + 2adz - d^2 = 0, \text{ whence } z \text{ will be found.}$$

CLIII.

Now let a and b be such that $a^2 + d = 3$, $2ad = 2$, and $d^2 = 3$, whereby the foregoing Equation will be transformed into $z^4 + 3z^2 + 2z - 3 = 0$, Comparing this Equation with the general Equation $z^4 + pz^2 + qz + r = 0$, we will have $p = 3$, $q = 2$, $r = -3$. And these Values being substituted in the reduced Equation $x^6 + 2px^4 + (pp - 4r)xx - qq = 0$, will transform it into $x^6 + 6x^4 + 21x^2 - 4 = 0$.

To solve this Equation, I put $x^2 = u - 2$, to make the second Term vanish, whereby it will be reduced to $u^3 + 9u - 30 = 0$, which according to Art. cxxxiv. is of the Number of those which have only one real Root, and consequently may be resolved by Means of the general Formula of Art. cxxvi. Whence it appears that the proposed Equation has two real Roots and two imaginary ones.

That the proposed Equation has imaginary Roots, might be perceived by observing that the Coefficient of its third Term $2z$ is positive, (Art. CLIII.

Resolving now the Equation $u^3 + 9u - 30 = 0$, by the Formula of Art. cxxvi. I find $u = \sqrt[3]{15 + 6\sqrt{7}} + \sqrt[3]{15 - 6\sqrt{7}}$ wherefore $\pm\sqrt{u-2}$, or $x = \pm\sqrt{\sqrt[3]{15+6\sqrt{7}} + \sqrt[3]{15-6\sqrt{7}} - 2}$.

This Value of x being substituted in the general Value, or Formula

$z = \pm \frac{1}{2}x \pm \sqrt{-\frac{1}{4}xx - \frac{p}{2} \mp \frac{q}{2x}}$, which in the present

Case is $x = \pm \frac{1}{2}x \pm \sqrt{-\frac{1}{4}x^2 - \frac{3}{2} \mp \frac{1}{x}}$ there will result for the

two real Roots, $x = -\frac{1}{2}\sqrt{\sqrt[3]{15+6\sqrt{7}} + \sqrt[3]{15-6\sqrt{7}} - 2}$

$\pm\sqrt{\left[-\frac{1}{4}\sqrt[3]{15+6\sqrt{7}} - \frac{1}{4}\sqrt[3]{15-6\sqrt{7}} - 1 + \frac{1}{\sqrt[3]{15+6\sqrt{7}} + \sqrt[3]{15-6\sqrt{7}} - 2}\right]}$

and the two imaginary Ones. $x = \frac{1}{2}\sqrt{\sqrt[3]{15+6\sqrt{7}} + \sqrt[3]{15-6\sqrt{7}} - 2}$

$\pm\sqrt{\left[-\frac{1}{4}\sqrt[3]{15+6\sqrt{7}} - \frac{1}{4}\sqrt[3]{15-6\sqrt{7}} - 1 - \frac{1}{\sqrt[3]{15+6\sqrt{7}} + \sqrt[3]{15-6\sqrt{7}} - 2}\right]}$

CLIV.

What two Numbers are those, if from the Sum of the Square of the Greater, and six Times the Lesser, be deducted the Cube of the Lesser, the Remainder will be 8; and their Product multiplied by the Greater gives 1.

If x be the Greater & z the Lesser of those Numbers, then $xx + 6 - z^3 = 8$,

Third
Problem
producing
an Equation
of the fourth
degree
solved by
The forego-
ing method.

or $xx = x^2 - 6x + 8$, and $x^2 x = 1$, or $xx = \frac{1}{x}$, then $x^3 - 6x + 8 = \frac{1}{x}$,
or $x^4 - 6x^2 + 8x - 1 = 0$, comparing this Equation with the general
One, I find $p = -6$, $q = 8$, $r = -1$, and consequently the reduced
Equation will be $x^6 - 12x^4 + 40x^2 - 64 = 0$, in which putting
 $x^2 = u + 4$ to make the second Term vanish, there results $u^3 - 8u - 32 = 0$,
which has only one real Root, expressed by

$$\sqrt[3]{16 + \frac{80}{3\sqrt{3}}} + \sqrt[3]{16 - \frac{80}{3\sqrt{3}}} = \frac{2\sqrt[3]{(6\sqrt{3}+10)} + 2\sqrt[3]{(6\sqrt{3}-10)}}{\sqrt{3}}$$

which by employing the Method of Art. LXXXV. is reduced to
 $u = \frac{2 \times (1 + \sqrt{3}) - 2 \times (1 - \sqrt{3})}{\sqrt{3}} = 4$, substituting this Value

of u in $x = \sqrt{u + 4}$, there results $x = \sqrt{8}$, whereby the general Equation
 $x = \pm \frac{1}{2}x \pm \sqrt{\left(-\frac{1}{2}xx - \frac{p}{2} \mp \frac{q}{2x}\right)}$ is transformed into
 $x = \pm \sqrt{2} \pm \sqrt{(1 \mp \sqrt{2})}$, which gives $-\sqrt{2} \pm \sqrt{(1 + \sqrt{2})}$
for the two real Roots of the proposed Equation, and $+\sqrt{2} \pm \sqrt{(1 - \sqrt{2})}$
for the two imaginary Ones.

CLV.

Let it be required to divide the Number 5 into two such Parts, that if
the Square of the Lesser be deducted from it, and the Remainder multiplied by
the Square of the Lesser, and four Times the Lesser deducted from the Product,
the last Remainder will be 29.

Fourth
problem
producing
an Equation
of the fourth
degree solved
by the fore-
going me-
thod.

Let $x =$ the lesser Part, then $(5 - x^2) \times x^2 - 4x = 29$, or
 $x^4 - 5x^2 + 4x + 29 = 0$. This Equation compared with
 $x^4 + px^2 + qx + r = 0$; gives $p = -5$, $q = 4$, $r = 29$, and conse-
quently the reduced Equation will be $x^6 - 10x^4 - 91x^2 - 16 = 0$.

which by putting $x^2 = u + \frac{10}{3}$, will be transformed into

$$u^3 - \frac{373}{3}u - \frac{10622}{27} = 0, \text{ Now, as this Equation cannot be solved}$$

by the Formula of Art. CXXVI. and consequently is of the Number of
those whose three Roots are real, it follows that the Roots required of the
Equation $x^4 - 5x^2 + 4x + 29 = 0$, are either all four real or imaginary,
but observing that when the Roots are real, the second Term of the reduced
Equation is negative and the third positive, I conclude that the proposed
Equation has all its Roots imaginary since the third Term $-91x^2$ of its
reduced Equation is negative,

CLVI.

A Number of Yards of two Sorts of Cloth were bought for 11 l. each Yard of the first Sort cost as many Pounds as there were Yards, and each Yard of the second Sort cost 1 l. Now, if the Number of Yards of the two Sorts of Cloth were interchanged, the Cost (regulated as before) would have been but 9 l. How many Yards of each were there.

Let the Number of Yards of each Sort of Cloth be expressed by x & y respectively, then $x^2 + y = 11$, or $x^2 = 11 - y$, and $y y + x = 9$, or $x = 9 - y y$, consequently $81 - 18 y^2 + y^4 = x^2$, whence $11 - y = y^4 - 18 y^2 + 81$, or $y^4 - 18 y^2 + y + 70 = 0$.
 To solve this Equation, I first find the reduced Equation $x^6 - 36 x^4 + 44 x^2 - 1 = 0$, or $u^3 - 388 u - 2929 = 0$, to which it is reduced by putting $x^2 = u + 12$.
 Now, as this Equation has its three Roots real, and that the second Term $36 x^4$ is negative, whilst the third $44 x^2$ is positive, it follows Art. cXLVIII. that the proposed Equation has all its four Roots real;

To find those Roots I employ the Method of Art. cXXXVII. to solve the Equation $u^3 - 388 u - 2929 = 0$, and substitute 22, 74, the Value of u , in the Equation $x = \sqrt{u + 12}$, which gives 5, 894 for x , and substituting this Value of x in $y = \pm \frac{1}{2} x - \sqrt{(-\frac{1}{4} x^2 - \frac{1}{2} p \mp \frac{q}{2x})}$ there will result 3,426; 2,467; -2,315, -3,579 for the four Roots of the proposed Equation, of which only 2,467 is to the Purpose.

CLVII.

Having shewn in the Resolution of Equations of the second, third, and fourth Degree, how the Analysts by Means of the radical Signs have been able to express the Value of the unknown Quantity in those Equations, it remains to explain how the Equation from whence any proposed radical Expression is derived may be found, for Example, how to find the Equation whose Root is $x = \sqrt[3]{ab^3} + \sqrt[3]{a^2b} + \sqrt[3]{a^2c}$, that, in which $x = \sqrt{(a^2 + b^3)} - \sqrt[3]{(a^3 - b^3)}$, &c.

To solve all Problems of this Kind, or which comes to the same, to take away any Number of Radicals out of an Equation, the Analysts operate as follows, they substitute in the Room of each Radical an unknown Quantity, whence there results 1^o. A new Equation clear of Radicals, 2^o. As many Equations consisting of two Terms as there were Radicals in the proposed Equation. Now each of those Equations consisting of two Terms, will be

cleared of Radicals by raising its two Members to the Power indicated by the Exponent of the radical Sign that one of its two Terms contains. Whence we have no more to do than to exterminate out of all those Equations cleared of Radicals, the unknown Quantities that have been introduced.

CLVIII.

The foregoing method illustrated by an example.

Let it be proposed to take away the Radicals out of the Equation $x = \sqrt[3]{a b^2} + \sqrt[3]{a d^2}$, by putting $\sqrt[3]{a b^2} = y$, and $\sqrt[3]{a d^2} = z$, there results the three Equations $x = y + z$, $y^3 = a b^2$, $z^3 = a d^2$, deducing from the first $y = x - z$, and substituting it in the Second, there results $x^3 - 3x^2z + 3xz^2 - z^3 = ab^2$, out of which, it remains to exterminate z by Means of the Equation $z^3 = a d^2$.

To effect which, I substitute in the first of those two Equations $x^3 - 3x^2z + 3xz^2 - z^3 = ab^2$ in the Room of z^3 , $a d^2$ given by the Second, and it becomes $x^3 - 3x^2z + 3xz^2 - a d^2 = ab^2$, whence is deduced $z^2 = \frac{a d^2 + a b^2 - x^3 + 3x^2z}{3x}$; multiplying afterwards the

two Members of this Equation by z , and substituting in the Room of z^3 its Value $a d^2$, there results a new Equation $a d^2 = \frac{a d^2 z + a b^2 z - x^3 z + 3x^2 z z}{3x}$

which gives $z z = \frac{3 a d^2 x - a b^2 z - x^3 z + 3 x^2 z z}{3 x^4}$

Equating those two Values of $z z$, I deduce an Equation in which z is only of one Dimension, and solving it, I find $z = \frac{x^4 + 2 a d^2 x - a b^2 x}{a b^2 + a d^2 + 2 x^3}$

which being substituted in one of the foregoing Equations, for Example, in $x^3 - 3x^2z + 3xz^2 - a d^2 = ab^2$, gives at Length $x^9 - 3 a d^2 x^6 - 3 a b^2 x^6 + 3 a^2 b^4 x^3 + 3 a^2 d^4 x^3 - 21 a^2 d^2 b^2 x^3 = a^3 b^6 + 3 a^3 b^4 d^2 + 3 a^3 d^4 b^2 + a^3 d^6$.

CLIX.

Sometimes Equations may be cleared of Radicals without having Recourse to the foregoing Method, by only transposing the Terms, and raising the two Members to the Power indicated by the Radical which then will be alone in one of the Members, for Example, if it was proposed to clear the Equation $x = y + \sqrt[3]{(a^3 + \sqrt{a^5 x})}$ of Radicals, transposing y into the first Member, $x - y = \sqrt[3]{(a^3 + \sqrt{a^5 x})}$, and raising both Members to the Cube, $x^3 - 3x^2y + 3xy^2 - y^3 = a^3 + \sqrt{a^5 x}$, and transposing a^3 we have $x^3 - 3x^2y + 3xy^2 - y^3 - a^3 = \sqrt{a^5 x}$, and squaring both Members there will result an Equation clear of Radicals.

C H A P. IV.

Of the Nature and the Number of the Roots of Equations of all Degrees; of the Method of finding the Equations of a lower Degree that are their Divisors, with the Methods of approximating to the Roots of both numeral and literal Equations of every Degree.

THE Resolution of an Equation of the fifth Degree might perhaps be made to depend on the Resolution of one of the Fourth, that of an Equation of the sixth Degree on the Resolution of one of the Fifth, &c. But this Method the Analysts have not pursued.

Having found that Equations of the second Degree have two Roots, those of the Third three, those of the Fourth four; they were inclined to think that in general an Equation has as many Roots as there are Units in the Number of its Dimensions, and to be assured of this Truth, instead of investigating the Roots of an Equation, they sought the Equation that would have given Quantities for its Roots, after the Manner we are about to explain in the Resolution of the following Problem.

The Sum (a), and the Sum of the m Powers (c) of four Numbers in continued geometrical Proportion being given, to determine the Numbers.

If u and y be assumed to represent the two middle Numbers, then from the Nature of continued Proportionals, the two Extrems will be expressed by $\frac{uu}{y}$ and $\frac{yy}{u}$, and so we shall have $\frac{u^2}{y} + u + y + \frac{y^2}{u} = a$, and $\frac{u^{2m}}{y^m} + u^m + y^m + \frac{y^{2m}}{u^m} = b$, now putting $u + y = x$, and $uy = z$, Problem producing an Equation the most complicated of the mth degree.

we will have $u^m + y^m = x^m - mx^{m-2}z + m \times \frac{m-3}{2} x^{m-4} z^2 -$

$m \times \frac{m-4}{2} \times \frac{m-5}{3} x^{m-6} z^3 + m \times \frac{m-5}{2} \times \frac{m-6}{3} \times \frac{m-7}{4} x^{m-8} z^4 - \&c.$

and because $\frac{u^2}{y} + \frac{y^2}{u} = a - x$, the Sum of the m Powers of the

two Extrems $= (a-x)^m - m \times (a-x)^{m-2} z + m \times \frac{m-3}{2} (a-x)^{m-4} z^2 \&c.$

we therefore have

$x^m + (a-x)^m - m \times [x^{m-4} + (a-x)^{m-4}] + \frac{m}{2} \times \frac{m-3}{2} z^2 \times [x^{m-6} + (a-x)^{m-6}] \&c.$

which Equation by writing $\frac{x^3}{2x+a}$ instead of its equal z , becomes

$$x^m + (a-x)^m - \frac{mx^3}{2x+a} \times [x^{m-2} + (a-x)^{m-2}] + \frac{m(m-3)}{2} \times \frac{x^6}{(2x+a)^2} \times [x^{m-4} + (a-x)^{m-4} \&c.]$$

which being solved, the Value of x will be found, from which the several Values of z, u, y will also become known.

II.

Let $x^m + px^{m-1} + qx^{m-2} + \&c. - - - r = 0$, express in general an Equation of the m th Degree clear of Radicals and Fractions. To find the Value of x in this Equation, is the same as to find a Quantity, positive, or negative, real or imaginary, which substituted in $x^m + px^{m-1} + qx^{m-2} + \&c. - - - r = 0$, in the Place of x , will make all the Terms vanish. Let us suppose this Quantity a to be found, I say that $x^m + px^{m-1} + qx^{m-2} + \&c. - - - r = 0$, will be divisible by $x-a$ for it is manifest 1^o. that since x is only of one Dimension in the Divisor, by the Rules of Division the Operation may be carried on until a Remainder is obtained that does not contain x . Let then Q be the Quotient, it is manifest, that if to the Product of the Quotient Q into the Divisor $x-a$, the Remainder R be added, there will result a Quantity equal to the Dividend, now by substituting in the Dividend, a for x , all the Terms by Hypothesis vanish, therefore the Quantity $(x-a) \times Q + R$ should also vanish, if x be put $= a$, but by putting $x = a$, this Quantity becomes $(a-a)Q + R$, and since $(a-a)Q + R = 0$, $R = 0$, wherefore the Division is performed without leaving a Remainder, wherefore $x^m + px^{m-1} + qx^{m-2} + \&c. - - - r$ is exactly divisible by $x-a$.

An Equation may be resolved into as many simple factors as there are units in the highest dimension of the unknown quantity.

Let a Quantity b when substituted in the Room of x in the Quotient Q resulting from the Division, make all the Terms of this Quotient vanish, it will be divisible by $x-b$; and it is manifest that if b when substituted in the Room of x makes all the Terms of the Quotient Q vanish, it will also make the Dividend vanish, for the Dividend, $= (x-a)Q$ therefore every Supposition that reduces Q to nothing, will also reduce the Dividend to nothing, wherefore $x-b$ will also divide the Dividend exactly.

In like Manner, if c be a Quantity which substituted in the Room of x , will make the Quotient Q divided by $x-b$ vanish, the new Quotient and consequently the Dividend will be divisible by $x-c$, we will therefore have as many simple Quantities $x-a, x-b, x-c, \&c.$ as there are Units in m , which multiplied by one another, will produce the proposed Equation, in the Room, therefore of the proposed Equation we may substitute $(x-a)(x-b)(x-c) \&c. = 0$.

III.

When a Quantity is supposed equal to nothing, one of its Factors must be equal to nothing, whence the proposed Equation is the Product of $x - a = 0$ into $(x - b)(x - c)$ &c. or of $x - b = 0$ into $(x - a)(x - c)$ &c. in each of those Cases x is the same Quantity in the same Case, and different in the different Cases, thus $x^2 - (a+b)x + ab$ is the Product of $x - a = 0$ into $(x - b)$ or of $x - b = 0$ into $x - a$ this Equation $x^2 - (a + b)x + ab$ represents these two $aa - aa - ab + ab = 0$ that results by substituting a for x , and $bb - ab - bb + ab = 0$ that results by substituting b for x . In the first of these Cases, x and its Powers, represent a and its Powers. in the second, the Letter x and its Powers represent b and its Powers, whence the Equation $x^m + px^{m-1} + qx^{m-2} - \dots - r = 0$ really represents as many simple Equations as there are Units in the highest Dimension m of the unknown Quantity.

An Equation admits of as many solutions as there are simple factors multiplied by one another that produce it.

IV.

The Product of $x - a$ into $x - b$ cannot be equal to another Product $x - e$ into $x - f$ for if it was then $\frac{x-a}{x-f} = \frac{x-e}{x-b}$ and consequently $x - a$ must be divisible by $x - f$ as also $x - e$ by $x - b$, which cannot be, or $x - f$ and $x - b$ must have a common Divisor, as also $x - a$ and $x - e$ which likewise is impossible, wherefore every Quantity $x^2 + px + q$ in which x rises to two Dimensions cannot be produced but by the Multiplication of two simple Factors $x - a$, $x - b$, and by no other than those two, wherefore in an Equation of the second Degree x cannot possibly have more than two different Values a and b .

No Equation can have more roots than it contains dimensions of the unknown quantity.

In like Manner it will appear, that the Product of $x - a$ into $x - b$ into $x - c$ cannot be equal to $x - e$ into $x - f$ into $x - g$, for then

$$\frac{x-a}{(x-f)(x-g)} = \frac{x-e}{(x-b)(x-c)}, \text{ wherefore the Denominators}$$

of those Factors should have a common Divisor as likewise their Numerators $x - a$, $x - e$, which cannot be, wherefore in an Equation of the third Degree, and in general in any Equation, the unknown Quantity cannot have more Values either real or imaginary than there are Units in the Degree of the Equation.

V.

Objection
against the
foregoing
demonstra-
tion solved.

Against the foregoing Demonstration the following Objection may be started. Let $a=4, b=17, c=7, e=8$ and $x=2$, then $(x-a)(x-b) = -2 \times -15 = -5 \times -6 = (x-7)(x-8) = (x-c)(x-e)$. it is true, that in some Cases by giving x a certain Value $(x-a)(x-b) = (x-c)(x-e)$, but as x is a general and undetermined Quantity, the foregoing Equation should take Place, whatever Value is given to x which is impossible, in Effect let $x=a$ then $(a-a)(a-b) = (a-c)(a-e)$ that is, $(a-c)(a-e) = 0$, which cannot be, since c and e differ from a and b .

VI.

General
form that
may be gi-
ven to every
imaginary
quantity
whatsoever.

Some of those Quantities a, b, c , &c. or all of them, may denote real Quantities, equal, or unequal, or simple imaginary Quantities, as $B\sqrt{-1}$, or mixed ones, as $A+B\sqrt{-1}$, to those two Forms all imaginary Quantities whatsoever being reducible, A and B expressing real Quantities.

1^o. $a+b\sqrt{-1} \pm g+b\sqrt{-1}$, may be reduced to the Form of $A+B\sqrt{-1}$, for $a+b\sqrt{-1} \pm g \pm b\sqrt{-1} = a \pm g + (b \pm b)\sqrt{-1} = A+B\sqrt{-1}$, by putting $a \pm g = A$, and $b \pm b = B$.

2^o. $(a+b\sqrt{-1}) \times (g+b\sqrt{-1})$ may be reduced to $A+B\sqrt{-1}$, for $(a+b\sqrt{-1}) \times (g+b\sqrt{-1}) = (ag-bb+bg+ab)\sqrt{-1}$, wherefore $ag-bb = A$, and $bg+ab = B$.

3^o. $\frac{a+b\sqrt{-1}}{g+b\sqrt{-1}} = A+B\sqrt{-1}$, for $\frac{a+b\sqrt{-1}}{g+b\sqrt{-1}} = \frac{(a+b\sqrt{-1}) \times (g-b\sqrt{-1})}{(g+b\sqrt{-1}) \times (g-b\sqrt{-1})} = \frac{(ag+bb+bg-ab)\sqrt{-1}}{gg+bb}$, wherefore $\frac{ag+bb}{gg+bb} = A$, and $\frac{bg-ab}{gg+bb} = B$.

4^o. $\frac{(a+b\sqrt{-1})}{g+b\sqrt{-1}} = A+B\sqrt{-1}$, for supposing it to be so, $\text{Log.} (a+b\sqrt{-1}) = \text{Log.} (A+B\sqrt{-1})$, or $(g+b\sqrt{-1}) \text{Log.} (a+b\sqrt{-1}) = \text{Log.} (A+B\sqrt{-1})$, wherefore since the limiting Ratio of the cotemporary Increments of $A+B\sqrt{-1}$, and of its Logarithm, is equal to the limiting Ratio of the

cotemporary Increments of $(a + b)$ $g + b\sqrt{-1}$ and of its *Logarithm*,
 $g + b\sqrt{-1}$ being supposed to be a permanent Quantity, we will have
 (N) $(g + b\sqrt{-1}) \times \left(\frac{da + db\sqrt{-1}}{a + b\sqrt{-1}} \right) = \frac{dA + dB\sqrt{-1}}{A + B\sqrt{-1}}$, and
 $(g + b\sqrt{-1}) \times \left(\frac{da + db\sqrt{-1}}{a + b\sqrt{-1}} \right) = (g + b\sqrt{-1}) \times \frac{(da + db\sqrt{-1})(a - b\sqrt{-1})}{(a + b\sqrt{-1})(a - b\sqrt{-1})}$
 $= \frac{gada + gbdb - abdb + bbda}{aa + bb} + \frac{(bada + bbdb + gadb - gbda)\sqrt{-1}}{aa + bb}$

and performing the same Operation upon the second Member of the Equation

(N), we will have $\frac{dA + dB\sqrt{-1}}{A + B\sqrt{-1}} = \frac{(dA + dB\sqrt{-1}) \times (A - B\sqrt{-1})}{(A + B\sqrt{-1}) \times (A - B\sqrt{-1})}$
 $= \frac{AdA + BdB}{AA + BB} \times \frac{(AdB - BdA)\sqrt{-1}}{AA + BB}$, and consequently
 $\frac{gada + gbdb - abdb + bbda + (bada + bbdb + gadb - gbda)\sqrt{-1}}{aa + bb}$
 $= \frac{AdA + BdB + (AdB - BdA)\sqrt{-1}}{AA + BB}$. Now because the real

Part of the first Member is equal to the real Part of the Second, and the imaginary Part to the imaginary Part, we will have the two following

Equations, (O) $\frac{gada + gbdb - abdb + bbda}{aa + bb} = \frac{AdA + BdB}{AA + BB}$

& (P) $\frac{(bada + bbdb + gadb - gbda)\sqrt{-1}}{aa + bb} = \frac{(AdB - BdA)\sqrt{-1}}{AA + BB}$

wherefore (Q) $\frac{AdA + BdB}{AA + BB} = g \times \left\{ \frac{ada + bdb}{aa + bb} \right\} - b \times \left\{ \frac{adb - bda}{aa + bb} \right\}$

I now observe, that if the *Logarithm* of $\sqrt{(AA + BB)}$ be expressed by x the limiting Ratio of the cotemporary Increments of $\sqrt{(AA + BB)}$

& of its *Logarithm* x will be expressed by the Equation $\frac{AdA + BdB}{AA + BB} = dx$.

In like Manner, if the *Logarithm* of $\sqrt{(aa + bb)}$ be expressed by y ,

the Equation $g \times \frac{ada + bdb}{aa + bb} = dy$ will express the limiting Ratio of

the cotemporary Increments of $\sqrt{(aa + bb)}$ and of its *Logarithm* y .

ELEMENTS OF

and if we put $-b \times \left\{ \frac{adb - bda}{aa + bb} \right\} = dz$, and consider this Equation as expressing the limiting Ratio of the cotemporary Increments of two variable Quantities, and $\int -b \times \left\{ \frac{adb - bda}{aa + bb} \right\} = z$

$$\int -b \left\{ \frac{adb - bda}{aa + bb} \right\}$$

that from which it has been deduced, or $\text{Log. } c = z$,

c expressing the Base of the *Logarithms* Art. XLVIII. Chap. III. we will have

$$\text{Log. } \sqrt{AA + BB} = \text{Log. } (aa + bb)^{\frac{z}{c}} + \text{Log. } c - b \int \left\{ \frac{adb - bda}{aa + bb} \right\}$$

$$\text{or } \sqrt{AA + BB} = \sqrt{aa + bb}^{\frac{z}{c}} \times c - b \int \left\{ \frac{adb - bda}{aa + bb} \right\}$$

Performing upon the Equation (P) the same Operation that was performed upon the Equation (O), this Equation (P) after dividing both Members by $\sqrt{-1}$, and reducing becomes

$$(R) \quad \frac{A dB - B dA}{AA + BB} = b \times \left\{ \frac{ada + bdb}{aa + bb} \right\} + g \times \left\{ \frac{adb - bda}{aa + bb} \right\}$$

Now, $\int b \times \left\{ \frac{ada + bdb}{aa + bb} \right\} = b \text{ Log. } \sqrt{aa + bb}$, as to

$\int \frac{A dB - B dA}{AA + BB}$ it Expresses an Arc whose Tangent is $\frac{B}{A}$, for let α

express an Arc whose Tangent is $\frac{B}{A}$, and let this Arc α receive the In-

crement $d\alpha$, then $d \text{ Tang. } \alpha = \text{Tang. } (\alpha + d\alpha) - \text{Tang. } \alpha$, but

$\text{Tang. } (\alpha + d\alpha) = \frac{\text{Tang. } \alpha + \text{tang. } d\alpha}{1 - \text{tang. } \alpha \times \text{tang. } d\alpha}$, consequently $d \text{ tang. } \alpha$

$$= \frac{\text{tang. } d\alpha + \text{tang.}^2 \alpha \times \text{tang. } d\alpha}{1 - \text{tang. } \alpha \times \text{tang. } d\alpha} \text{ and } \frac{\text{tang. } d\alpha}{d \text{ tang. } \alpha} = \frac{1 - \text{tang. } \alpha \times \text{tang. } d\alpha}{1 + \text{tang.}^2 \alpha}$$

which Ratio is less than $\frac{1}{1 + \text{tang.}^2 \alpha}$, but continually approaches to it

when $d\alpha$ decreases, and coincides with it when $d\alpha$ vanishes, and because in this Case $d\alpha$ and $\text{tang. } d\alpha$ continually approach to a Ratio of Equality,

$da = \frac{d. \text{tang. } a}{1 + \text{tang.}^2 a}$ will express the limiting Ratio of the cotemporary Increments of the Arc and its Tangent, in which Expression substituting for $\text{tang. } a$ its Value $\frac{B}{A}$, and for $d. \text{tang. } a$ its Value $\frac{A dB - B dA}{AA}$, there re-

sults $d a = \frac{A dB - B dA}{AA + BB}$. In like Manner it will appear that

$\int. g \times \frac{a db - b da}{a a + b b}$ expresses the Product arising from the Multiplication of g into an Arc whose Tangent is $\frac{b}{a}$.

$$g + b\sqrt{-1}$$

Whence to reduce $(a + b\sqrt{-1})$ to the Form $A + B\sqrt{-1}$, it suffices to describe a Circle with a Radius equal to the Value found for

$\sqrt{(AA + BB)}$, that is, equal to $\sqrt{(a a + b b)} \times c$

and take on the Circumference of this Circle, an Arc equal to the Value

also found of $\int \left\{ \frac{A dB - B dA}{AA + BB} \right\}$ that, is, equal to $h \text{Log.} \sqrt{(a a + b b)} + g \times$

$\int \left\{ \frac{a db - b da}{a a + b b} \right\}$; $\frac{B}{A}$ will be the Tangent of this Arc; B will be

the Sinus and A the Cosinus.

VII.

If one of the Terms of a Multinomial is an imaginary Radical, as for Example, $x - a - b\sqrt{-1}$, the Radical may be made to vanish, by multiplying it by another Multinomial that differs from it only by the Sign that precedes the Radical. Whence it is only the Product of $x - a - b\sqrt{-1}$ into $x - a + b\sqrt{-1}$, that can make the Radical in the proposed Quantity disappear, by giving the Product $x^2 - 2ax + a^2 + b$. Because it is only in this Case that the particular Products of each real Term into $b\sqrt{-1}$ destroy each other by contrary Signs, and in the same Case it is manifest that the Term b that contains the Product of the two Radicals $b\sqrt{-1}$ and $-b\sqrt{-1}$ is necessarily positive. Whence there are never in any Equation whose Coefficients are real Quantities, single imaginary Roots, or an odd Number of imaginary Roots; but the Roots become imaginary in Pairs, and an Equation of an odd Number of Dimensions, has always one real Root.

The imaginary Expressions cannot disappear in the Equation produced but when their Number is even.

VIII.

Let a, b, c, d, e, f, g , &c. express real and positive Quantities, a being greater than b , b greater than c , c greater than d , d greater than e , &c. The Systems of Factors of the Formula's of the second Degree.

Method of finding the Systems of factors corresponding to the Formula's of the second, third, &c. degree.

I. $x^2 + mx + n = 0$, II. $x^2 - mx + n = 0$, III. $x^2 + mx - n = 0$, IV. $x^2 - mx - n$, V. $x^2 + n = 0$, VI. $x^2 - n = 0$.

In which m and n denote real and positive Numbers, will be as follows,
 $(x + a)(x + a)$; $(x + a)(x - a)$; $(x - a)(x - a)$; $(x + a)(x + b)$;
 $(x + a)(x - b)$; $(x - a)(x + b)$; $(x - a)(x - b)$; $(x + a\sqrt{-1})(x - a\sqrt{-1})$;
 $(x + a + a\sqrt{-1})(x + a - a\sqrt{-1})$; $(x - a + a\sqrt{-1})(x - a - a\sqrt{-1})$;
 $(x + a + b\sqrt{-1})(x + a - b\sqrt{-1})$; $(x - a + b\sqrt{-1})(x - a - b\sqrt{-1})$;
 $(x + b + a\sqrt{-1})(x + b - a\sqrt{-1})$; $(x - b + a\sqrt{-1})(x - b - a\sqrt{-1})$;

In like Manner the Systems of Factors of the Formula's of the third, fourth, fifth, &c. Degree will be found.

IX.

To discover the Nature, that is, the Sign of each Coefficient of each System of Factors corresponding to Equations of the second, third, fourth, &c. Degree, the Factors of those Systems are to be multiplied into one another.

When several Systems of Factors produce the same Formula, to determine the nature of the Roots of any proposed Equation contained in this Formula, it will be requisite to find the System of Factors that will produce it, how this may be effected by Means of the Coefficients of its Terms, we shall proceed to explain.

When several Systems of Factors correspond to the same Formula how to find the system of factors corresponding to any proposed Equation contained in this Formula.

I observe that there are as many Quantities a, b, c, d , in each System of Factors, as there are Coefficients m, n, p , &c. Whence there will result as many Equations between a, b, c, d , &c. as there are Coefficients. Consequently if some of the Quantities a, b, c , &c. are equal, there will be more Equations than unknown Quantities, by which Means an Equation will be obtained, out of which the unknown Quantities a, b, c , &c. have been exterminated, which will serve to determine whether the proposed Equation has equal Roots or not, and in Case it has, the Form and Signs of its Roots.

But if all the Quantities a, b, c, d , &c. are unequal, there will result only as many Equations as there are unknown Quantities, and consequently they cannot all be exterminated. But by putting some of them equal, a Quantity greater than 0 in some Cases, and less than 0 in other Cases will be obtained, which will serve to determine whether the Roots of any proposed Equation are all unequal, and in Case they are, the form and Signs of those Roots.

X.

Thus, 1^o. let $x^2 + mx + n = (x+a)(x+a) = x^2 + 2ax + a^2 = 0$; then $m = 2a$, $n = a^2$, consequently $m^2 - 4n = 0$.

2^o. Let $x^2 + mx + n = (x+a+a\sqrt{-1})(x+a-a\sqrt{-1}) = x^2 + 2ax + 2a^2 = 0$, then $m = 2a$, $n = 2a^2$. Consequently $m^2 - 2n = 0$,

3^o. Let $x^2 + mx + n = (x+a)(x+b)$, if b be put equal to a the Quantity $m^2 - 4n = 0$, wherefore if b be less than a , the Quantity $m^2 - 4n$ will be greater or less than 0, to find which of the two should take Place, let $a = 2$ and $b = 1$, then $m = 3$, $n = 2$, which Values being substituted in the Quantity $m^2 - 4n$ there will result $3^2 - 2^2 \cdot 2 > 0$. Whence I conclude that $m^2 - 4n > 0$ is the Condition for this System.

4^o. Let $x^2 + mx + n = (x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$ if b be put equal 0, this System will coincide with the first System, and consequently $m^2 - 4n = 0$, but if b be greater or less than 0, this Quantity will be greater or less than 0, and on Examination it will be found to be less than 0, whence the Condition for this System will be $m^2 - 4n < 0$.

If b be put equal a , this System will coincide with the Second, and consequently $m^2 - 2n = 0$, but when b is less than a this Quantity will be greater or less than 0, and on Examination it will be found to be less than 0, whence the Conditions of this System are $m^2 - 4n < 0$, $m^2 - 2n > 0$.

5^o. Let $x^2 + mx + n = (x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$, b being put equal to a this System will coincide with the Second, and consequently $m^2 - 2n = 0$, wherefore when b is less than a , this Quantity is greater or less than 0, it is easy to perceive that in order to distinguish this System from the foregoing one, that the Condition for this System should be $m^2 - 2n < 0$.

It is thus the following Tables were constructed, which when completed, will serve to determine in any proposed Equation, the Nature and Number of its Roots, that is, whether they are real or imaginary, equal or unequal, positive or negative.

Tables whereby are discovered the nature and number of the roots of any proposed Equation.

I. $\{x^2 + mx + n = 0\}$			II. $\{x^2 - mx + n = 0\}$		
$(x+a)(x+a)$	- - - - -	$m^2 - 4n = 0$	$(x-a)(x-a)$	- - - - -	$m^2 - 4n = 0$
$(x+a+a\sqrt{-1})(x+a-a\sqrt{-1})$	- - - - -	$m^2 - 2n = 0$	$(x-a+a\sqrt{-1})(x-a-a\sqrt{-1})$	- - - - -	$m^2 - 2n = 0$
$(x+a)(x+b)$	- - - - -	$m^2 - 4n > 0$	$(x-a)(x-b)$	- - - - -	$m^2 - 4n > 0$
$(x+a+b\sqrt{-1})(x+a-b\sqrt{-1})$	$\begin{cases} m^2 - 4n < 0 \\ m^2 - 2n > 0 \end{cases}$		$(x-a+b\sqrt{-1})(x-a-b\sqrt{-1})$	$\begin{cases} m^2 - 4n < 0 \\ m^2 - 2n > 0 \end{cases}$	
$(x+b+a\sqrt{-1})(x+b-a\sqrt{-1})$	- - - - -	$m^2 - 2n < 0$	$(x-b+a\sqrt{-1})(x-b-a\sqrt{-1})$	- - - - -	$m^2 - 2n < 0$
III. $x^2 + mx - n = (x+a)(x-b)$			IV. $x^2 - mx - n = (x-a)(x+b)$		
V. $x^2 + n = (x+a\sqrt{-1})(x-a\sqrt{-1})$			VI. $x^2 - n = (x+a)(x-a)$		

$$\text{I. } \{ x^3 + m x^2 + n x + p = 0 \}$$

$$\begin{aligned}
 & (x+a)(x+a)(x+a) \quad - \quad - \quad - \quad - \quad - \quad m^2 - 3n = 0, m^3 - 27p = 0 \\
 & (x+a)(x+a\sqrt{-1})(x-a\sqrt{-1}) \quad - \quad - \quad - \quad - \quad m^2 - n = 0, m^3 - p = 0 \\
 & (x+a)(x+a+a\sqrt{-1})(x+a-a\sqrt{-1}) \quad - \quad - \quad - \quad - \quad 4m^2 - 9n = 0, 2m^3 - 27p = 0 \\
 & (x+a)(x+b)(x+b) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 4(m^2-3n)(n^2-3mp) - (mn-9p)^2 = 0 \begin{cases} 2m^3-9mn+27p > 0 \\ 2m^3-9mn+27p < 0 \end{cases} \\
 & (x+a)(x+a)(x+b) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 4(m^2-3n)(n^2-3mp) - (mn-9p)^2 = 0 \begin{cases} 2m^3-9mn+27p > 0 \\ 2m^3-9mn+27p < 0 \end{cases} \\
 & (x+a)(x+b\sqrt{-1})(x-b\sqrt{-1}) \quad - \quad - \quad - \quad - \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} m n - p = 0 \begin{cases} m^2 - n > 0 \\ m^2 - n < 0 \end{cases} \\
 & (x+b)(x+a\sqrt{-1})(x-a\sqrt{-1}) \quad - \quad - \quad - \quad - \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} m n - p = 0 \begin{cases} m^2 - n > 0 \\ m^2 - n < 0 \end{cases} \\
 & (x+a)(x+b+b\sqrt{-1})(x+b-b\sqrt{-1}) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 2(m^2-n)(n^2-mp) - (mn-p)^2 = 0 \begin{cases} 4m^2-9n > 0 \\ 4m^2-9n < 0 \end{cases} \\
 & (x+b)(x+a+a\sqrt{-1})(x+a-a\sqrt{-1}) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 2(m^2-n)(n^2-mp) - (mn-p)^2 = 0 \begin{cases} 4m^2-9n > 0 \\ 4m^2-9n < 0 \end{cases} \\
 & (x+b)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1}) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 100n(m^2+n)^2 - (2m^3+17mn+p)^2 = 0 \begin{cases} 4m^2-9n > 0 \\ 4m^2-9n < 0 \\ m^2-n > 0 \\ m^2-n < 0 \end{cases} \\
 & (x+a)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1}) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 100n(m^2+n)^2 - (2m^3+17mn+p)^2 = 0 \begin{cases} 4m^2-9n > 0 \\ 4m^2-9n < 0 \\ m^2-n > 0 \\ m^2-n < 0 \end{cases} \\
 & (x+a)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1}) \quad - \quad - \quad - \quad - \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 2m^3-9mn+27p = 0 \begin{cases} 4m^2-9n < 0 \\ 4m^2-9n > 0 \end{cases} \\
 & (x+b)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1}) \quad - \quad - \quad - \quad - \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 2m^3-9mn+27p = 0 \begin{cases} 4m^2-9n < 0 \\ 4m^2-9n > 0 \end{cases} \\
 & (x+a)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1}) \quad - \quad - \quad - \quad - \quad - \quad 4(m^2-3n)(n^2-3mp) - (mn-9p)^2 > 0 \\
 & (x+a)(x+b+c\sqrt{-1})(x+b-c\sqrt{-1}) \quad - \quad - \quad - \quad - \quad \begin{cases} 4(m^2-3n)(n^2-3mp) - (mn-9p)^2 < 0 \\ 2(m^2-n)(n^2-mp) - (mn-p)^2 > 0 \\ 2m^3-9mn+27p > 0, 4m^2-9n > 0 \end{cases} \\
 & (x+a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1}) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{cases} mn-p > 0, 2(m^2-n)(n^2-mp) - (mn-p)^2 < 0 \\ 100n(m^2+n)^2 - (2m^3+17mn+p)^2 < 0, m^2-n > 0 \end{cases} \\
 & (x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}) \quad \begin{cases} mn-p < 0, 100n(m^2+n)^2 - (2m^3+17mn+p)^2 < 0 \\ 4(m^2-3n)(n^2-3mp) - (mn-9p)^2 < 0 \\ 100n(m^2+n)^2 - (2m^3+17mn+p)^2 < 0 \\ 2m^3-9mn+27p < 0, 4m^2-9n > 0 \end{cases} \\
 & (x+b)(x+a+c\sqrt{-1})(x+a-c\sqrt{-1}) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{cases} mn-p > 0, 2m^3-9mn+27p > 0 \\ 100n(m^2+n)^2 - (2m^3+17mn+p)^2 > 0, 4m^2-9n < 0 \end{cases} \\
 & (x+b)(x-c+a\sqrt{-1})(x-c-a\sqrt{-1}) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{cases} mn-p < 0, 100n(m^2+n)^2 - (2m^3+17mn+p)^2 > 0 \\ m^2-n < 0 \end{cases} \\
 & (x+c)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1}) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{cases} 100n(m^2+n)^2 - (2m^3+17mn+p)^2 > 0 \\ 2(m^2-n)(n^2-mp) - (mn-p)^2 > 0 \end{cases} \\
 & (x+c)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1}) \quad - \quad - \quad - \quad - \quad \begin{cases} 2m^3-9mn+27p < 0 \\ 2(m^2-n)(n^2-mp) - (mn-p)^2 < 0 \\ 4m^2-9n < 0 \end{cases}
 \end{aligned}$$

$$\text{II. } \{ x^3 + m x^2 + n x - p = 0 \}$$

$$\begin{aligned}
 & (x+a)(x+a)(x-b) \quad - \quad - \quad - \quad - \quad - \quad 4(m^2-3n)(n^2+3mp) - (mn+9p)^2 = 0 \\
 & (x-b)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1}) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 100n(m^2+n)^2 - (2m^3+17mn-p)^2 = 0 \begin{cases} 2m^3-p > 0 \\ 2m^3-p < 0 \end{cases} \\
 & (x-a)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1}) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} 100n(m^2+n)^2 - (2m^3+17mn-p)^2 = 0 \begin{cases} 2m^3-p > 0 \\ 2m^3-p < 0 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 (x-b)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1}) &= 2m^3+mn-p=0 \\
 (x-b)(x+a+a\sqrt{-1})(x+a-a\sqrt{-1}) &= 2(m^2-n)(n^2+mp)-(mn+p)^2=0 \\
 (x+a)(a+b)(x-c) &= 4(m^2-3n)(n^2+3mp)-(mn+9p)^2>0 \\
 (x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1}) &= \begin{cases} 100n(m^2+n)^2-(2m^3+17mn-p)^2<0 \\ 2m^3-p<0 \end{cases} \\
 (x-b)(x+a+c\sqrt{-1})(x+a-c\sqrt{-1}) &= \begin{cases} 4(m^2-3n)(n^2+3mp)-(mn+9p)^2<0 \\ 100n(m^2+n)^2-(2m^3+17mn-p)^2<0 \\ 2m^3-p>0 \end{cases} \\
 (x-b)(x+c+a\sqrt{-1})(x+c-a\sqrt{-1}) &= \begin{cases} 100n(m^2+n)^2-(2m^3+17mn-p)^2>0 \\ 2m^3+mn-p<0 \end{cases} \\
 (x-c)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1}) &= \begin{cases} 100n(m^2+n)^2-(2m^3+17mn-p)^2>0 \\ 2(m^2-n)(n^2+mp)-(mn+p)^2>0 \\ 2m^3-p>0 \end{cases} \\
 (x-c)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1}) &= \begin{cases} 2m^3+mn-p>0 \\ 2(m^2-n)(n^2+mp)-(mn+p)^2<0 \end{cases}
 \end{aligned}$$

$$\text{III. } \{x^3 + mx^2 - nx + p = 0\}$$

$$\begin{aligned}
 (x+a)(x-b)(x-b) &= 4(m^2+3n)(-3mp+n^2)-(mn+9p)^2=0 \\
 (x+a)(x-b+b\sqrt{-1})(x-b-b\sqrt{-1}) &= 2(m^2+n)(-mp+n^2)-(mn+p)^2=0 \\
 (x+a)(x-b)(x-c) &= 4(m^2+3n)(-3mp+n^2)-(mn+9p)^2>0 \\
 (x+a)(x-b+c\sqrt{-1})(x-b-c\sqrt{-1}) &= \begin{cases} 4(m^2+3n)(-3mp+n^2)-(mn+9p)^2<0 \\ 2(m^2+n)(-mp+n^2)-(mn+p)^2>0 \end{cases} \\
 (x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}) &= 2(m^2+n)(-mp+n^2)-(mn+p)^2<0
 \end{aligned}$$

$$\text{IV. } \{x^3 - mx^2 + nx + p = 0\}$$

$$\begin{aligned}
 (x-a)(x-a)(x+b) &= 4(m^2-3n)(n^2+3mp)-(mn+9p)^2=0 \\
 (x+b)(x-a+b\sqrt{-1})(x-a-b\sqrt{-1}) &= \begin{cases} 100n(m^2+n)^2-(2m^3+17mn-p)^2=0 \\ 2m^3-p>0 \\ 2m^3-p<0 \end{cases} \\
 (x+a)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1}) &= 2m^3+mn-p=0 \\
 (x+b)(x-a+a\sqrt{-1})(x-a-a\sqrt{-1}) &= 2(m^2-n)(n^2+mp)-(mn+p)^2=0 \\
 (x-a)(x-b)(x+c) &= 4(m^2-3n)(n^2+3mp)-(mn+9p)^2>0 \\
 (x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}) &= \begin{cases} 100n(m^2+n)^2-(2m^3+17mn-p)^2<0 \\ 2m^3-p>0 \end{cases} \\
 (x+b)(x-a+c\sqrt{-1})(x-a-c\sqrt{-1}) &= \begin{cases} 4(m^2-3n)(n^2+3mp)-(mn+9p)^2<0 \\ 100n(m^2+n)^2-(2m^3+17mn-p)^2<0 \\ 2m^3-p<0 \end{cases}
 \end{aligned}$$

$$(x+b)(x-c+a\sqrt{-1})(x-c-a\sqrt{-1}) - \begin{cases} 100n(m^2+n)^2 - (2m^3-17mn-p)^2 > 0 \\ 2m^3+mn-p < 0 \end{cases}$$

$$(x+c)(x-a+b\sqrt{-1})(x-a-b\sqrt{-1}) - \begin{cases} 100n(m^2+n)^2 - (2m^3+17mn-p)^2 > 0 \\ 2(m^2-n)(n^2+mp) - (mn+p)^2 > 0 \\ 2m^3-p > 0 \end{cases}$$

$$(x+c)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1}) - \begin{cases} 2m^3+mn-p > 0 \\ 2(m^2-n)(n^2+mp) - (mn+p)^2 < 0 \end{cases}$$

$$V. \{x^3+mx^2-nx-p=0\}$$

$$(x+a)(x+a)(x-a) - - - - - m^2-n=0, m^3-p=0$$

$$(x+a)(x+b)(x-b) - - - - - \begin{cases} mn-p=0 \\ m^2-n > 0 \\ m^2-n < 0 \end{cases}$$

$$(x+a)(x-a)(x+b) - - - - - \begin{cases} 4(m^2+3n)(n^2+3mp) - (-mn+9p)^2 = 0 \\ m^2-n < 0 \\ m^2-n > 0 \end{cases}$$

$$(x-a)(x+b)(x+b) - - - - - \begin{cases} 4(m^2+3n)(n^2+3mp) - (-mn+9p)^2 = 0 \\ m^2-n < 0 \\ m^2-n > 0 \end{cases}$$

$$(x+a)(x+b+b\sqrt{-1})(x+b-b\sqrt{-1}) - 2(m^2+n)(n^2+mp) - (-mn+p)^2 = 0$$

$$(x-a)(x+a+b\sqrt{-1})(x+a-b\sqrt{-1}) - 2m^3-mn-p=0$$

$$(x+a)(x+b)(x-c) - - - - - \begin{cases} 4(m^2+3n)(n^2+3mp) - (-mn+9p)^2 > 0 \\ mn-p < 0, m^2-n > 0 \end{cases}$$

$$(x+a)(x-b)(x+c) - - - - - mn-p > 0$$

$$(x-a)(x+b)(x+c) - - - - - \begin{cases} 4(m^2+3n)(n^2+3mp) - (-mn+9p)^2 > 0 \\ mn-p < 0, m^2-n < 0 \end{cases}$$

$$(x-a)(x+b+c\sqrt{-1})(x+b-c\sqrt{-1}) - \begin{cases} 4(m^2+3n)(n^2+3mp) - (-mn+9p)^2 < 0 \\ 2(m^2+n)(n^2+mp) - (-mn+p)^2 > 0 \\ 2m^3-mn-p < 0, m^2-n < 0 \end{cases}$$

$$(x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1}) - 2(m^2+n)(n^2+mp) - (-mn+p)^2 < 0$$

$$(x-b)(x+a+c\sqrt{-1})(x+a-c\sqrt{-1}) - \begin{cases} 4(m^2+3n)(n^2+3mp) - (-mn+9p)^2 < 0 \\ 2m^3-mn-p > 0, m^2-n > 0 \end{cases}$$

$$VI. \{x^3-mx^2+nx-p=0\}$$

$$(x-a)(x-a)(x-a) - - - - - m^3-3n=0, m^3-27p=0$$

$$(x-a)(x+a\sqrt{-1})(x-a\sqrt{-1}) - - - - - m^2-n=0, m^3-p=0$$

$$(x-a)(x-a+a\sqrt{-1})(x-a-a\sqrt{-1}) - 4m^2-9n=0, 2m^3-27p=0$$

$$(x-a)(x-b)(x-b) - - - - - \begin{cases} 4(m^2-3n)(n^2-3mp) - (mn-9p)^2 = 0 \\ 2m^3-9mn+27p > 0 \\ 2m^3-9mn+27p < 0 \end{cases}$$

$$(x-a)(x+b\sqrt{-1})(x-b\sqrt{-1}) - - - - - \begin{cases} mn-p=0 \\ m^2-n > 0 \\ m^2-n < 0 \end{cases}$$

$$(x-b)(x+a\sqrt{-1})(x-a\sqrt{-1}) - - - - - \begin{cases} mn-p=0 \\ m^2-n > 0 \\ m^2-n < 0 \end{cases}$$

$$(x-a)(x-b+b\sqrt{-1})(x-b-b\sqrt{-1}) - - - - - \begin{cases} 2(m^2-n)(n^2-mp) - (mn-p)^2 = 0 \\ 4m^2-9n > 0 \\ 4m^2-9n < 0 \end{cases}$$

$$(x-b)(x-a+a\sqrt{-1})(x-a-a\sqrt{-1}) - - - - - \begin{cases} 2(m^2-n)(n^2-mp) - (mn-p)^2 = 0 \\ 4m^2-9n > 0 \\ 4m^2-9n < 0 \end{cases}$$

$$(x-b)(x-a+b\sqrt{-1})(x-a-b\sqrt{-1}) - - - - - \begin{cases} 100n(m^2+n)^2 - (2m^3+17mn+p)^2 = 0 \\ 4m^2-9n > 0 \\ 4m^2-9n < 0 \end{cases}$$

$$(x-a)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1}) - - - - - \begin{cases} 100n(m^2+n)^2 - (2m^3+17mn+p)^2 = 0 \\ m^2-n > 0 \\ m^2-n < 0 \end{cases}$$

$$(x-a)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1}) - - - - - \begin{cases} 100n(m^2+n)^2 - (2m^3+17mn+p)^2 = 0 \\ m^2-n > 0 \\ m^2-n < 0 \end{cases}$$

VIII. $x^3 + mx^2 - nx + p = 0$.

$$\begin{array}{ll}
 (x+a)(x-b)(x-b) & - - - - - 4(m^2+3n)(n^2-3mp) - (mn+9p)^2 = 0 \\
 (x+a)(x-b+b\sqrt{-1})(x-b-b\sqrt{-1}) & - - - 2(m^2+n)(n^2-mp) - (mn+p)^2 = 0 \\
 (x+a)(x-b)(x-c) & - - - - - 4(m^2+3n)(n^2-3mp) - (mn+9p)^2 > 0 \\
 (x+a)(x-b+c\sqrt{-1})(x-b-c\sqrt{-1}) & - - - \begin{cases} 4(m^2+3n)(n^2-3mp) - (mn+9p)^2 < 0 \\ 2(m^2+n)(n^2-mp) - (mn+p)^2 > 0 \end{cases} \\
 (x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}) & - - - 2(m^2+n)(n^2-mp) - (mn+p)^2 < 0
 \end{array}$$

IX. $x^3 - mx^2 - nx - p = 0$.

$$\begin{array}{ll}
 (x-a)(x+b)(x+b) & - - - - - 4(m^2+3n)(n^2-3mp) - (mn+9p)^2 \\
 (x-a)(x+b+b\sqrt{-1})(x+b-b\sqrt{-1}) & - - - 2(m^2+n)(n^2-mp) - (mn+p)^2 = 0 \\
 (x-a)(x+b)(x+c) & - - - - - 4(m^2+3n)(n^2-3mp) - (mn+9p)^2 > 0 \\
 (x-a)(x+b+c\sqrt{-1})(x+b-c\sqrt{-1}) & - - - \begin{cases} 4(m^2+3n)(n^2-3mp) - (mn+9p)^2 < 0 \\ 2(m^2+n)(n^2-mp) - (mn+p)^2 > 0 \end{cases} \\
 (x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1}) & - - - 2(m^2+n)(n^2-mp) - (mn+p)^2 < 0
 \end{array}$$

$$x^3 + mx^2 + p = (x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}) = 0$$

$$\begin{array}{ll}
 x^3 + mx^2 - p = \begin{cases} (x-a)(x+a+a\sqrt{-1})(x+a-a\sqrt{-1}) & - - - 2m^3 - p = 0 \\ (x+a)(x+a)(x-b) & - - - - - 4m^3 - 27p = 0 \\ (x+a)(x+b)(x-c) & - - - - - 4m^3 - 27p > 0 \\ (x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1}) & - - - 2m^3 - p < 0 \\ (x-b)(x+a+c\sqrt{-1})(x+a-c\sqrt{-1}) & - - - 4m^3 - 27p < 0, 2m^3 - p > 0 \end{cases} \\
 x^3 - mx^2 + p = \begin{cases} (x+a)(x-a+a\sqrt{-1})(x-a-a\sqrt{-1}) & - - - 2m^3 - p = 0 \\ (x-a)(x-a)(x+b) & - - - - - 4m^3 - 27p = 0 \\ (x-a)(x-b)(x+c) & - - - - - 4m^3 - 27p > 0 \\ (x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}) & - - - 2m^3 - p < 0 \\ (x+b)(x-a+c\sqrt{-1})(x-a-c\sqrt{-1}) & - - - 4m^3 - 27p < 0, 2m^3 - p > 0 \end{cases}
 \end{array}$$

$$x^3 - mx^2 - p = (x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1})$$

$$\begin{array}{ll}
 x^3 + nx + p = \begin{cases} (x+a)(x-b+a\sqrt{-1})(x-b-a\sqrt{-1}) & - - - 100n^3 - p^2 = 0 \\ (x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}) & - - - 100n^3 - p^2 < 0 \\ (x+b)(x-c+a\sqrt{-1})(x-c-a\sqrt{-1}) & - - - 100n^3 - p^2 > 0 \end{cases}
 \end{array}$$

$$\begin{array}{ll}
 x^3 - nx + p = \begin{cases} (x+a)(x-b)(x-b) & - - - - - 4n^3 - 27p^2 = 0 \\ (x+a)(x-b+b\sqrt{-1})(x-b-b\sqrt{-1}) & - - - 2n^3 - p^2 = 0 \\ (x+a)(x-b)(x-c) & - - - - - 4n^3 - 27p^2 < 0 \\ (x+a)(x-b+c\sqrt{-1})(x-b-c\sqrt{-1}) & - - - \begin{cases} 4n^3 - 27p^2 < 0 \\ 2n^3 - p^2 > 0 \end{cases} \\ (x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}) & - - - 2n^3 - p^2 < 0 \end{cases}
 \end{array}$$

$$\begin{array}{ll}
 x^3 + nx - p = \begin{cases} (x-a)(x+b+a\sqrt{-1})(x+b-a\sqrt{-1}) & - - - 100n^3 - p^2 = 0 \\ (x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1}) & - - - 100n^3 - p^2 < 0 \\ (x-b)(x+c+a\sqrt{-1})(x+c-a\sqrt{-1}) & - - - 100n^3 - p^2 > 0 \end{cases}
 \end{array}$$

$$\begin{array}{ll}
 x^3 - nx - p = \begin{cases} (x-a)(x+b)(x+b) & - - - - - 4n^3 - 27p^2 = 0 \\ (x-a)(x+b+b\sqrt{-1})(x+b-b\sqrt{-1}) & - - - 2n^3 - p^2 = 0 \\ (x-a)(x+b)(x+c) & - - - - - 4n^3 - 27p^2 > 0 \\ (x-a)(x+b+c\sqrt{-1})(x+b-c\sqrt{-1}) & - - - \begin{cases} 4n^3 - 27p^2 < 0 \\ 2n^3 - p^2 > 0 \end{cases} \\ (x-a)(x+c+b\sqrt{-1})(x+c-b\sqrt{-1}) & - - - 2n^3 - p^2 < 0 \end{cases}
 \end{array}$$

$$x^3 + p = (x+a)(x-c+b\sqrt{-1})(x-c-b\sqrt{-1}); x^3 - p = (x-a)(x+c+b\sqrt{-1})(x-c-b\sqrt{-1})$$

XX.
If one of the Roots of $x^m + px^{m-1} + qx^{m-2} + \dots + r = 0$, Method of finding the commensurable roots of an equation. is an integer Number a , either positive or negative, it will be a Divisor of the last Term r ; for $a^m + pa^{m-1} + qa^{m-2} + \dots + na + r = 0$, consequently $a^m + pa^{m-1} + qa^{m-2} + \dots + na = -r$, wherefore $a^{m-1} + pa^{m-2} + qa^{m-3} + \dots + n = -\frac{r}{a}$. The first Member of

which Equation is an Integer, as being composed of Integers, wherefore $\frac{r}{a}$ is an Integer, wherefore a is one of the Divisors of r . Whence is derived a Method for investigating all the commensurable Roots of an Equation; for they should all be found by trying the Division of the Equation, by x more or less each of the Divisors of the last Term.

For Example, suppose the Equation $x^3 - 5x^2 + 7x - 3 = 0$ is to be resolved, the Divisors of the last Term -3 , being either -1 , or -3 , or $+1$, or $+3$; I try the Division by $x-1$, $x-3$, $x+1$, $x+3$; the Division by $x-1$ and $x-3$ succeeding, I perceive that the Equation may be wrote thus, $(x-1) \times (x-1) \times (x-3) = 0$, whereby it appears that the proposed Equation has three commensurable Roots, one of which is $+3$, and the other two are equal, and expressed each by $+1$.

When an Equation is not divisible by a simple Equation composed of $x +$ or $-$ some one of the Divisors of the last Term, it will be a Sign that this Equation has no commensurable Root.

XXI.

It may be objected against the foregoing Method of finding the commensurable Roots of an Equation, that if any of the Roots of this Equation is a Fraction, it will not be possible to find it among the Divisors of the last Term: because if fractional Divisors be admitted in a Number, such Divisors may be found without End. But this Objection is easily removed, by observing that in an Equation clear of Fractions, and having the Coefficient of its highest Term Unit, the Value of the unknown Quantity cannot be a Fraction. In an equation whose coefficients are Integers the unknown quantity cannot be a fraction.

known Quantity cannot be a Fraction ($\frac{a}{b}$), whose Numerator and Denominator are rational and integer Numbers.

Let for Example $x^3 + px^2 + qx + r = 0$, and let $\frac{a}{b}$ be supposed to be a Root of this Equation, then $\frac{a^3}{b^3} + \frac{pa^2}{b^2} + \frac{qa}{b} + r = 0$, and $a^3 + pba^2 + qb^2a^2 + b^3r = 0$, consequently by the foregoing Article the Integer Number a should be a Divisor of the last Term b^3r . Now,

since, a and b have no common Divisor, the Fraction $\frac{a}{b}$ being by Hypothesis, reduced to its lowest Terms, it follows that a and b^3 have no common Divisor, Art. xxix. Chap. ii. Wherefore x must be a Divisor of r , wherefore $r = na$, n being an integer Number; whence we have $a^3 + pba^2 + qb^2a + nb^3a = 0$, wherefore $a^2 + pba + qb^2 + nb^3 = 0$, therefore, by the foregoing Article, a should be a Divisor of the last Term $qb^2 + nb^3$, and consequently of $q + bn$, wherefore $q + bn = ma$, whence $a^2 + pba + b^2ma = 0$, and $a + pb + b^2m = 0$, and of course $\frac{a}{b} = -p - mb$, that is, = an integer Number which is contrary to the Hypothesis.

XXII.

Transformation by means of which an equation may be cleared of fractions.

When a Problem produces an Equation whose Coefficients are Fractions, the commensurable Roots it may contain, cannot be found by the foregoing Method; however the Problem by a simple Transformation may be changed into another, in which the Equation to be resolved is clear of Fractions, and having the Coefficient of its highest Term, Unit.

Suppose the Equation proposed is $x^3 + \frac{a}{b}x^2 + \frac{c}{d}x + \frac{e}{f} = 0$, by putting the unknown Quantity x equal to another y divided by some undetermined Number m , I transform the Equation into a new one, $\frac{y^3}{m^3} + \frac{ay^2}{bm^2} + \frac{cy}{dm} + \frac{e}{f} = 0$, or $y^3 + \frac{am}{b}y^2 + \frac{cm^2}{d}y + \frac{em^3}{f} = 0$, in which I perceive, that if m is divisible by b , by d , and by f ; $\frac{am}{b}$, $\frac{cm^2}{d}$ and $\frac{em^3}{f}$ will be Integers, we have therefore no more to do than to assume for m the Product of the Numbers b , d , f , and if those Numbers are not prime to each other, it will be easy to find a Number less than their Product that will be divisible by all the three.

The Equation being transformed into another clear of Fractions, the commensurable Roots of this last Equation are to be investigated by the foregoing Method; and if none are found, neither will the first have any, since x supposed commensurable, cannot give an incommensurable Quantity when divided by the Number m , which is also commensurable.

XXIII.

Inconveniency to which the foregoing method is liable.

The foregoing Method is liable to this Inconveniency, that when the last Term has very many Divisors, the Labour will be excessive to try all the Divisions that this Method prescribes; how the Analysts have abridged it, by limiting to a small Number the Divisors that are to be tried, we shall now proceed to explain.

XXIV.

They observed first, that if in the Root $x + a$ of any Equation, or if in Observations that have served to render the Divisor $x + a$ of any Quantity, x be put equal to a given Number, the Number into which the Divisor will then be transformed, will be a Divisor of the proposed Quantity, in which x has been put equal to the same Number, that is, for Example, if in the Quantity $x^3 - 2x - 21$ of the foregoing method more perfectly which $x - 3$ is a Divisor, x be put $= 5$, the Number 94 into which $x^3 - 2x - 21$ is transformed by this Supposition is necessarily divisible by the Number 2 into which $x - 3$ is transformed by the same Supposition.

They then supposed x in the Quantity whose Divisor they sought, successively equal to several Numbers, such, for Example, as $+1, 0, -1$; Suppositions that should be first made, because they are the easiest substituted, they afterwards sought the Divisors of the Numbers into which the proposed Quantity is transformed by those Substitutions, and observed:

1°. That among all the Divisors of the Number arising from the Supposition of $x = +1$ in the proposed Quantity, the Number $1 + a$ should be found, since $x + a$ is the Divisor sought.

2°. That among all the Divisors of the Number arising from the Supposition of $x = 0$, which are no other than the Divisors of the last Term of the proposed Quantity, the Number a should be found.

3°. That among all the Divisors of the Number arising from the Supposition of $x = -1$ should be the Number $-1 + a$.

XXV.

Now as the Numbers $1 + a, a, -1 + a$ are necessarily such, that the first exceeds the second by Unity, and that the second exceeds the third also by Unity; it follows that of all the Divisors of the Number arising from the Supposition of $x = 0$, those only can express a , that are exceeded by Unity by some one of the Divisors of the Number arising from the Supposition of $x = 1$, and exceed at the same Time by Unity some one of the Divisors of the Number arising from the Supposition of $x = -1$. Fundamental principle for finding the commensurable roots of an equation.

If several Numbers are found among the Divisors arising from the Supposition of $x = 0$, that have the above mentioned Conditions, to reduce the Divisors of the last Term to still more narrow Limits, x is to be put $= 2$, and then observe if among the Divisors of the Number arising from this Supposition, there be found any that exceed by Unity those arising from the Supposition of $x = 1$, and so on.

It is easy to perceive that the Trial that is to be made of all those Divisors should be double, that is, that each of them should be taken as well negatively as positively.

To illustrate this Method and to render the Application of it more easy, we shall proceed to give some Examples shewing the Order to be observed in the Calculation to avoid mistakes, and to abridge, as much as possible, the Labour of the Calculator.

Let it be proposed to find the commensurable Roots of the Equation $x^3 - 2x^2 - 13x + 6 = 0$; or which is the same, let it be proposed to find the Divisors of one Dimension of the Quantity

$$\begin{array}{r|l}
 x^3 - 2x^2 - 13x + 6 \\
 \hline
 \begin{array}{r} 1 \\ 0 \\ -1 \\ -2 \end{array} \left| \begin{array}{l} 8 \\ 6 \\ 16 \\ 16 \end{array} \right. & \begin{array}{r} 1 \cdot 2 \cdot 4 \cdot 8 \\ 1 \cdot 2 \cdot 3 \cdot 6 \\ 1 \cdot 2 \cdot 4 \cdot 8 \cdot 16 \\ \end{array} \left| \begin{array}{l} +4 \\ +3 \\ +2 \\ +1 \end{array} \right. \begin{array}{l} -2 \\ -3 \\ -4 \\ \end{array}
 \end{array}$$

Application
of the fore-
going me-
thod to an
example.

I first set down one under the other the Suppositions 1, 0, -1, to be made for x ; I set down afterwards beside each of those Numbers, the Numbers -8, +6, +16, or simply 8, 6, 16, (because the Signs do not affect the Divisors) into which the Quantity $x^3 - 2x^2 - 13x + 6$ is transformed by those Suppositions, and I separate them from the first Numbers by a vertical Line. I set down in a third Column the Numbers 1, 2, 4, 8; 1, 2, 3, 6; 1, 2, 4, 8, 16, which are the Divisors of the foregoing Numbers; the four first beside 8, of which they are the Divisors; the four that follow beside 6, and the five last beside 16.

This being done, to find among the Divisors, 1, 2, 3, 6, of the Number 6 arising from the Supposition of $x = 0$, that which is to be added or deducted from x to obtain the Divisor sought, or rather to exclude from all those Divisors, those that have not the required Conditions; I observe 1^o. that 1, which is the first of those Divisors, cannot be admitted, whether it be taken with the Sign + or with the Sign -; for if it be taken with the Sign +, that is, if $x + 1$ be considered as the Divisor sought, this Divisor becoming 2 by the Supposition of $x = +1$, and 0 by the Supposition of $x = -1$, we should find 2 among the Numbers of the first Line, and 0 among those of the third: Now the second of those Conditions is not answered. As to -1, it cannot be admitted either; that is, $x - 1$ is not the Divisor sought, for this Divisor becoming 0 by the Supposition of $x = +1$, and -2 by the Supposition of $x = -1$, we should find 0 among the Numbers of the first Line, and the Number 2 among those of the second. Now only the second of those two Conditions is answered. I next perceive that the Divisor 2 is also to be rejected, because if it be taken with the Sign +, that is, if $x + 2$ be considered as the Divisor sought, we would have +3 by the Supposition of $x = +1$, and +1, by the Supposition of $x = -1$, which would require that the Number 3 should be found in the first Line, and 1 in the third: Now the first of those two Conditions is not fulfilled. And if 2 be taken with the Sign -, that is, if $x - 2$ be considered as the Divisor, we would have -1 and -3 for the Suppositions of $x = +1$ and of $x = -1$, which

would require that 1 should be found in the first Line and in the third: the first of which Conditions is only fulfilled.

Having excluded 1 and 2, I assume the Divisor 3, and I perceive that taking it with the Sign +, that is, considering $x + 3$ as the Divisor sought, we should find + 4 by the Supposition of $x = + 1$, and + 2 by the Supposition of $x = - 1$: and in effect I find 4 in the first Line and 2 in the third, wherefore + 3 has the Conditions required, I therefore set it down in the second Line, that is, opposite the Number of which it is a Divisor. I set down at the same Time the Numbers + 4 and + 2 in the Line above it, and below it, not that those Numbers are to be joined to x to serve as Divisors to the proposed Quantity, but because the Trial of the Divisors not being compleated, other Numbers besides + 3 might be found having the required Conditions, and in this Case new Suppositions are to be made, to discover among those Numbers those that are to be excluded. I now try if 3 taken with the Sign — will succeed, that is, if $x - 3$ has the Conditions required to be a Divisor of the proposed Quantity; in this Case we should have — 2 and — 4 by the Suppositions of $x = + 1$, and of $x = - 1$, which happens in effect, whence $x - 3$ has the Conditions required to be a Divisor as well as $x + 3$, I therefore set down in a fifth vertical Column — 2, — 3, — 4.

Lastly, I try 6, and I perceive that if it be taken with the Sign + we should find + 7 and + 5 in the Lines above it and below it, which does not happen; and if it be taken with the Sign — we should find — 5 and — 7 in the same Lines, which likewise does not happen. I conclude therefore that the proposed Quantity can have but $x - 3$ and $x + 3$, for Divisors of one Dimension.

To discover if there is the same Foundation for trying the Division by $x - 3$ as by $x + 3$; I observe that if a fourth Line was made by supposing $x = - 2$, we should find — 5 for the fourth Term of the Column — 2, — 3, — 4; and + 1 for the fourth Term of the Column + 4, + 3, + 2; for it is manifest that the Divisor $x - 3$, would become — 5, by the Supposition of $x = - 2$, and that the Divisor $x + 3$ would become + 1 by the same Supposition. But by putting $x = - 2$ in the proposed Quantity $x^3 - 2x^2 - 13x + 6$, it becomes 16 which is not divisible by 5 and is by 1. Wherefore $x - 3$ cannot be a Divisor of this Quantity, whence if it has one, it must be $x + 3$, I therefore divide $x^3 - 2x^2 - 13x + 6$ by $x + 3$ which succeeds, and gives for exact Quotient $xx - 5x + 2$.

In Numbers so simple as 8, 6, 16, it was an easy Matter not to omit any of their Divisors, because those Numbers have but few Divisors, but in Numbers that have many Divisors, to obtain them all, they must be sought with a certain Order, we shall therefore proceed to explain by an Example how the Analysts perform this Operation.

Method of
finding all
the divi-
sors of any
proposed
number.

Let it be proposed to find all the Divisors of the number 120. I first draw a vertical Line to the left Hand of this Number, and to the left Hand of this Line opposite to 120 I set down 1, as being its first Divisor. I afterwards try whether 2 will divide 120, and as the Division succeeds, I set down 2 to the left Hand of the vertical Line, opposite to 60 the Quotient of the Division, which I set down to the right Hand of the same Line.

1	120
2	60
4	30
8	15
24	5
120	1

I afterwards try the Division by 2, which succeeds, and gives 30 for Quotient, I then set down the new Divisor 2 under the first, and 30 under 60. I multiply at the same Time the new Divisor 2 by the one above it 2, and set down the Product 4 to the left Hand of the second 2, as being a new Divisor of the proposed Number 120.

As 30 is divisible by 2, I set down 2 again to the left Hand of the vertical Line in the fourth Row, and the Quotient 15 to the right Hand opposite to it in the same Row. I multiply at the same Time the new Divisor 2 by 4, which gives 8 for a new Divisor of the proposed Number. But omit multiplying it by the first Divisors, because there would result 4, which has been already set down.

15 not being divisible by 2, I try the Division by 3, which succeeds, and gives 5 for the Quotient, which I set down to the right Hand in the fifth Row, and the Divisor 3 to the left; I afterwards multiply 3 by 2, by 4 and by 8, which I find in the superior Lines, and set down to the left Hand of 3, the Products 6, 12, 24, which are, as is manifest, new Divisors of the proposed Number.

5 having no Divisor I set it down to the left of the vertical Line in the fifth Row, and I set down at the same Time the Product of this Number into all the foregoing Divisors 2, 3, 4, 6, 8, 12, 24, and there results 10, 15, 20, 30, 40, 60, 120, which I set down in the same Row to the left Hand of 5.

Which being done, all the Numbers which are to the left Hand of the vertical Line from 1 to 120 are the Divisors of 120. After the same Manner all the Divisors of any other Number may be found by dividing it by its least Divisor that exceeds Unit, and the Quotient again by its least Divisor, and so on, till you have a Quotient that is not divisible by any Number greater than Unit. This Quotient, with these Divisors, will be the first or simple Divisors of the Number, and the Products of the Multiplication of any 2, 3, 4, &c. of them will be the compound Divisors.

XXVIII.

Let it now be propos'd to find the commensurable Roots of the Equation.

$$x^5 - 12x^4 + 5x^3 - 61x^2 + 22x - 120 = 0.$$

I	165	I . 3 . 5 . 11 . 15 . 33 . 55 . 165	+3	-11
O	120	I . 2 . 3 . 4 . 5 . 6 . 8 . 10 . 12 . 15 . 20 . 24 . 30 . 40 . 60 . 120	+2	-12
-I	221	I . 13 . 27 . 221	+1	-13

Having set down in a first vertical Column 1, 0, — 1 for the Values to be successively given to x ; and in another vertical Column the Numbers 165, 120, 221 resulting from the Substitution of those Values in the Quantity $x^5 - 12x^4 + 5x^3 - 61x^2 + 22x - 120$. I place in a third Column the Divisors of those three Numbers, each opposite the Number that produced them. This being done, I examine first if among the Divisors of 120 the Number 2 has the required Conditions, and I perceive that when taken with the Sign + it agrees with the Numbers 3 and 1 above and below it. I therefore set down in the fourth vertical Column + 3, + 2, + 1. But the same Number taken with the Sign — does not succeed, as — 3 is not found below it.

Another example of finding the commensurable roots of an equation by the foregoing method.

Examining in like Manner all the other Divisors of 120, I observe, that the Number 12 taken with the Sign — has the required Conditions, provided the Numbers 11 and 13 that are above and below it are also taken with the Sign —, I therefore set down in a fifth vertical Column the three Numbers — 11, — 12, — 13.

To discover now by which of the two Quantities $x + 2$ or $x - 12$ the Division is to be tried, I observe, that if it was by the first, -2 substituted for x in the Equation should make it vanish, and as this does not happen, I conclude that the Division by $x - 12$ is the only one to be tried, which succeeds and gives $x^4 + 5x^2 - x + 10$ for Quotient, whence 12 is one of the Values of x in the given Equation, and the only commensurable one.

XIX.

Let it be proposed to find the commensurable Roots of the Equation.

$$x^6 - 4x^5 + 3x^4 - 12x^3 - 5x^2 + 11x + 36 = 0$$

2	74					— 1	— 1	the method
I	30	I . 2 . 3 . 5 . 10 . 15 . 30	+ 3	— 2	— 3	+ 10	the corn-	of finding
O	36	I . 2 . 3 . 4 . 6 . 9 . 12 . 18 . 36	+ 2	— 3	— 4	+ 9	menfurable	roots in an
— I	40	I . 2 . 4 . 5 . 8 . 10 . 20 . 40	+ 1	— 4	— 5	+ 8	equation.	

Having set down in a first vertical Column the Values 1, 0, — 1 to be given to x ; in a second, the Numbers 30, 36, 40, into which the proposed Quantity is transformed by these Suppositions, and in a third, all the Divisors 1, 2, 3, 5, 6, 10, 15, 30 of the Number 30; the Di-

visors 1, 2, 3, 4, 6, 9, 12, 18, 36 of the Number 36, the Divisors 1, 2, 4, 5, 8, 10, 20, 40 of the Number 40; I find among all those Divisors four Columns that have the required Conditions, the first $+3, +2, +1$, the second $-2, -3, -5$, the third $-3, -4, -5$, the fourth $+10, +9, +8$.

To discover which of those Progressions are to be rejected, I first put $x = 2$, and set down 2 above 1 in the first Column, I afterwards set down in the second Column 74, the Number into which the proposed Quantity is transformed by supposing $x = 2$. I now perceive without taking the Trouble of investigating the Divisors of this Number, that the two Columns $+3, +2, +1$, and $+10, +9, +8$, are to be rejected, for if the first could be admitted, we should find $+4$ among the Divisors of 74, and if the second could be admitted, we should find $+11$ among the same Divisors.

But I perceive, that the Numbers -1 and -2 , that the Columns $-2, -3, -4$, and $-3, -4, -5$ require, are Divisors of 74. I therefore set down the Numbers -1 and -2 above -2 and -3 in the second and third Columns, I afterwards try to exclude one of these two Columns $-1, -2, -3, -4$; and $-2, -3, -4, -5$; which is easily effected, since if the first was to be admitted, the Equation should be reduced to 0 by putting $x = 3$, which not happening, there remains only the Column $-1, -2, -3, -4, -5$ to be tried, that is, the only Divisor to be tried is $x - 4$, which succeeds, and gives for Quotient $x^5 + 3x^3 - 5x - 9$. whence $+4$ is one of the Values of x in the proposed Equation, and the only commensurable one.

xxx.

Advantage
of investi-
gating the
commensu-
rable divi-
sors in the
reduced ra-
ther than in
the propos-
ed equation
of the
fourth de-
gree.

When an Equation of the fourth Degree has been found to have its four Roots real, before its reduced Equation is attempted to be solved by Approximation, in order to obtain the Value of x to substitute it in that of z (Art. cL. Chap. III.); it will be proper to examine by the foregoing Method whether any of its Roots are commensurable if it has but one commensurable Root, the other three cannot be found otherwise than by Approximation, viz. by solving an Equation of the third Degree, whose three Roots are real; but if an Equation of the fourth Degree has two commensurable Roots, its Resolution may be compleatly effected, since when these two Roots are found, there will only remain to be solved an Equation of the second Degree.

xxxI.

When an Equation of the fourth Degree has two commensurable Roots it will be easier to find them by Means of the reduced Equation than by itself; for it is manifest that in this Case, in the four Values of z represented in general by $i + k; i - k; -i + l; -i - l$; i can only be a commensurable Quantity, and consequently in the Root $xx - 4i$

of the reduced Equation, 4 *ii* will represent a commensurable and Square Quantity, whence we are to select out of the Divisors of the last Term, only square Quantities, affected with the Sign —.

Likewise if the Equation $z^4 + pzz + qz + r = 0$ can be resolved into two Equations of the second Degree the Coefficients of whose Terms are commensurable Quantities, we are to investigate the Divisors of the reduced Equation, and select out of the Divisors of the last Term only square Quantities, and affected with the Sign —.

XXXII.

Tho' an Equation of the fourth Degree has not two commensurable Roots, or is not resolvable into two Equations of the second Degree, the Coefficients of whose Terms are commensurable Quantities, it may be resolvable into two Equations of the second Degree, the Coefficients of whose Terms are square Radicals, in this Case the Part $\pm \frac{1}{2}x$ of the Expression $\pm \frac{1}{2}x + \sqrt{\left(-\frac{1}{4}xx - \frac{1}{2}p \mp \frac{q}{2x}\right)}$ of the Value of z , or which is the same, the Part i common to the four Roots $i + k, i - k, -i + l, -i - l$, should be a simple Radical of the second Degree, because in the reduced Equation x is every where of an even Number of Dimensions, consequently in all Equations of this Nature, their reduced Equation is divisible by $xx \pm$ a commensurable Quantity, and of course Equations of the fourth Degree are resolvable by the foregoing Method, when their reduced Equation has Divisors of this Form.

How to distinguish the equations of the fourth degree whose roots are only affected by square radicals.

XXXIII.

Let, for Example, the Equation of Art. CLI. Chap. III. $u^4 + 16u^3 + 99u^2 + 228u + 144 = 0$ be proposed to be solved, to discover whether it has two rational Roots, I select out of the Divisors of the last Term 2704 of its reduced Equation, $x^4 + 6x^3 - 183x^2 - 2704 = 0$, those that are Squares, and that are affected with the Sign —, and I find that this reduced Equation has for Divisor $xx - 16 = 0$, whence the Roots of the proposed Equation are found as in the above mentioned Article.

The foregoing observations illustrated by an example.

In like Manner, if the Equation of Art. CLIV. Chap. III. $z^4 - 6z^2 + 8z - 1 = 0$ was proposed to be solved, it would appear by this Method that its reduced Equation $x^4 - 12x^2 + 40x^2 - 64 = 0$, has for Divisor $xx - 8$, whence are obtained the Roots of the proposed Equation.

To show the Use of the foregoing Rules in the Solution of Problems, here follow some Examples:

XXXIV.

Two Masons A and B jointly perform a Piece of Work in 12 Days; now, if the Sum of the Days in which they could each have separately performed the same, be multiplied by the Days in which A alone (he working quicker than B) could have done it, the Product will be 1000, in what Time could each do it?

Another problem solved by the foregoing method.

Suppose A could do it alone in x Days, and B in y Days, then $x : 1 = 12 : \frac{12}{x} =$ the Work done by A, and $y : 1 = 12 : \frac{12}{y} =$ the

Work done by B, in 12 Days: then $\frac{12}{y} + \frac{12}{x} = 1$, or $12x + 12y = xy$,

or $12x = xy - 12y$, then $y = \frac{12x}{x - 12}$, consequently $\frac{12xx}{x - 12} = xy$,

but $(x + y) \times x = xx + xy = 1000$, that is, $xx + \frac{12xx}{x - 12} = 1000$,

or $x^3 - 12xx + 12xx = 1000x - 12000$, then $x^3 - 1000x + 12000 = 0$.

Now, by the foregoing Method, there will be found the Divisors 21, 20, 19, differing by Unity, but $x - 20$ will divide the Equation, wherefore $x = 20$, and $y = 30$.

XXXV.

A Mason had two cubical Pieces of Marble, containing between them 1072 solid Inches, and being severally placed in his Yard, they stand upon 130 square Inches of the Ground, what are the Sides of those Cubes?

Third problem solved by the foregoing method.

Suppose $x =$ the Side of the greater, and y the Side of the lesser Cube, then $x^3 + y^3 = 1072$; or $x^3 = 1072 - y^3$ and $xx + yy = 130$, and $x^2 = 130 - yy$; but $x^6 = (1072 - y^3)^2 = (130 - yy)^3$; hence $y^6 - 195y^4 - 1072y^3 + 25350y^2 - 523908 = 0$; and by the foregoing Method, there will be found among the Divisors 9, 8, 7, 6, 5, differing by Unity, but $y - 7$ will divide the Equation, wherefore $y = 7$ and $x = 9$.

XXXVI.

If the proposed Equation has no simple Divisor, then we are to enquire if it has not some quadratick Divisor, (if itself is an Equation of more than three Dimensions) the Solution of Equations of the second Degree being as compleat as that of the first. How the Analysts have proceeded in this Research we shall therefore continue to explain.

Investigation of the method of finding the quadratick commensurable factors of any proposed equation.

Let $xx + bx + c = 0$ express the Equation of the second Degree that may be one of the Factors of a given Equation, or what amounts to the same, let $xx + bx + c$ be the Divisor sought of a given Quantity, by putting $x = 0$ in this Divisor, it is manifest, that it will be reduced to the Number c , and that this Number will be one of the Divisors of the last Term of the given Quantity.

If we afterwards put $x = +1$ in the Divisor $x^2 + b + x c$, it will be transformed into $1 + b + c$, which will be one of the Divisors of the Number that results, by putting likewise in the proposed Quantity $x = 1$. Wherefore, if all the Divisors of this Number be investigated, and Unity be deducted from each of them affected as well with the Sign $+$ as with the Sign $-$, among the Numbers positive and negative, resulting from this Operation, the Number expressing $b + c$ will be found.

If lastly we put $x = -1$, as well in the proposed Quantity as in the Divisor $x x + b x + c$, which then becomes $1 - b + c$, it is easy to perceive that if all the Divisors of the Number that results be investigated, and Unity be deducted from each of them affected as well with the Sign $+$ as with the Sign $-$, among the Numbers positive and negative resulting from this Operation, the Number expressing $-b + c$ will be found.

Now, as c is an arithmetical Mean between $b + c$ and $-b + c$, it follows that out of the three Series of Numbers expressing $b + c$, c , $-b + c$ those only that are in arithmetical Progression are to be selected. When three Numbers in arithmetical Progression are found, it is manifest the one corresponding to the Supposition of $x = 0$ is to be assumed for c , and as the Number that corresponds to the Supposition of $x = 1$, will represent $b + c$, it is manifest, that by deducting the first from the second, the Quantity b will be obtained; substituting afterwards those two Numbers in the Room of b and c in $x^2 + b x + c$, there will result a Divisor for the proposed Quantity to be tried, and the only one to be tried, if but one arithmetical Progression has been found among all the Series of Numbers furnished by the Divisors of the Quantity in which x has been made successively $= 1, 0, -1$. If several arithmetical Progressions have been found, those that are to be rejected will be found much in the same Manner as in the Case of simple Divisors, by making new Suppositions for x , as $-2, -3$, or $+2, +3$, &c.

For if in the proposed Quantity we put, for Example, $x = -2, x = -3$, &c. it is manifest that all the Divisors, as well positive as negative of the Number that results, will be expressed by $4 - 2b + c$, or $9 - 3b + c$, &c. into which $x^2 + b x + c$ is transformed, when $x = -2$, or $x = -3$, &c. and deducting from all those same Divisors $4, 9$, &c. the remainders will be expressed by $-2b + c, -3b + c$, &c. Now, $-2b + c, -3b + c$, &c. being Terms of the arithmetical Progression $b + c, c, -b + c$, no more remains to be done than to search among the Progressions already found, for one that is not altered by these new Values of x , which will serve to determine the Numbers b and c .

Let it be proposed, for Example, to find a Divisor of two Dimensions of the Quantity

$$x^5 + 3x^4 + 2x^3 + 8x^2 - 36x + 21.$$

1	1	1	1	1	-2, 0	-2	-2	0	0	0
0	21	1. 3. 7. 21	0	0	-21, -7, -3, -1, 1, 3, 7, 21	-1	1	-7	-3	-1
-1	65	1. 5. 13. 65	1	1	-66, -14, -6, -2, 0, 4, 12, 64	0	4	-14	-6	-2
-2	125	1. 5. 25. 125	4	4	-129, -29, -9, -5, -3, 1, 21, 124	1			-9	-3
-3	147	{ 1. 3. 7. 21. }	9	9	{ -156, -58, -30, -16, -12, -10, }				-12	
		{ 49. 147 }			{ -8, -6, -2, 12, 40, 138 }					

I first set down in a vertical Column the Values 1, 0, -1 that are proposed to be given to x , I set down afterwards in a second vertical Column the Numbers 1, 21, 65, into which the proposed Quantity is transformed by those Suppositions.

Application
of the fore-
going me-
thod to an
example.

I set down in like Manner in a third Column, beside the first Number, its only Divisor 1; beside 21 its Divisors 1, 3, 7, 21; and beside 65 its Divisors 1, 5, 13, 65.

This being done, I set down in a fourth Column the Numbers 1, 0, 1, Squares of 1, 0, -1 wrote in the first Column, and consequently the Values of xx in the same Suppositions of 1, 0, -1. Lastly, I form a fifth Column from those Conditions:

1°. That the first Line may be -2, 0, found by deducting 1 from 1 taken with the Sign - and with the Sign +.

2°. That the second Line may be composed of the Numbers -21, -7, -3, -1, +1, +7, +21, the same as the Divisors which are set down beside them, but taken with the Sign - and with the Sign +.

3°. That the Numbers of the third Line may be -66, -14, -6, -2, 0, +4, +12, +64, the first of which -66, -14, -6, -2 found by deducting 1 from the Numbers 65, 13, 5, 1 affected with the Sign -, and the others 0, +4, +12, +64 found by deducting 1 from the same Numbers 1, 5, 13, 65 affected with the Sign +.

To determine afterwards the arithmetical Progressions contained in those three series of Numbers. I begin by taking in the first Line the Number -2 for the first Term of a Progression, I take successively for second Terms all those of the second Line; and seek at the same Time the third Terms which these two first would require, and I examine which of those third Terms are found in the third Line: Now, -2 and -21 should give for third Term -40, which is not found in the third Line, I therefore reject 21, and next assume -7 for second Term, and as it should give -12 for third Term, and that -12 is not found in the third Line, I reject also -7, as likewise -3, because this last should give -4, which is not found in the third Line.

As to -1 , since it gives 0 for third Term, and 0 is found in the third Line, I set down in a sixth Column the Progression $-2, -1, 0$, likewise $+1$ taken for second Term, giving $+4$ which is also found in the third Line, I set down the second Progression $-2, +1, +4$, and as $+7$ and $+21$ should give each a third Term, which is not found in the third Line, I reject them. Having determined the Progressions that can begin by 2, I pass to those whose first Term will be 0, and to find them I take, as I did for 2, all the Numbers of the second Line one after the other.

I perceive first, that -21 should give for third Term -42 , which is not found in the third Line; I next perceive that -7 gives -14 which is found there, I therefore set down the Progression $0, -7, -14$; likewise -3 and -1 , giving -6 and -2 , which are also found in the third Line; I set down the Progressions $0, -3, -6$, and $0, -1, -2$. As to the Numbers $+1, +7, +21$, because the third Terms they give are not found in the third Line, I reject them.

To discover now which of those Progressions are to be rejected, I put $x = -2$, and set down -2 in the first Column; observing at the same Time to place, 1°. In the second Column, 125 Number into which the proposed Quantity is transformed, by substituting in it this Value of x . 2°. In the third, the Numbers 1, 5, 25, 125, Divisors of 125. 3°. In the fourth Column, the Number 4 square of -2 , and Value of xx in this Supposition. 4°. In the fifth Column, the Numbers $-129, -29, -9, -5$, found by deducting 4 from the Numbers 125, 25, 5, 1, taken with the Sign $-$, and the Numbers $-3, +1, +21, +121$, found by deducting 4 from the same Numbers 1, 5, 25, 125, taken with the Sign $+$.

By this Means, I perceive that the two Progressions $-2, +1, +4$, and $0, -7, -14$, are to be rejected, because they should give for fourth Term $+7$ and -21 which are not found in the fourth Line.

But the Progressions $-2, -1, 0$; $0, -3, -6$; $0, -1, -2$, giving for fourth Terms $+1, -9, -3$, which are found in this fourth Line, to exclude at least one of those Progressions I have need of a new Supposition. I therefore put $x = -3$, whereby the proposed Quantity is transformed into 147. I therefore set down 147 in the second Column, and beside it, in the third, its Divisors 1, 3, 7, 21, 49, 147; I likewise set down in the fourth Column 9 Square of -3 , or Value of xx in the new Supposition; lastly, I set down in the fifth Column the Numbers $-156, -58, -30, -16, -12, -10, -8, -6, -2, +12, +40, +138$, found by deducting 9 from the Numbers 1, 3, 7, 21, 49, 147, taken with the Sign $-$ and with the Sign $+$.

By this Means, I find that the Progressions $-2, -1, 0, +1$; $0, -1, -2, -3$, should be rejected, but that the Progression $0, -3, -6, -9$, should be retained; because the two fifth Terms of the

first Progressions should be $+2$ and -4 , which are not found in the fifth Line, whilst the fifth Term of the Progression $0, -3, -6, -9$ is -12 which is found there.

After having reduced all the Progressions to one, viz. $0, -3, -6$, &c. I take in this Progression the Number -3 , which, in the sixth Column corresponds to the Supposition of $x = 0$, to express the Term c of the Divisor sought $xx + bx + c$. I afterwards take in the sixth Column likewise, 0 which corresponds to the Supposition of $x = 1$, and which according to the foregoing Principles should express $b + c$; whence deducting c or -3 from 0 , or from $b + c$, I find $+3$ for b , and consequently the Divisor sought $x^2 + bx + c$ is $xx + 3x - 3$, if the proposed Quantity has one of two Dimensions; I therefore try the Division of this Quantity $x^5 + 3x^4 + 2x^3 + 8x^2 - 36x + 21$ by $xx + 3x - 3$, which succeeds, and gives $x^3 + 5x - 7$ for Quotient.

XXXVIII.

Let it be proposed to find a Divisor of two Dimensions of the Quantity

$$\begin{array}{r|rrrrrr}
 & 2 & 133 & 1.7.19.133 & 4 & -137, -23, -11, -5, -3, +3, +15, +129 & +15 & +3 \\
 & 1 & 33 & 1.3.11.33 & 1 & -34, -12, -4, -2, 0, +2, +10, +32 & +10 & +2 \\
 & 0 & 5 & 1.5 & 0 & -5, -1, +1, +5 & +5 & +1 \\
 & -1 & 1 & 1 & 1 & -2, 0 & 0 & 0 \\
 & -2 & 3 & 1.3 & 4 & -7, -5, -3, -1 & -5 & -1
 \end{array}$$

Application
of the fore-
going me-
thod to a-
nother ex-
ample.

I set down in a first vertical Column the Values $2, 1, 0, -1, -2$, which are proposed to be given to x ; I afterwards set down in the second vertical Column the Numbers $133, 33, 5, 1, 3$, into which the proposed Quantity is transformed by those Suppositions; I set down in the third Column opposite to those Numbers all their Divisors. In the fourth the Values $4, 1, 0, 1, 4$ of xx , resulting from the Suppositions made for x in the first Column; lastly, in the fifth Column I set down, for the first Line, the Numbers $-137, -23, -11, -5, -3, +3, +15, +129$ found by deducting 4 from the Numbers $133, 19, 7, 1$, taken first with the Sign $-$ and afterwards with the Sign $+$. Likewise in the second Line the Numbers $-34, -12, -4, -2, 0, +2, +10, +32$ produced by deducting 1 from the Numbers $33, 11, 3, 1$, taken first with the Sign $-$ and afterwards with the Sign $+$. I proceed in like Manner to find the other Lines. All those Numbers being set down, I take in the fifth Line of the fifth Column, the first Number -7 for first Term of a Progression, and taking at the same Time -2 in the fourth Line for second Term, I perceive that the third should be $+3$, and is not found in the third Line; I therefore pass to 0 , which by assuming -7 for first Term, will give $+7$ for third Term, and as $+7$ is not found in the third Line,

I conclude that among the Numbers in the fifth Column, there is no Progression that can begin by -7 .

I therefore take -5 for first Term, and I perceive that assuming -2 for second Term, the third will be $+1$ which is found in the third Line, but gives for fourth Term $+4$, which is not found in the second Line; I therefore reject -2 , and assume 0 for second Term, and there results $+5, +10, +15$, for third, fourth, and fifth Terms; and as all these Numbers are found in the third, second, and first Lines, I set down in the sixth Column the Numbers $15, 10, 5, 0, -5$.

Afterwards taking -3 for first Term, I perceive that neither -2 nor 0 can be assumed for second Terms, because the first would produce the Progressions $-3, -2, -1, 0, +1$, the last Term of which is not found in the first Line; and the second would produce the Progression $-3, 0, +3$, &c. which fails counting from the third Term $+3$ which is not found in the third Line.

It remains to take -1 for first Term, I first give it -2 for second Term which does not succeed, but assuming 0 for second Term, there result for third, fourth, and fifth Terms, the Numbers $+1, +2, +3$ which are in the third, second, and first Lines; I therefore set down in sixth Column the Numbers $+3, +2, +1, 0, -1$.

This being done, I perceive that there is no further Occasion of giving x new Values to exclude one of the two Progressions, because the given Quantity being of four Dimensions, must necessarily have two Divisors of two Dimensions, or none at all; for when a Quantity of four Dimensions has been found to have a Divisor of two Dimensions, the Quotient which is likewise a Divisor, will be also of two Dimensions.

According to this Observation, I take indifferently either of the two foregoing Progressions, the first, for example, in which 5 corresponding to the Supposition of $x = 0$, and 10 to the Supposition of $x = 1$, there results $c = 5$ and $b + c = 10$, that is, $b = 5$. Whence the Divisor furnished by this Progression is $xx + 5x + 5$. I divide therefore the proposed Quantity by $xx + 5x + 5$, and the Division succeeding, gives for Quotient $xx + x + 1$, which is the Divisor that the other Progression would have furnished.

XXXIX.

If it be proposed to find the Divisors of an Equation, such as $6y^4 - y^3 - 21y^2 + 3y + 20 = 0$, whose first Term has a Coefficient different from Unity, they may be obtained by employing the foregoing Principles without being at the Trouble of transforming it into one that shall have the Coefficient of the first term Unit, as may be effected by the Method of Art. XXII.

To make this appear, we shall first examine what concerns Divisors of one Dimension. Let $mx + a$ represent the simple Divisor of any pro-

Method of finding the divisors of one dimension when the highest term of the equation

has a coefficient different from unity.

posed Equation. It is manifest, that if x be put successively equal to 2, 1, 0, — 1, — 2, &c. This Divisor will become in all those Cases $2m + a, m + a, a, -m + a, -2m - a$, &c. Quantities that are in arithmetical Progression in which it is easy to observe,

1°. That the Difference m of all the Terms is the Coefficient of x in the Divisor.

2°. That this same Difference m is a Divisor of the Coefficient of the first Term of the proposed Quantity.

3°. That the Term a corresponding to the Supposition of $x = 0$, is the Part of the Divisor clear of x .

4°. That the same Quantities in arithmetical Progression will be Divisors of the given Quantity, in which x has been successively put equal to 2, 1, 0, — 1, — 2, &c.

From which it follows, that in Order to discover such a simple Divisor when there is any, the same Process is to be pursued as above for the three first Columns. Then among all those Numbers a Progression is to be looked out for, whose common Difference is a Divisor of the Coefficient of the first Term of the proposed Quantity. Lastly, to employ this Progression, in the Room of a in $mx + a$ the Term of the Progression corresponding to the Supposition of $x = 0$ is to be substituted, and in the Room of m the Number found by deducting any Term of the Progression from that immediately above it.

XL.

Let it be required to find, for Example, the Divisors of the Quantity

$6x^4 - x^3 - 21x^2 + 3x + 20$			
2	30	1 . 2 . 3 . 5 . 6 . 10 . 15 . 30	10
1	7	1 . 7	7
0	20	1 . 2 . 4 . 5 . 10 . 20	4
— 1	3	1 . 3	1
— 2	34	1 . 2 . 17 . 34	— 2

I range as usual in the first Column the Suppositions 2, 1, 0, — 1, — 2 to be made for x , in the second, the Numbers 30, 7, 20, 3, 34 into which the proposed Quantity is transformed by those Suppositions. Lastly, in the third all the Divisors of those Numbers.

This being done, in order to discover among all the Numbers of the third Column some Progression that will serve to find the Divisor sought; I try the first Number 1 of the first Line, and I perceive that assuming for second Term the first Number 1 of the second Line, the Progression 1, 1, 1, &c. that results cannot be admitted, because it cannot represent the Divisor sought $mx + a$ which should necessarily vary according to the different Values given to x . I perceive likewise, that 7 cannot be

assumed for second Term, because the third Term produced by this Supposition would be 13, which is not found in the third Line.

Taking afterwards 1 with the Sign —, I perceive that the 1 of the second Line cannot be assumed for second Term, because the third would be 3, which is not found in the third Line. As to 7, it is useless to try whether the third Term it gives is found in the third Line, since the Difference of — 1, to 7 is 8, which is not a Divisor of the Coefficient of the first Term of the proposed Quantity. Wherefore 1 whether taken with the Sign +, or with the Sign —, is to be rejected.

Taking in like Manner all the other Numbers of the first Line, I find only 10 that can have the required Conditions. Assuming 7 for second Term, it gives the Progression 10, 7, 4, 1, — 2 whose common Difference is 3, Divisor of the Coefficient of $6x^3$.

Setting down therefore this Progression in the fourth Column, I take the Term + 4 corresponding to the Supposition of $x = 0$ to express the Part a of the Divisor sought, and deducting the same Term + 4 from the Term above it + 7 which represents $m + a$, I find 3 for the Expression of m , that is, the Divisor sought if there is one, must be $3x + 4$. I therefore try the Division by this Quantity, which succeeds, and gives $2x^3 - 3x^2 - 3x + 5$ for Quotient.

In like Manner, if it was required to find the Divisors of the Quantity

$$24x^5 - 50x^4 + 49x^3 - 140x^2 + 64x + 30$$

2	42	1	2	3	6	7	14	21	42	+ 3	+ 3	+ 7			
1	23	1	23										+ 1	- 1	+ 1
0	30	1	2	3	5	6	10	15	30	- 1	- 5	- 5			
-1	297	1	3	9	11	27	33	99	297	- 3	- 9	- 11			

Here the Divisors to be tried are $2x - 1$, $4x - 5$ and $6x - 5$; of which $6x - 5$ succeeds, there coming out $4x^4 - 5x^3 + 4x^2 - 20x - 6$.

XII.

To shew the Use of the foregoing Rules in the Solution of Problems, here follow some Examples:

A Mason had 2 cubical Pieces of Marble, the Side of the one exceeded the Side of the other by 7 Inches, and the solid Inches in both made 873; what were their Dimensions?

Problem
solved by
the fore-
going rules.

If the Side of the lesser Cube be expressed by x , then the Side of the greater will be $x + 7$; and $x^3 + (x + 7)^3 = 873$ by Quest. Hence $2x^3 + 21x^2 + 147x - 530 = 0$: Now, by the foregoing Method I find the Progression 7, 5, 3, whose common Difference is 2 a Divisor of the Coefficient of $2x^3$, and taking the Term 5 corresponding to the Supposition of $x = 0$, to express a and 2 for m , I find that $2x^3 + 21x^2 + 147x - 530$ is divisible by $2x - 5$, giving for Quotient $x^2 + 13x + 106$, whence $2x = 5$, and $x = \frac{5}{2}$.

XLII.

Another
problem
solved by
the fore-
going rules.

There are two Numbers, the Difference of whose Cubes is 604, and if their Difference be multiplied by the greater, the Product will be 36: What are those Numbers?

If x = the greater and y the lesser of these Numbers; consequently $(x - y) \times x = 36$, or $x - y = \frac{36}{x}$. Also $x^3 - y^3 = 604$, by Quest. and dividing the second Equation by the first, $x^2 + xy + y^2 = \frac{604x}{36} = \frac{151x}{9}$, also by second $x^2 - 2xy + y^2 = \frac{36 \times 36}{xx} = \frac{1296x}{xx}$. The Difference of the two last is $3xy = \frac{151x}{9} - \frac{1296}{xx}$, therefore

$$\frac{11664}{151 - 27y} - y^3 = 604, \text{ whence } 27y^4 - 151y^3 + 16308y - 79540 = 0.$$

Now by the foregoing Method there will be found the Progression 7, 6, 5, 4, 3, whose common Difference is 1 a Divisor of 27 the Coefficient of y^4 the highest Power of y , and taking the Term 5 corresponding to the Supposition of $y = 0$ to express a , and 1 for m , I find that $27y^4 - 151y^3 + 16308y - 79540 = 0$ is divisible by $y - 5$, giving for Quotient $27y^3 - 16y^2 - 80y + 15908$; Wherefore $y = 5$.

XLIII.

Method of
finding the
divisors of
two dimen-
sions when
 x has a
coefficient.

We shall now proceed to explain how Divisors of two Dimensions, when the highest Power of x has a Coefficient, are investigated. Let $mx^2 + bx + c$ represent the Divisor sought of a given Quantity, it is manifest, that the last Term c will be a Divisor of the last Term of the given Quantity, and that m will be a Divisor of the Coefficient of the highest Power of x in the same Quantity.

Assuming therefore one of the Divisors of the Coefficient of the first Term of the proposed Quantity to express m , and performing the same Operation as in Art. xxxvi. with this difference, that deducting instead of the Squares 16, 9, 4, &c. the Product of these Squares into the Number taken for m , in like Manner b and c will be found.

If the Divisor that results, by substituting in $mx^2 + bx + c$, for m the assumed Number, and for b and c those that have been determined in consequence of this Assumption, does not succeed, another of the Divisors of the Coefficient of the first Term of the given Quantity is to be assumed to express m , and the Calculation completed as before. If after having tried all the Divisors of the Coefficient of the first Term, no Divisor is found by this Method, you may conclude that the proposed Quantity has no commensurable Divisor.

XLIV.

Suppose, for Example, you are to find a Divisor of two Dimensions of the Quantity

$$4x^5 + 16x^4 - 22x^3 - 14x^2 - 56x + 77$$

2 117 1.3.9.13.39.117	8 -125, -47, -21, -17, -11, -9, -7, -5, 1, 5, 31, 109	+ 5
1 5 1.5	2 -7, -3, -1, 3	- 3
0 77 1.7.11.77	0 -77, -11, -7, -1, 1, 7, 11, 77	- 11
-1 153 1.3.9.17.51.153	2 -155, -53, -19, -11, -5, -3, -1, 1, 7, 15, 49, 151	- 19
-2 437 1.19.23.437	8 -445, -31, -27, -9, -7, 11, 15, 429	- 27

Having placed, as usual, in the first Column the Numbers 2, 1, 0, Application of this method to an example.
 - 1, - 2, &c. to which x is supposed successively equal; in the second, the Numbers 117, 5, 77, 153, 437 into which the proposed Quantity is transformed by substituting in it those Values of x ; in the third, all the Divisors of the Numbers in the second: I first try if it has a Divisor of two Dimensions, the Coefficient of whose first Term is Unit, and not finding any, I assume for m , that is, for the Coefficient of the first Term of the Divisor, the Number 2 which is one of the Divisors of the Coefficient 4 of the first Term of the given Quantity.

I then set down in the fourth Column, instead of the Squares of the Numbers of the first, the Product of those same Squares into 2 supposed Value of m , that is, I set down in the fourth Column the Numbers 8, 2, 0, 2, 8. I afterwards deduct these Numbers from all those of the third Column taken with the Sign - and with the Sign +, which gives for the first Line of the fifth Column - 125, - 47, - 21, - 17, - 11, - 9, - 7, - 5, + 1, + 5, + 31, + 109; for the second Line - 7, - 3, &c.

All those Numbers being set down, I seek for arithmetical Progressions among those Numbers, and I find but the Progression + 5, - 3, - 11, - 19, - 27 which I set down in the sixth Column; now taking - 11 corresponding to the Supposition of $x = 0$, to express the Number c , and deducting this Number from - 3 which is above it, I find the Remainder 8 for to express b . The Divisor therefore that results from the Supposition of $m = 2$ is $2x^2 + 8x - 11$; and trying the Division it succeeds, giving for Quotient $2x^3 - 7$, and without taking the Trouble of going thro' the Calculation that the Supposition of $m = 4$ would require, I perceive that it would not succeed, because then the Quotient $2x^3 - 7$ could be resolved into Factors of two Dimensions, which is impossible. Whence the proposed Quantity has no other Divisor of two Dimensions but $2x^2 + 8x - 11$. Every quantity of less than six dimensions that has divisors must have some of a less number of dimensions than three.

When the Quantity whose Divisors are to be investigated does not exceed the fifth Degree, they may be found by the foregoing Methods; for if this Quantity is found by those Methods to have no simple or quadratick Divisor, we may also conclude that it has no cubick Divisor.

XLV.

If the quantity has six or more dimensions it may have only divisors of three or more dimensions.

But if the Quantity rises to six or more Dimensions, it may possibly not be resolvable but into Factors of more Dimensions than two, the Method to be pursued for finding those Divisors is founded upon the same Principles as the preceding ones.

Thus if it be a cubick Divisor that is required to be found, let it be $m x^3 - n x^2 + r x - s$, by supposing x equal to the Terms of the arithmetical Progression, it will be as follows:

$x = 3$	$27m - 9n + 3r - s$	$27m - 9n - 3r + s$	$5n - r$	$2n$
$x = 2$	$8m - 4n + 2r - s$	$8m - 4n - 2r + s$	$3n - r$	$2n$
$x = 1$	$m - n + r - s$	$m - n - r + s$	$n - r$	$2n$
$x = 0$	$-s$	$+s$	$-n - r$	
$x = -1$	$-m - n - r - s$	$-m - n + r + s$		

Method for finding divisors of three or more dimensions.

Where the first Differences are not themselves in arithmetical Progression, as in the last Case, but the Differences of its Terms, or the second Differences, are in arithmetical Progression, the common Difference being $2n$, whence n is known. The Quantity r is found in the Column of the second Differences, and s is always to be assumed some Divisor of the last Term of the proposed Equation, as m is of the Coefficient of the first Term; whence all the Coefficients of a Divisor $m x^3 - n x^2 + r x - s$, with which Trial is to be made, may be determined.

If it is a Divisor of four Dimensions that is required, by proceeding in like Manner, you will obtain a Series of Differences whose second Differences are in arithmetical Progression. If it is a Divisor of five Dimensions that is required, you will obtain in the same Manner a Progression whose third Differences will be in arithmetical Progression, and by observing these Progressions, you may discover Rules for determining the Coefficients of the Divisor required.

XLVI.

Hitherto we have only shewn how to find the Divisors of Numeral Equations, but as literal Equations may also have commensurable Divisors, we shall proceed to explain how they are to be investigated.

Suppose, 1^o. That the Equation includes only one known Letter with x , and that this Equation, to use the Language of the Analysts, is homogeneous, that is, has all its Terms raised to the same Dimension, such for Example, as the Equation $x^3 - a x^2 - 10 a^2 x + 6 a^3$. It suffices to substitute Unity in the Room of the given Letter a of this Equation, and find its Divisors by the preceding Rules. These Divisors being found, if they are of one Dimension, the Letter a is to be substituted again beside the Number which serves for second Term. If the Divisor

is of two Dimensions, a is to be placed after the Coefficient of the second Term, and aa after the Number that serves for third Term.

Let the Quantity $x^3 + 4ax^2 - 17a^2x - 12a^3$ be proposed, by putting $a = 1$, that Quantity becomes $x^3 + 4x^2 - 17x - 12$ whose Divisor is $x - 3$, whence $x - 3a$ is a Divisor of the proposed Quantity.

In like Manner if $2x^5 + 5ax^4 - 3a^2x^3 - 8a^3x^2 - 20a^4x + 12a^5$ was proposed, by putting $a = 1$, it will become $2x^5 + 5x^4 - 3x^3 - 8x^2 - 20x + 12$, whose Divisor of two Dimensions is found to be $2x^2 + 5x - 3$. Substituting again in this Divisor a beside 5, and aa beside 3, there results $2x^2 + 5ax - 3aa$ for the Divisor of two Dimensions of $2x^5 + 5ax^4 - 3a^2x^3 - 8a^3x^2 - 20a^4x + 12a^5$.

XLVI.

The same Rules may serve for discovering the Divisors in an homogeneous Equation involving three Letters, but by Help of some Observations of Calculation which naturally occur, they may be found after a much easier Manner.

Suppose, 1°. That the Quantity given which involves three Letters a, b, x , should have for Divisor a Quantity that includes only two Letters x and a , for Example. Since this Divisor, be it what it will, tho' it does not contain b , divides the proposed Quantity in which b enters, the Value of the Letter b must not affect the Division, and consequently this Division may be performed even when $b = 0$. Wherefore if b be put $= 0$ in the given Quantity, the Quantity arising from this Supposition must have for common Divisor with the whole Quantity, the Divisor sought, whence by the Rules of Art. LXXIII. Chap. I. the Divisor sought of whatever Number of Dimensions it, be provided it contains but two Letters, may be found.

Method of finding all the divisors involving two letters in a quantity involving three.

XLVII.

To shew the Application of this Method, let, 1°. The Quantity $x^4 + ax^3 + 2a^2x^2 + 3a^3x + abbx + a^4 + aabb$ be proposed, of which it is required to find a Divisor involving only the Letters a, x .

By putting $b = 0$, there results $x^4 + ax^3 + 2a^2x^2 + 3a^3x + a^4$ whose common Divisor with the entire Quantity, or which is the same, with the Remainder $abbx + aabb$ is $x + a$, which is therefore necessarily a Divisor of the given Quantity, and the greatest it can admit of that does not contain b .

Example.

XLVIII.

Let $x^5 - 4ax^4 + 6aax^3 - abx^3 + abbx^2 + 2aabx^2 - 4a^3x^2 - 2aabbx - 2a^3bx + 2a^3bb$, be proposed; by putting $b = 0$ in this Quantity, there results $x^5 - 4ax^4 + 6aax^3 - 4a^3x^2$ whose greatest common Divisor with the proposed Quantity, or which comes to the same, with the Remainder $-abx^3 + abbx^2 + 2aabx^2$

Another Example.

$-2 a a b b x - 2 a^3 b x + 2 a^3 b b$, that is, the greatest common Divisor of the Quantities $x^3 - 4 a x^2 + 6 a a x - 4 a^3$, and $-x^3 + b x^2 + 2 a x^2 - 2 a b x - 2 a a x + 2 a a b$. Now, if those two Quantities have a common Divisor that does not contain b , it will also be common to the two Parts $-x^3 + 2 a x^2 - 2 a^2 x$ and $b x^2 - 2 a b x + 2 a^2 b$ of the last of those two Quantities; but the common Divisor of those two Parts can only be $x x - 2 a x + 2 a^2$, I try therefore if it divides also $x^3 - 4 a x^2 + 6 a a x - 4 a^3$, and as it divides it in effect, I conclude that it is the Divisor sought of the proposed Quantity.

XLIX.

Method of
finding the
divisors of
one dimen-
sion involv-
ing three
letters.

Suppose now that the proposed Quantity, involving three Letters whose Divisors are required has no Divisor, composed only of two Letters, or if it has, that they have been found and set apart. To find afterwards the Divisors involving three Letters and of one Dimension, that it may contain, I express this Divisor by $m x + n a + p b$, m , n , p denoting Numbers, I now observe that if a , x , b be put successively $= 0$ in this Divisor, there will result the three Quantities $m x + p b$, $n a + p b$, $m x + n a$, such, that the two Terms that each of them contain is found repeated in the two other Quantities. $m x + p b$, for Example, given by the Supposition of $a = 0$, is composed of $m x$ which is found in $m x + n a$ arising from the Supposition of $b = 0$, and of $p b$ which is found in $n a + p b$ arising from the Supposition of $x = 0$; I perceive at the same Time, that the Sum of these three Quantities $m x + n a$, $m x + p b$, $n a + p b$ is the double of the Divisor $m x + n a + p b$.

Now, as those three Quantities are necessarily Divisors of those to which the proposed Quantity is reduced by the same Suppositions of a , x , b equal 0, it follows that to find the Divisors of this Quantity which are of one Dimension and involve three Letters, we must first set down separately the three Quantities into which the proposed one is transformed, by the Supposition of a , x , $b = 0$, then set down beside each of those new Quantities all its Divisors involving two Letters of one Dimension. This being done, we must select three Divisors out of those three Classes of Divisors involving two Letters, having the Conditions above-mentioned, viz. that the two Terms of which each of those Divisors consists, is found in the other two Divisors. Those three Divisors being thus found, the Half of their Sum will be the Divisor of the proposed Quantity, if it has one.

If it be found necessary in order to have in one of those three Divisors of two Letters, the two Terms which should be the Repetition of those that are in the two others, to change their Signs, it is easy to perceive that this Change is admissible, since in general a Quantity that divides another, will still divide it, when the Signs of all its Terms are changed.

L.

To shew the Application of this Method, let it be proposed to find the Divisors involving three Letters of the Quantity

$$\begin{array}{r|l}
 2x^3 + 7ax^2 - 3bx^2 + 5a^2x - 3abx + 4b^2x + 10ab^2 - 6b^3 & 5a - 3b, 10a - 6b \\
 10ab^2 - 6b^3 & 5a - 3b \\
 2x^3 - 3bx^2 + 4b^2x - 6b^3 & 2x - 3b \\
 2x^3 + 7ax^2 + 5a^2x & 2x + 5a \\
 \hline
 & 5a + 2x - 3b
 \end{array}$$

Having set down in a vertical Column the three Quantities $10ab^2 - 6b^3$; $2x^3 - 3bx^2 + 4b^2x - 6b^3$; $2x^3 + 7ax^2 + 5a^2x$, into which this Quantity is transformed by the Supposition of x, a, b equal 0; I set down in a second vertical Column opposite each of these three Quantities, their Divisors of one Dimension involving two Letters. The first gives $5a - 3b$ and $10a - 6b$; the second only $2x - 3b$, and the third $2x + 5a$. I now perceive that if of the two Divisors $5a - 3b, 10a - 6b$, the first $5a - 3b$ be taken; it will have with the two Divisors $2x - 3b, 2x + 5a$, the Conditions required. For this first Divisor $5a - 3b$ contains $5a$, which is repeated in the Divisor $2x + 5a$, and $-3b$ which is repeated in $2x - 3b$; likewise $2x - 3b$ contains $2x$, which is repeated in $2x + 5a$, and $-3b$ which is repeated in $5a - 3b$; lastly, $2x + 5a$ is composed of $2x$ and of $+5a$, which are repeated in the two others $5a - 3b, 2x - 3b$. I therefore, according to the foregoing Rule, add these three Divisors together, and there results $4x - 6b + 10a$, the one half of which $2x - 3b + 5a$ is the Divisor sought. In effect trying the Division, I find for Quotient $x^2 + ax + 2bb$.

LI.

Let it be proposed to find the Divisors of one Dimension involving three Letters, of the Quantity

$$\begin{array}{r|l}
 8x^4 - 2ax^3 - 10bx^3 - 3a^2x^2 - 5abx^2 - 12ab^2x + 9a^2b^2 + 15ab^3 & 3a + 5b, 9a + 15b \\
 9a^2b^2 + 15ab^3 & 3a + 5b \\
 8x^4 - 10bx^3 & 4x - 5b, 8x - 10b \\
 8x^4 - 2ax^3 - 3a^2x^2 & 4x^3 - 3a, 2x + a \\
 \hline
 & 4x - 3a - 5b
 \end{array}$$

Putting in this Quantity x, a, b successively = 0, there results the three Quantities $9a^2b^2 + 15ab^3, 8x^4 - 10bx^3, 8x^4 - 2ax^3 - 3a^2x^2$, which I set down one under the other in a vertical Column. I set down

Application of the foregoing method to an example.

Application of the foregoing method to another example.

in another vertical Column beside each of these Quantities, their Divisors of one Dimension involving two Letters; those of the first are $3a + 5b$ and $9a + 15b$; those of the second $4x - 5b$, $8x - 10b$; and those of the third $4x - 3a$ and $2x + a$.

It is now easy to perceive that the three Divisors $3a + 5b$, $4x - 5b$, and $4x - 3a$ have the Conditions required, provided the Signs of the first be changed, that is, by setting it down thus $-3a - 5b$; I therefore set apart those three Divisors in the fourth Column, I add them together, and I take the one half of their Sum, which gives $4x - 3a - 5b$ for the Divisor sought, and trying the Division, it succeeds, giving for exact Quotient $2x^3 + ax^2 - 3ab^2$.

LII.

In those two Examples the Divisors of one Letter of each of the three Quantities of the first Column were not set down, because those Divisors could never be the Quantities into which the Divisor involving three Letters is transformed by the Supposition of x, a, b equal 0, and besides it has been supposed that it has been found by the Method of Art. XLVI. that the proposed Quantity has no Divisor involving two Letters, but if the proposed Quantity has Divisors of this Sort, they may be found at the same Time as those involving three Letters, by the foregoing Method, provided they be but of one Dimension.

Suppose, for Example, you are to find the Divisors of the quantity

$$\begin{array}{l}
 16x^3 + 16bx^2 - 48ax^2 + 35a^2x - 16abx - 6a^3 + 3a^2b \\
 - 6a^3 + 3a^2b \\
 16x^3 + 16bx^2 \\
 16x^3 - 48ax^2 + 35a^2x - 6a^3
 \end{array}
 \begin{array}{l}
 | a, 3a, -2a+b, -6a+3b \\
 \{ x, 2x, 4x, 8x, 16x, x+b, 2x+2b \\
 \{ 4x+4b, 8x+8b, 16+16b \\
 | a-4x, 3a-4x, x-2a
 \end{array}
 \begin{array}{l}
 \left| \begin{array}{c|c|c} a & 2a & -2a+b \\ -4x & -4x & x+b \\ a-4x & 3a-4x & x-2a \end{array} \right. \\
 \hline
 a-4x \mid 3a-4x \mid x-2a+b
 \end{array}$$

Having set down in a first Column the three Quantities $-6a^3 + 3a^2b$, $16x^3 + 16bx^2$, $16x^3 - 48ax^2 + 35a^2x - 6a^3$ into which this Quantity is transformed by the Supposition of x, a, b equal 0; I set down in the second Column, and in the first Line $a, 3a, -2a+b, -6a+3b$ Divisors of one Dimension involving one and two Letters of the Quantity $-6a^3 + 3a^2b$, likewise I set down in the second Line the Divisors $x, 2x, 4x, 8x, 16x, x+b, 2x+2b, 4x+4b, 8x+8b, 16x+16b$, of the second Quantity $16x^3 + 16bx^2$; and in the third Line, $a-4x, 3a-4x, x-2a$ Divisors of the third Quantity $16x^3 - 48ax^2 + 35a^2x - 6a^3$.

Now on account of the great Number of these Divisors, in order to omit none of them that may have the Conditions required, I observe

much the same Order as in trying the numerical Divisors. Comparing the first Divisor of the first Line with all those of the other Lines, and performing afterwards the same Operation for each of the other Divisors of the first Line, I perceive first, That if a is a Part of a Divisor of the proposed Quantity, it must be a Divisor containing only a and x , because if it contained another Term b , this Divisor could not be reduced to a by the Supposition of $x = 0$. In this Case therefore we have only to chuse among the five first Divisors $x, 2x, 4x, 8x, 16x$. Now, as among all those Divisors, only $4x$ is repeated in the third (provided that this Divisor is affected with the Sign $-$), and at the same Time of all the Divisors of the third Line, there is only the first $a - 4x$ which includes the same Term a of the first Line. I conclude that if a be a Part of a Divisor, this Divisor must be $a - 4x$; I therefore set it down a Part. I afterwards pass to $3a$, and as I find it repeated in the Divisor $3a - 4x$ of the third Line, and that the other Term $4x$ of the same Divisor is found among the Divisors of the second Line, after changing the Sign of this Divisor; I conclude that $3a - 4x$ may be also a Divisor of the proposed Quantity, and I set it apart likewise in order to try it.

As to $-2a + b$ it cannot alone be a Divisor of the proposed Quantity, because then among the Divisors of $16x^3 + 16bxx$ the Term b should be found, to which $-2a + b$ is reduced by the Supposition of $a = 0$. It remains therefore to try if it be not a Part of a Divisor in which x enters; I observe that of all the Divisors of the second Line, only $x + b$ can be compared with it, because it is the only Divisor that has the Term b in common with it. I perceive likewise, that of all the Divisors of the third Line only $x - 2a$ can be compared with the same Divisor $-2a + b$, because it is the only one that contains the Term $-2a$. I observe, lastly, that as the two Terms of the Divisor $-2a + b$ are repeated in the two other Divisors $x + b, x - 2a$, so likewise the two Terms of the Divisor $x + b$ are repeated in the two others $-2a + b, x - 2a$, and reciprocally that the two Terms of the Divisor $x - 2a$ are repeated in the two others $x + b, -2a + b$. Whence I conclude that the three Divisors $-2a + b, x + b, x - 2a$, have the required Conditions to form a Divisor, I therefore add them together, and set apart the one Half $x - 2a + b$ of their Sum for a Divisor to be tried. As to the Divisor $6a + 3b$, I perceive at once that neither of its two Terms are repeated among the Divisors of the other Lines, and consequently it is to be rejected.

Having thus found the three Divisors $a - 4x, 3a - 4x, x - 2a + b$ with which Trial is to be made, I try the Division by the last, which succeeds, and gives for Quotient $3a - 16ax + 16xx$, which I divide afterwards by $3a - 4x$. The Division likewise succeeds, and gives for

Quotient the first Divisor $a - 4x$, whence the proposed Quantity is the Product of those three Divisors.

LIII.

Method of finding the divisors of two dimensions involving three letters.

If the proposed Quantity has no Divisor of one Dimension, we are to enquire if it has not some quadratic Divisor. Let $mx^2 + nax + pbx + qa^2 + rab + sbb$ express this Divisor; putting successively $x = 0$, $a = 0$, $b = 0$, in this Divisor, there results $qa^2 + rab + sbb$; $mx^2 + pbx + sbb$; $mx^2 + nax + qa^2$; which are all three Divisors of the Quantities into which the proposed one is transformed by the same Suppositions of x , a , b equal 0. Moreover, each of those Divisors is such, that the Terms affected with square Letters are repeated in the two other Divisors, whilst the Terms, that contain a Product of two Letters are alone of their Species. When, therefore, among the Divisors of two Dimensions and involving two Letters of the Quantities to which the proposed Quantity is reduced by the foregoing Suppositions, three are found having the above-mentioned Conditions, by adding them together, and taking the one half of all the Terms affected with Squares, and leaving those that are rectangles entire, a Divisor of two Dimensions to be tried will be obtained.

LIV.

Application of this method to an example.

To show the Application of this Method, let it be proposed to discover whether the following Quantity, which has no simple Divisor, has any quadratick Divisors:

$$\begin{array}{r}
 x^5 - 4ax^4 - 5a^2x^3 + 4b^2x^3 - a^3x^2 - 14ab^2x^2 + 3b^4x - a^4x - 3a^3b^2 \\
 - 3a^2b^2 \\
 x^5 + 4b^2x^3 + 3b^4x \\
 x^5 - 4ax^4 - 5a^2x^3 - a^3x^2 - a^4x
 \end{array}
 \left|
 \begin{array}{l}
 a^2, ab, b^2, 3a^2, 3ab, 3b^2 \\
 x^2 + 3b^2, x^2 + b^2 \\
 x^2 + ax
 \end{array}
 \right|
 \begin{array}{l}
 -b^2 \\
 x^2 + b^2 \\
 x^2 + ax
 \end{array}
 \left|
 \begin{array}{l}
 3b^2 \\
 x^2 + 3b^2 \\
 x^2 + ax
 \end{array}
 \right|$$

$$\begin{array}{l}
 x^2 + b^2 + ax \quad x^2 + 3b^2 + ax
 \end{array}$$

I set down in a first Column the three Quantities $-3a^3b^2$; $x^5 + 4b^2x^3 + 3b^4x$; $x^5 - 4ax^4 - 5a^2x^3 - a^3x^2 - a^4x$; into which the proposed Quantity is transformed by the Suppositions of x , a , b equal 0. I afterwards set down beside these Quantities their Divisors of two Dimensions; the first gives a^2 , ab , b^2 , $3a^2$, $3ab$, $3b^2$; the second $x^2 + 3b^2$, $x^2 + b^2$; the third only $x^2 + ax$. In this Method, the Divisors consisting only of one Term are not to be rejected, even tho' it has been found that the proposed Quantity has no Divisor involving two Letters, because a Divisor consisting of three Terms and of two Dimensions, may be reduced to one Term, by supposing one of the Letters equal 0.

It remains now to try all the Divisors of the first Line; I perceive, 1^o. That the first a^2 is to be rejected, because this Square is not repeated in the other Lines. I pass afterwards to $a b$, and as this Divisor does not contain any Term affected with $a a$ or with $b b$, I conclude that the Divisor, of which it might be a Part, cannot have, besides this Term, any other but $x x$, or $a x$, or $b x$, whereby the Comparison of $a b$ with the Divisors $x^2 + 3 b^2$, $x^2 + b^2$ being excluded, it follows that if $a b$ be a Part of a Divisor of the proposed Quantity, this Divisor must be $x x + a x + b a$; but at the same Time, I perceive that $x x + a x + b a$ cannot be a Divisor of the proposed Quantity, since it would be reduced to $x x$ by the Supposition of $a = 0$, and that $x x$ is not one of the Divisors of the second Line. Wherefore the Divisor $a b$ is likewise to be rejected.

As to the Divisor $b b$, I find it repeated in the Divisor $x^2 + b^2$ of the second Line, and finding that the same Divisor $x^2 + b^2$, has x^2 in common with the Divisor of the third Line, I conclude that $x^2 + b^2 + a x$ has the Conditions required. I afterwards pass to the other Divisors of the first Line, and I perceive, 1^o. That $3 a a$ and $3 a b$ are to be rejected, in like Manner as a^2 and $a b$; I observe afterwards that $3 b b$ is repeated in the Divisor $x^2 + 3 b^2$ and x^2 in $x x + a x$, whence I conclude that $x x + 3 b b + a x$ has also the Conditions required. Trying those two Divisors, I find that the second only succeeds, giving for Quotient $x^3 - 5 a x^2 + b^2 x - a^3$.

Instead of trying all the Divisors of the first Line, we might, by trying that one alone which is contained in the last Line, find much more readily that $x^2 + a x + b^2$ and $x^2 + a x + 3 b^2$ are the only Divisors to be tried. For observing that the Divisor $x^2 + a x$ contains x^2 which is repeated in $x^2 + 3 b^2$ and in $x^2 + b^2$, and that those two last contain, the one $3 b^2$, and the other b^2 , which are each found among the Divisors of the first Line, it is easy to conclude that $x^2 + a x + b^2$ and $x^2 + a x + 3 b^2$ have the required Conditions, and that they are the only ones, since if there were others, they would give different Quantities from $x x + a x$ by supposing $b = 0$, or different Quantities from $x^2 + 3 b^2$ and $x^2 + b^2$ by supposing $a = 0$.

LV.

If some Letter in the proposed Quantity is only of one Dimension, it is easy to perceive that only one of the Divisors of this Quantity can contain it, whence there will be at least one Divisor that does not contain it, and consequently to find this Divisor (Art. XLVI.) you may seek for the greatest common Divisor of the Terms in which that Letter is found, and of the remaining Terms in which it is not found. Thus, in the Quantity $x^4 - 3 a x^3 - 8 a a x x + 18 a^3 x + c x^3 - a c x x - 8 a a c x + 6 a^3 c - 8 a^4$, the common Divisor of the Terms $c x^3 - a c x x - 8 a a c x + 6 a^3 c$ and of $x^4 - 3 a x^3 - 8 a a x x + 18 a^3 x - 8 a^4$, viz. $x x + 2 a x - 2 a a$, will divide the whole quantity.

Case in which the divisors are found much easier than by the foregoing method.

LVI.

Let it now be proposed to find the Divisors of $2x^5 + 3ax^4 + b^2x^3 - a^2x^3 + 4ab^2x^2 + 6a^2b^2x + 2ab^4 - 2a^3b^2$ involving two or three Letters, either of one or two Dimensions,

Another example.

	1	$2ab^4 - 2a^3b^4$
	2	$ab^4 - a^3bb$
	$2a, a$	$b^4 - a^2b^2$
	$2ab, ab, 2b, b$	$b^3 - a^2b$
	$2ab^2, ab^2, 2b^2, b^2b$	$b^2 - a^2$
		$b + a$

$b^3 - ab^2, 2ab^2 - 2a^2b, ab^2 - a^2b, 2b^2 - 2ab, b^2 - ab, 2ba - 2a^2, ba - a^2, 2b - 2a, b - a$
 $2b^3 - 2ab^2, ab^3 - a^2b^2, 2ab^3 - 2a^2b^2)$
 $b^3 + ab^2, 2ab^2 + 2a^2b, ab^2 + a^2b, 2b^2 + 2ab, b^2 + ab, 2ba + 2a^2, ba + a^2, 2b + 2a, b + a,$
 $2b^3 + 2ab^2, ab^3 + a^2b^3, 2ab^3 + 2a^2b^2)$
 $2ab^3 - 2a^3b, ab^3 - a^3b, 2b^3 - 2a^2b, b^3 - a^2b, 2b^2a - 2a^3, b^2a - a^3, 2b^2 - 2a^2, b - a$
 $b^4 - a^2b^2, 2b^4 - 2a^2b^2, ab^4 - a^3b^2, 2ab^4 - 2a^3b^2)$

$$2x^5 + 3ax^4 + b^2x^3 - a^2x^3 + 4ab^2x^2 + 6a^2b^2x + 2ab^4 - 2a^3b^2$$

$2ab^4 - 2a^3b^2$	$a, 2a, b, 2b, b - a, 2b - 2a, b + a, 2b + 2a$ $bb, 2bb, ab, 2ab, bb - aa, 2bb - 2aa, ba - aa,$ $ba + aa, bb - ab, bb + ab, 2ba - 2aa,$ $2ba + 2aa, 2bb - 2ab, 2bb + 2ab$	$ab \quad 2ab$	$bb - aa$
$2x^5 + b^2x^3$	$x, xx, 2xx + bb$	$xx \quad xx$	$2xx + bb$
$x^5 + 3ax^4 - a^2x^3$	$x, x^2, 2x^2 + 3ax - aa$	$xx \quad xx$	$2x^2 + 3ax - aa$
	$x^2 + a, x^2 + 2a$ $x^2 - ab, x^2 - 2ab$		$2x^2 + b^2 + 3ax - a^2$

$2ab^4 - 2a^3b^2$, $2x^5 + b^2x^3$, and $2x^5 + 3ax^4 - a^2x^3$ being the Quantities to which the proposed one is reduced by the Supposition of x, a, b equal 0, opposite each of these Quantities must be ranged all the Divisors they may admit of, as well of one Dimension as of two. As the first of those three Quantities admits of a great Number, in order to omit none of them, I follow the same Order as was observed in the investigation of the numerical Divisors.

Application of the method of Art. xxiv. for finding all the divisors of literal quantities.

Having set down this first Quantity $2ab^4 - 2a^3bb$, and drawn a Line to the left Hand of it, I write down Unity, as being its first Divisor; I afterwards set down 2 under 1, because it is after 1 the simplest Divisor that this Quantity will admit of, and I set down to the right Hand of the same Line $ab^4 - a^3b^2$. I afterwards divide this Quantity by a , and set down a to the left Hand of the vertical Line, writing down at the same Time to

the right Hand the Quotient $b^4 - a^2 b^2$. I next multiply a by 2, and set down $2a$ to the left Hand of a , as being a new Divisor of the proposed Quantity; I afterwards divide $b^4 - a^2 b^2$ by b , and set down the Divisor b to the left Hand, and the Quotient $b^3 - a^2 b$ to the right. Lastly, I multiply b by 2, by a , by $2a$; and set down to the left Hand of b , the Products $2b$, ab , $2ab$, as being new Divisors of the proposed Quantity.

The proposed Quantity being reduced to $b^3 - a^2 b$, I divide it again by b , which I set down to the left Hand, and the Quotient $b^2 - a^2$ to the right; I omit multiplying b by 2, or by a , or by $2a$, because there would result the Divisors already found; but I multiply it by b and by $2b$, which produces the new Divisors of two Dimensions bb and $2bb$; if Divisors of three Dimensions were to be admitted, as may be necessary on other Occasions, then b should be likewise multiplied by ab and $2ab$.

After having reduced the Quantity to $bb - aa$, I perceive that it is divisible by $b - a$, and the Quotient is $b + a$. I therefore set down one to the left and the other to the right Hand, and I multiply $b - a$ by 2, by a , by $2a$, by b , and by $2b$, which give for new Divisors, $2b - 2a$, $ba - aa$, $bb - ab$, $b^2 - 2b^2$, $2ba - 2aa$, $2bb - 2ab$. And if Divisors of three or four Dimensions were required, I should have also multiplied $b - a$ by ab , $2ab$, abb , $2abb$. The Quantity remaining $b + a$ admitting of no Divisor, I set it down to the left Hand, and I multiply it by 2, by a , by b , by $2b$, $2a$, $b - a$, $2b - 2a$, which give for new Divisors of one and two Dimensions, $2b + 2a$, $ba + aa$, $2ba + 2aa$, $bb + ab$, $2bb + 2ab$, $bb - aa$, $2bb - 2aa$. If Divisors of three, four, five Dimensions, that is, all the Divisors that the proposed Quantity can admit of were required, I would have also multiplied $b + a$ by ab , $2ab$, bb , $2bb$, abb , $2abb$, $ba - aa$, $2ba - 2aa$, $bb - ab$, $2bb - 2ab$, $abb - aab$, $2abb - 2a^2 b$, $b^3 - ab^2$, $2b^3 - 2ab^2$, $ab^3 - a^2 b^2$, $2ab^3 - 2a^2 b^2$.

This being done, I set down all the Divisors of one and two Dimensions, a , $2a$, b , $2b$, $b - a$, $2b - 2a$, $b + a$, $2b + 2a$, ab , $2ab$, $ab - aa$, $2ba - 2aa$, $b^2 - ab$, $2b^2 - 2ab$, $ba + aa$, $2ba + 2aa$, $b^2 + ba$, $2bb + 2ab$, beside the Quantity $2ab^4 - 2a^3 b^2$ which produced them. I afterwards set down beside the Quantity $2x^5 + b^2 x^3$ its Divisors of one and two Dimensions x , $2xx$, $xx + bb$, and beside the Quantity $2x^5 + 3ax^4 - a^2 x^3$ its Divisors of one and two Dimensions x , x^2 , $2x^2 + 3ax - a^2$.

Now, trying all those Divisors to discover those that are to be rejected, I soon perceive that all those of one Dimension are of this Sort; for x being the only Quantity of one Dimension in the second and third Lines, I conclude that it must be the only Divisor of one Dimension that the proposed Quantity can admit of; since if the Divisor of one Dimension

Another
example.

Let it now be proposed to find the Divisors of $2x^5 + 3ax^4 + b^2x^3 - a^2x^3 + 4ab^2x^2 + 6a^2b^2x + 2ab^4 - 2a^3b^2$ involving two or three Letters, either of one or two Dimensions,

	1	$2ab^4 - 2a^3b^2$
	2	$ab^4 - a^3bb$
	$2a, a$	$b^4 - a^2b^2$
	$2ab, ab, 2b, b$	$b^3 - a^2b$
	$2ab^2, ab^2, 2b^2, b^2b$	$b^2 - a^2$
		$b + a$

$b^3 - ab^2, 2ab^2 - 2a^2b, ab^2 - a^2b, 2b^2 - 2ab, b^2 - ab, 2ba - 2a^2, ba - a^2, 2b - 2a, b - a$
 $2b^3 - 2ab^2, ab^3 - a^2b^2, 2ab^3 - 2a^2b^2)$
 $b^3 + ab^2, 2ab^2 + 2a^2b, ab^2 + a^2b, 2b^2 + 2ab, b^2 + ab, 2ba + 2a^2, ba + a^2, 2b + 2a, b + a,$
 $2b^3 + 2ab^2, ab^3 + a^2b^3, 2ab^3 + 2a^2b^2)$
 $2ab^3 - 2a^3b, ab^3 - a^3b, 2b^3 - 2a^2b, b^3 - a^2b, 2b^2a - 2a^3, b^2a - a^3, 2b^2 - 2a^2, b - a$
 $b^4 - a^2b^2, 2b^4 - 2a^2b^2, ab^4 - a^3b^2, 2ab^4 - 2a^3b^2)$

$2x^5 + 3ax^4 + b^2x^3 - a^2x^3 + 4ab^2x^2 + 6a^2b^2x + 2ab^4 - 2a^3b^2$			
$2ab^4 - 2a^3b^2$	$a, 2a, b, 2b, b - a, 2b - 2a, b + a, 2b + 2a$ $bb, 2bb, ab, 2ab, bb - aa, 2bb - 2aa, ba - aa,$ $ba + aa, bb - ab, bb + ab, 2ba - 2aa,$ $2ba + 2aa, 2bb - 2ab, 2bb + 2ab$	ab xx xx	$bb - aa$
$2x^5 + b^2x^3$	$x, xx, 2xx + bb$	xx xx	$2xx + bb$
$x^5 + 3ax^4 - a^2x^3$	$x, x^2, 2x^2 + 3ax - aa$	xx xx	$2x^2 + 3ax - aa$
		$x^2 + a$ $x^2 - ab$	$x^2 + 2ab$ $x^2 - 2ab$
			$2x^2 + b^2 + 3ax - a^2$

$2ab^4 - 2a^3b^2$, $2x^5 + b^2x^3$, and $2x^5 + 3ax^4 - a^2x^3$ being the Quantities to which the proposed one is reduced by the Supposition of x, a, b equal 0, opposite each of these Quantities must be ranged all the Divisors they may admit of, as well of one Dimension as of two. As the first of those three Quantities admits of a great Number, in order to omit none of them, I follow the same Order as was observed in the investigation of the numerical Divisors.

Application
of the
method of
Art. xxiv.
for finding
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sors of lite-
ral quan-
tities.

Having set down this first Quantity $2ab^4 - 2a^3b^2$, and drawn a Line to the left Hand of it, I write down Unity, as being its first Divisor; I afterwards set down 2 under 1, because it is after 1 the simplest Divisor that this Quantity will admit of, and I set down to the right Hand of the same Line $ab^4 - a^3b^2$. I afterwards divide this Quantity by a , and set down a to the left Hand of the vertical Line, writing down at the same Time to

the right Hand the Quotient $b^4 - a^2 b^2$. I next multiply a by 2, and set down $2a$ to the left Hand of a , as being a new Divisor of the proposed Quantity; I afterwards divide $b^4 - a^2 b^2$ by b , and set down the Divisor b to the left Hand, and the Quotient $b^3 - a^2 b$ to the right. Lastly, I multiply b by 2, by a , by $2a$; and set down to the left Hand of b , the Products $2b$, ab , $2ab$, as being new Divisors of the proposed Quantity.

The proposed Quantity being reduced to $b^3 - a^2 b$, I divide it again by b , which I set down to the left Hand, and the Quotient $b^2 - a^2$ to the right; I omit multiplying b by 2, or by a , or by $2a$, because there would result the Divisors already found; but I multiply it by b and by $2b$, which produces the new Divisors of two Dimensions bb and $2bb$; if Divisors of three Dimensions were to be admitted, as may be necessary on other Occasions, then b should be likewise multiplied by ab and $2ab$.

After having reduced the Quantity to $bb - aa$, I perceive that it is divisible by $b - a$, and the Quotient is $b + a$. I therefore set down one to the left and the other to the right Hand, and I multiply $b - a$ by 2, by a , by $2a$, by b , and by $2b$, which give for new Divisors, $2b - 2a$, $ba - aa$, $bb - ab$, b^2 , $2b^2$, $2ba - 2aa$, $2bb - 2ab$. And if Divisors of three or four Dimensions were required, I should have also multiplied $b - a$ by ab , $2ab$, abb , $2abb$. The Quantity remaining $b + a$ admitting of no Divisor, I set it down to the left Hand, and I multiply it by 2, by a , by b , by $2b$, $2a$, $b - a$, $2b - 2a$, which give for new Divisors of one and two Dimensions, $2b + 2a$, $ba + aa$, $2ba + 2aa$, $bb + ab$, $2bb + 2ab$, $bb - aa$, $2bb - 2aa$. If Divisors of three, four, five Dimensions, that is, all the Divisors that the proposed Quantity can admit of were required, I would have also multiplied $b + a$ by ab , $2ab$, bb , $2bb$, abb , $2abb$, $ba - aa$, $2ba - 2aa$, $bb - ab$, $2bb - 2ab$, $abb - aab$, $2abb - 2a^2 b$, $b^3 - ab^2$, $2b^3 - 2ab^2$, $ab^3 - a^2 b^2$, $2ab^3 - 2a^2 b^2$.

This being done, I set down all the Divisors of one and two Dimensions, a , $2a$, b , $2b$, $b - a$, $2b - 2a$, $b + a$, $2b + 2a$, ab , $2ab$, $ab - aa$, $2ba - 2aa$, $b^2 - ab$, $2b^2 - 2ab$, $ba + aa$, $2ba + 2aa$, $b^2 + ba$, $2bb + 2ab$, beside the Quantity $2ab^4 - 2a^3 b^2$ which produced them. I afterwards set down beside the Quantity $2x^5 + b^2 x^3$ its Divisors of one and two Dimensions x , $2xx$, $xx + bb$, and beside the Quantity $2x^5 + 3ax^4 - a^2 x^3$ its Divisors of one and two Dimensions x , x^2 , $2x^2 + 3ax - a^2$.

Now, trying all those Divisors to discover those that are to be rejected, I soon perceive that all those of one Dimension are of this Sort; for x being the only Quantity of one Dimension in the second and third Lines, I conclude that it must be the only Divisor of one Dimension that the proposed Quantity can admit of; since if the Divisor of one Dimension

included either a Term affected with a , or one affected with b ; the Term affected with a , would remain in the Divisors given by the Supposition of $b = 0$, and the one affected with b would remain in the Divisors given by the Supposition of $a = 0$; but x is not a Divisor of the proposed Quantity, consequently it has no Divisor of one Dimension.

I pass next to the Divisors of two Dimensions, beginning by x^2 which I take in the third Line; finding it also in the second Line, I conclude that if it makes a Part of a Divisor, it can only want a Term affected with the Rectangle ab ; in effect if there were Terms in this Divisor affected with aa , with bb , with ax , or with bx , these affected with bb or with bx would not have disappeared by the Supposition of $a = 0$; and those affected with aa , or with ax , would not have vanished by the Supposition of $b = 0$. But I find in the first Line ab , $2ab$; wherefore $xx + ab$, $xx + 2ab$, $xx - ab$, $xx - 2ab$, are the Divisors to be tried.

I afterwards pass to the Divisor $2x^2 + 3ax - aa$, and I find the Term $2x^2$ repeated in the Divisor $2xx + bb$ above it, as likewise the Term $-aa$ repeated in several Divisors that are in the upper Lines; but of all the Divisors in which it is repeated there is only $bb - aa$, which has the Term bb common with the Divisor $2xx + bb$. Whence it is only with $2xx + bb$, and $bb - aa$, that $2x^2 + 3ax - aa$ can concur to form a Divisor that will have the Conditions required, and this Divisor which is $2x^2 + 3ax - aa + bb$ is consequently to be tried.

I perceive that it is useless to seek for another Divisor, because if there could be found one that could not be discovered by trying the Divisors of the third Line, it would consist of one Term affected with ab : Now it is easy to perceive that the proposed Quantity admits of no Divisor of this kind. I therefore attempt the Division by $2x^2 + 3ax - a^2 + bb$, which succeeds, and gives for Quotient $x^3 + 2abb$, whereby it appears that none of the Quantities $xx + ab$, $xx - ab$, $xx + 2ab$, $xx - 2ab$ can divide the proposed one.

LVII.

If the proposed Quantity is of more than five Dimensions, and having found that it has no Divisor of one or of two Dimensions, then we are to enquire by a similar Process if it has not some Divisor of three or four Dimensions.

LVIII.

How the
divisors of
quantities
that are not
homogene-
ous are
found.

When the proposed Quantity is not homogeneous, that is, when all its Terms are not of the same Dimensions, it is to be rendered homogeneous by means of a new Letter, and the Divisors of this new Quantity being found by the preceding Rules, the new Letter introduced is to be exterminated by putting it equal to Unit, and by this Means the Divisors required of the proposed Quantity will be obtained. Suppose, for Example,

the Quantity to be $y^6 + by^5 - by^4 + y^2 + by - b$; I multiply the Term by^4 by a , to make it of six Dimensions. For the same Reason I multiply y^2 by a^4 , by by a^4 , and b by a^5 , whence there results the Quantity $y^6 + by^5 - aby^4 + a^4yy + a^4by - a^5b$, which by the foregoing Method is found to be the Product of $yy - ab + by$ into $y^4 + a^4$. I suppose $a=1$ in those Factors, and there results $yy - b + by$, and $y + 1$ for the two Factors of the proposed Quantity $y^6 + by^5 - by^4 + y^2 + by - b$.

LIX.

To find the incommensurable Roots of Equations of every Degree, Method of
 1^o. Substitute successively in the proposed Equation 1, 2, 3, 4, &c. in place approxima-
 of the unknown Quantity, until the Quantity that results after each Sub- ting to the
 stitution changes its Sign, then the Root y will be a positive Number roots of
 greater than that which immediately precedes the Change of the Sign of numeral
 the Result, and less than that where this Change happens. equations
 involving
 one un-
 known
 quantity.

2^o. Instead of putting the first Member of the Equation equal 0, put it $=z$; then find the limiting Ratio of the cotemporary Increments of y and z , and let it be expressed by A . Substitute the Value of y , estimated pretty near the Truth by the foregoing Method, in the Equation, as also in the Value of A , and let the Error or resulting Number in the former, be divided by this numerical Value of A , and the Quotient be subtracted from the said former Value of y , and from thence will arise a new Value of that Quantity much nearer to the truth than the former, wherewith proceeding as before, another new value may be had, and so another, &c. till we arrive to any Degree of Accuracy desired.

LX.

Let it be proposed to find a Value of y in the Equation $y^4 + 2y^3 - 36yy + 5y - 116 = 0$.

$\left. \begin{array}{l} \text{If } y = 1 \\ \text{If } y = 2 \\ \text{If } y = 3 \\ \text{If } y = 4 \\ \text{If } y = 5 \\ \text{If } y = 6 \end{array} \right\} \text{ then}$	$\left\{ \begin{array}{l} 1 + 2 - 36 + 5 - 116 = -144 \\ 16 + 16 - 144 + 10 - 116 = -218 \\ 81 + 54 - 324 + 15 - 116 = -290 \\ 256 + 128 - 576 + 20 - 116 = -288 \\ 625 + 250 - 900 + 25 - 116 = -116 \\ 1296 + 432 - 1296 + 30 - 116 = +346 \end{array} \right.$
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Whence I conclude that one of the Values of y will be greater than 5, but less than 6. Let it be supposed to be 5,5, to find a more approximate Value of it, I investigate the limiting Ratio of the cotemporary Increments dy and dz of y and z in the Equation $y^4 + 2y^3 - 36yy + 5y - 116 = z$, which I find to be $4y^3 + 6yy - 72y + 5$, then because $y = \frac{y^4 + 2y^3 - 36yy + 5y - 116}{4y^3 + 6yy - 72y + 5} = y$. I substitute in the first Member of this Equation, in Place of y , its Value 5,5 already found, whence we have $5,5 - \frac{70,3125}{456} = 5,340$ for a new Value of y , with which pro-

ceeding as before the next Value of y will be found to be 5,3354, and from thence the third Value of $y = 5,335438$, which is true to the last Decimal Place.

LXI.

Application
of the fore-
going me-
thod to an-
other ex-
ample.

Let $\sqrt{(1-y)} + \sqrt{(1-2yy)} + \sqrt{(1-3y^3)} - 2 = 0$, be proposed. one of the Values of y in this Equation will be nearly 0,5. To find a more approximate Value of it, I investigate the limiting Ratio of the cotemporary Increments dy and dz of y and z in the Equation $\sqrt{(1-y)} + \sqrt{(1-2yy)} + \sqrt{(1-3y^3)} - 2 = z$, which I find to be $-\frac{1}{2\sqrt{(1-y)}} - \frac{2y}{\sqrt{(1-2yy)}} - \frac{9yy}{2\sqrt{(1-3y^3)}}$; whence we have $0,5 + \frac{0.204}{3.545} = 0,557$ for the next Value of y , from whence by proceeding as before, the next following will be found 0.5516, &c.

LXII.

When
there are
two equa-
tions given,
and as
many
Quantities
 x and y
how to
determine
those
Quantities.

When there are two Equations given, and as many Quantities (x and y) to be determined, find the Equation expressing the limiting Ratio of the cotemporary Increments dx and dy of x and y in both Equations, and in the former collect all the Terms, affected with dx , under their proper Signs, and having divided by dx put the Quotient $= A$, and let the remaining Terms, divided by dy be represented by B : in like Manner, having divided the Terms in the latter affected with dx , by dx , let the Quotient be put $= a$, and the Rest divided by dy , $= b$. Assume the Values of x and y pretty near the Truth, and substitute in both the Equations, marking the Error in each, and let these Errors, whether positive or negative, be signified by A and r respectively, substitute likewise in the Values of A, B, a, b , and let $\frac{Br-bR}{Ab-aB}$ and $\frac{aR-Ar}{Ab-aB}$ be converted into Numbers, and respectively added to the former Values of x and y , and thereby new Values of those Quantities will be obtained; from whence, by repeating the Operation, the true Values may be approximated *ad Libitum*.

LXIII.

Applica-
tion of the
foregoing
method to
an ex-
ample.

Let there be given the Equations $y + \sqrt{(yy - xx)} - 10 = 0$, and $x + \sqrt{(yy + x)} - 12 = 0$; to find x and y .

The Equations expressing the Relation of the cotemporary Increments dx , and dy , of x , and y , being here $dy + \frac{y dy - x dx}{\sqrt{(yy - xx)}} = 0$, and $dx + \frac{y dy + \frac{1}{2} dx}{\sqrt{(yy + x)}} = 0$, or $dy + \frac{y dy}{\sqrt{(yy - xx)}} - \frac{x dx}{\sqrt{(yy - xx)}} = 0$, and $dx + \frac{\frac{1}{2} dx}{\sqrt{(yy + x)}} + \frac{y dy}{\sqrt{(yy + x)}} = 0$, we have A equal $\frac{x}{\sqrt{(yy - xx)}}$, B equal $1 + \frac{y}{\sqrt{(yy + x)}}$, $a = 1 + \frac{\frac{1}{2}}{\sqrt{(yy + x)}}$, and $b = \frac{y}{\sqrt{(yy + x)}}$.

Let x be supposed equal 5, and $y = 6$; then will R equal $-.68$, $r = -.6$, A equal -1.5 , B equal 2.8 , a equal 1.1 , b equal 9 ; therefore $\frac{Br - bR}{Ab - aB} = .23$, and $\frac{aR - Ar}{Ab - aB} = .37$, and the new Values of x and y equal to 5.23 , and 6.37 respectively; the next Values will come out $5, 23263$, and $6, 36898$.

LXIV.

Let $49 \times (x - \frac{x}{(x+y)^2}) - 25 \times (1 - \frac{xx}{(1+y)^2}) = 0$, and $81 \times (1 - \frac{xx}{(1+y)^2}) - 49 \times (\frac{x}{y} - \frac{xy}{(1+x)^2}) = 0$. Here we have A equal $49 \times (1 + \frac{x-y}{(x+y)^2} + \frac{50x}{(1+y)^2})$, $B = \frac{98x}{(x+y)^2} - \frac{50x^2}{(1+y)^2}$, $a = -\frac{162x}{(1+y)^2} + 49 \times [\frac{y}{1+x} - \frac{1}{y} - \frac{2xy}{(1+x)^2}]$, and $b = \frac{162x^2}{(1+y)^2} + 49x \times [\frac{1}{yy} + \frac{1}{(1+x)^2}]$.

Application of the foregoing method to another Example.

Suppose $x = .8$, and $y = .6$, then will be found $R = .45$, $r = 2.66$, $A = 68$, $B = 20.7$, $a = -131$, $b = 146$, and the next Values of x and y equal to $.799$, and $.582$, with which repeating the Operation, the next following will come out $.79912$ and $.58138$.

LXV.

When an Equation contains two unknown Quantities x and y , and it is proposed to express the Root y in a Series of Terms including the Quantity x with the other known Quantities; to find the Value of y in such a converging Series, it is necessary to consider x either 1^o. as very small, or 2^o as very great, or 3^o. as differing very little from some given Quantity; that so by supposing x to be very small the Series may converge wherein the Root y is expressed by a Progression of Terms, in which the Dimensions of x increase in the Numerators, 2^o by supposing x to be very great that Series may converge, in which the Dimensions of x continually increase in the Denominators of the Terms, or that 3^o by supposing x to differ but very little from some given Quantity, some Letter as z being substituted for that Difference, it may come in the Place of x considered as very small.

LXVI.

Thus if the Equation to be resolved was $x^2y + ay^2 - ax = 0$, if x be considered as very little, the Value of y may exhibited in a Series having its Terms composed of the Powers of x divided by those of a with their respective Coefficients. Since when x is very little in respect of a , the Terms $x, \frac{x^2}{a}, \frac{x^3}{a^2},$

$\frac{x^4}{a^3}$, $\frac{x^3}{a^4}$, &c. decrease very quickly. If x vanishes in respect of a , the second Term will vanish in respect of the first, since $\frac{x^2}{a} : x = x : a$, and after the same Manner $\frac{x^3}{a^2}$ vanishes in respect of the Term immediately preceding it.

But if x be considered as very great, the Value of y may be exhibited in a Series whose Terms are composed of the Powers of a divided by those of x with their respective Coefficients, since when x is very great in respect of a , then a is very great in respect of $\frac{a^2}{x}$, and $\frac{a^2}{x}$ in respect of $\frac{a^3}{x^2}$, so that the Terms a , $\frac{a^2}{x}$, $\frac{a^3}{x^2}$, $\frac{a^4}{x^3}$, $\frac{a^5}{x^4}$, &c. in this Case decrease very swiftly; in either Case, the Series converge swiftly that consist of such Terms; and a few of the first Terms will give a near Value of the Root required.

Division of
Series into
ascending
and descend-
ing ones.

Whence it appears that the Roots of Equations involving two unknown Quantities may be exhibited by two Sorts of Series, whereof one converges so much sooner as the unknown Quantity it contains is less, and the other converges so much sooner as the unknown Quantity is greater. In the Terms of the former, which are called ascending Series, the Exponents of the unknown Quantity continually increase, but continually decrease in the Terms of the latter.

LXVII.

An Equati-
on loses
some of its
terms
when one of
its unknown
quantities
is supposed
indefinitely
great or in-
definitely
little.

To determine the first Term of the Series, exhibiting the Value of an unknown Quantity, in an Equation that involves two unknown Quantities, as of y in the Equation $x^2 y + a y^2 - a^2 x = 0$, the Analysts assume together such Terms of the Equation as will be found to become vastly greater than the other Terms, that is, which give a Value of y which substituted for it in all the Terms of the Equation, shall raise the Dimensions of the other Terms all above or all below the Dimensions of the assumed Terms, according as x is supposed to be vastly little or vastly great in respect of a .

LXVIII.

Method of
finding the
greatest
terms of an
equation by
way of ex-
clusion.

To determine the Terms in any proposed Equation that the Supposition of x or of y being indefinitely great or indefinitely little, will render vastly greater than the other Terms: I observe first, that one Term of the Equation cannot be supposed vastly greater than all the rest, for all the other Terms would vanish in respect of it, and this Term then alone would be equal to nothing, which is absurd. The greatest Terms therefore of an Equation are at least two in Number, but there is no Reason

why any two Terms may not be supposed the greatest of the Equation, unless the Consequences arising from this Supposition destroy the Supposition itself.

Thus in the proposed Equation $x^2 y + a y^2 - a^2 x = 0$, x being considered as indefinitely great, if $x^2 y$ and $a y^2$ be supposed to be the greatest Terms of the Equation, then $x^2 y + a y^2 = 0$, or $x^2 y = -a y^2$, and dividing by $-a y$, $y = -\frac{x^2}{a}$ which substituted for x , the Equation be-

Application
of this me-
thod to an
example.

comes $\frac{x^4}{a} + \frac{x^4}{a} - a^2 x = 0$, where the assumed Terms $x^2 y + a y^2$ are of more Dimensions than the other $a^2 x$, and consequently may be supposed to be the greatest Terms of the Equation.

If $x^2 y$ and $-a^2 x$ be supposed to be the greatest Terms of the Equation, then $x^2 y - a^2 x = 0$, which by transposing and dividing by x^2 gives $y = \frac{a^2}{x}$, which substituted for y , then $a^2 x + \frac{a^5}{x^2} - a^2 x = 0$, where the assumed Terms $x^2 y$, $-a^2 x$ are of more Dimensions than $a y^2$, and consequently may be supposed to be the greatest Terms of the Equation.

Finally, if $a y^2$ and $-a^2 x$ be supposed to be the greatest Terms of the Equation, then $a y^2 - a^2 x = 0$, wherefore $y = a^{\frac{1}{2}} x^{\frac{1}{2}}$

which substituted for y , the Equation becomes $a^{\frac{1}{2}} x^{\frac{5}{2}} + a^2 x - a^2 x$ where the assumed Terms are of less Dimensions than the other $x^2 y$, and consequently cannot be supposed to be the greatest Terms of the Equation.

x being supposed indefinitely little, if $a y^2$ and $-a^2 x$ be supposed to be the greatest Terms of the Equation, then $a y^2 - a^2 x = 0$, which gives $y = a^{\frac{1}{2}} x^{\frac{1}{2}}$ which substituted for y , the Equation becomes $a^{\frac{1}{2}} x^{\frac{5}{2}} + a^2 x - a^2 x = 0$, where the assumed Terms $a y^2 - a^2 x$ are of less Dimensions than $x^2 y$, and consequently may be supposed to be the greatest Terms of the Equation.

If $x^2 y + a y^2$ be supposed to be the greatest Terms of the Equation, then $x^2 y + a y^2 = 0$, which gives $y = -\frac{x^2}{a}$, which substituted for y , the Equation becomes $\frac{x^4}{a} + \frac{x^4}{a} - a^2 x = 0$, where the assumed Terms $x^2 y$, $a y^2$ are of more Dimensions than $-a^2 x$, and consequently cannot be supposed to be the greatest Terms of the Equation.

If $x^2 y$ and $-a^2 x$ be supposed to be the greatest Terms of the Equation, then $y = \frac{a^2}{x}$ which substituted for y , the Equation becomes

Of this Figure or Triangle we are to observe, that being placed upon the Side AB , when x is supposed indefinitely great, or indefinitely little, (and on the Side AC when y is supposed indefinitely great or indefinitely little) of all the Terms in the same vertical Column, that only can be considered as one of the greatest Terms of the Equation, which is placed in the highest Square of this Column, when x is supposed indefinitely great, or that which is placed in the lowest Square of this Column, when x is supposed indefinitely little; for y having the same Exponent in all the Terms in the same Column, their Subordination entirely depends on the Exponent of x ; wherefore x being supposed indefinitely great, the greatest Term in the Column is that in which x has the greatest Exponent, which is placed in the highest Square of this Column, and x being supposed indefinitely little, the greatest Term in the Column is that in which x has the least Exponent, or which is placed in the lowest Square of this Column.

Property of
the analytick
triangle.

LXX.

The foregoing Observation, as is easy to perceive, serves to diminish the Number of Comparisons requisite to be made, for discovering the greatest Terms of an Equation, but in order to avoid every useless Comparison, we are further to observe, that if every where for y , be substituted the Value expressed in the Powers of x , that arises for it, by supposing any two Terms of the Equation equal, the Dimensions of x in all the Terms that are found in the same straight Line will be equal, but the Dimensions of x in the Terms above that Line will be greater than in those in that Line, and the Dimensions of x in the Terms below the said Line will be less than its Dimensions in that Line.

The exponents of the terms which are in the same straight line are in arithmetical progression.

Thus if the two Terms x^2 and $x^3 y^2$ be supposed equal, we find $y^2 = \frac{1}{x}$, and $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$, and substituting this Value for y in all the Squares, the Dimensions of x in the Terms, $x^4 y^4$, $x^5 y^6$, $x^6 y^8$, &c. which are found in the same straight Line with x^2 and $x^3 y^2$, will be 2, but the Dimensions of x in all the Terms above that Line will be more than 2, and in all the Terms below that Line will be less than 2. For Example, substituting the foregoing Value of y in the Term $x^4 y^3$, which is above the Line, it will be transformed into $x^{\frac{5}{2}}$, whose exponent exceeds 2, and if it be substituted in the Term $x^3 y^3$ below the Line, this Term will be transformed into $x^{\frac{3}{2}}$ whose Exponent is less than 2.

LXXI.

The foregoing Property of the Analytick Triangle is founded upon this Principle, that the Squares whose Centers are in a straight Line contain Terms in which the Exponents of x and y are in Arithmetical Progression: which is manifest with respect to the Terms in the vertical Column AC , or horizontal Line AB , and their Parallels, in the Terms of the same Column the Exponents of y are the same, and those of x form

Demonstration of the foregoing property of the analytic triangle

the Arithmetical Progression 1, 2, 3, 4, &c. in the Terms upon the same Line, the Exponents of x are the same, and those of y form the Arithmetical Progression 1, 2, 3, 4, &c. As to the Terms taken in any oblique straight Line, let us suppose that this straight Line parting from the Center of a Square, traverses k Lines and l Columns before it passes through the Center of another Square, it is manifest, that since the Squares are ranged uniformly, it must traverse k Lines and l Columns before it passes thro' the Center of a third Square, and as many more before it can attain the fourth, and so on. Wherefore, since the Exponent of x is increased by Unity in ascending a Line, and the Exponent of y is increased by Unity in traversing a Column, from right to left, if the

Term placed in the first Square be $x^m y^n$, that placed in the second will be $x^{m+k} y^{n+l}$, that in the third $x^{m+2k} y^{n+2l}$, and so on, in which the Exponents of x form an Arithmetical Progression, $m, m+k, m+2k$, &c. and those of y the Progression $n, n+l, n+2l$, &c.

LXXII.

Conversely, Terms as $x^m y^n$, $x^{m+k} y^{n+l}$, $x^{m+2k} y^{n+2l}$, &c. in which the Exponents both of x and of y are in Arithmetical Progression whose Differences are k and l , those Terms are placed in Squares whose Centers are under a Ruler traversing at the same Time k Lines and l Columns, which determines the Inclination of this Ruler to the Sides of the Triangle.

The ratio of the differences of those progressions depend on the inclination of the ruler to the sides of the triangle.

For the Inclination of the Ruler to the Sides of the Triangle, and the Ratio of k to l depend entirely of each other, since k and l are the Number of Lines and the Number of Columns which the Ruler at the same Time traverses. If k surpasses l , the Ruler is more inclined to the Columns than to the Lines, and cuts off a greater Portion of the Side AC of the Triangle, than of the Side AB . In general, since the Ruler traverses k Lines in traversing l Columns, it will traverse, from Column to Column a Number of Lines expressed by $\frac{k}{l}$, whether $\frac{k}{l}$ denotes an Integer or a Fraction, if the Ruler traverses two Columns in traversing one Line, it will traverse but Half a Line in traversing one Column.

LXXIII.

The terms which are in two parallel straight lines have exponents which form arithmetical progressions

Therefore if on the Surface of the Analytic Triangle there be drawn two parallel straight Lines, and consequently inclined as much one as the other to the Lines and Columns, the Squares whose Centers those straight Lines traverse, contain Terms in which the Exponents of x and y are in Arithmetical Progression, which in both one and the other straight Line have the same Difference. Those under the first straight Line being, for Example, $x^m y^n$, $x^{m+k} y^{n+l}$, $x^{m+2k} y^{n+2l}$, &c. those

under the second may be expressed by $x^{\frac{p}{m}} y^{\frac{q}{n}}$, $x^{\frac{p+k}{m}} y^{\frac{q+l}{n}}$, $x^{\frac{p+2k}{m}} y^{\frac{q+2l}{n}}$, &c. having the same common difference.

Conversely, if two Series of Terms be assumed as $x^{\frac{m}{m+2k}} y^{\frac{n}{n+2l}}$, &c. and $x^{\frac{p}{p+k}} y^{\frac{q}{q+l}}$, $x^{\frac{p+2k}{p+2k}} y^{\frac{q+2l}{q+2l}}$, &c. in each of which the Exponents of x and y are in Arithmetical Progression, having the same common Difference, the straight Lines which pass through the Centers of the Cells of those two Series of Terms will be parallel.

LXXIV.

Such being the Disposition of the Terms in the analytick Triangle, if any two Terms be supposed equal, and a Ruler be applied to the Centers of the Cells of those two Terms. I say, that if in all the Terms placed in the Cells through whose Centers the Ruler passes, be substituted for y , the Value that arises for it from the foregoing Supposition, the Dimensions of x in those Terms will be all equal.

For as their Exponents are in Arithmetical Progression, they may be represented by the Series $x^{\frac{m}{m+4k}} y^{\frac{n}{n+4l}}$, $x^{\frac{m+k}{m+5k}} y^{\frac{n+l}{n+5l}}$, $x^{\frac{m+2k}{m+6k}} y^{\frac{n+2l}{n+6l}}$, $x^{\frac{m+3k}{m+7k}} y^{\frac{n+3l}{n+7l}}$, &c. Now let any two of them as $x^{\frac{m+2k}{m+2k}} y^{\frac{n+2l}{n+2l}}$ and $x^{\frac{m+5k}{m+5k}} y^{\frac{n+5l}{n+5l}}$ be supposed equal, then $\frac{x^{\frac{m+2k}{m+2k}} y^{\frac{n+2l}{n+2l}}}{x^{\frac{m+5k}{m+5k}} y^{\frac{n+5l}{n+5l}}} = 1$.

$\frac{x^{\frac{m+5k}{m+2k}} y^{\frac{n+5l}{n+2l}}}{x^{\frac{m+2k}{m+2k}} y^{\frac{n+2l}{n+2l}}}$, or $1 = x^{\frac{3k}{m+2k}} y^{\frac{3l}{n+2l}}$, consequently $x^{\frac{k}{m+2k}} y^{\frac{l}{n+2l}} = 1$, &c.

whence the Dimensions of x , in all the Terms under the Ruler $x^{\frac{m}{m+2k}} y^{\frac{n}{n+2l}}$, $x^{\frac{m+k}{m+3k}} y^{\frac{n+l}{n+3l}}$, &c. resulting from the Multiplication of $x^{\frac{m}{m+2k}} y^{\frac{n}{n+2l}}$ into 1 , $x^{\frac{k}{m+2k}} y^{\frac{l}{n+2l}}$, $x^{\frac{2k}{m+2k}} y^{\frac{2l}{n+2l}}$, &c. is $m - nk : l$, and the Dimensions of x in the Terms $x^{\frac{p}{p+k}} y^{\frac{q}{q+l}}$, $x^{\frac{p+2k}{p+2k}} y^{\frac{q+2l}{q+2l}}$, &c. which are all found in the same straight Line parallel to the Ruler will be $p - qk : l$.

LXXV.

The Dimension of x in all the Terms that are under the Ruler may be also found by examining where the Ruler cuts the first Column. If it passes through the Center of some Cell of this Column, it is manifest that the Exponent of x in the Term placed in this Cell will represent the Dimensions of x in all the Terms under the Ruler, and if the Ruler

Ratio of the orders of the two undetermined quantities in this supposition.

ler passes between the Centers of two Cells, the Point through which it passes will still serve to determine the Dimensions of x in those Terms, whose Exponent in this Case is a Fraction. We are to conceive this Column, and in general each Column, as a straight Line divided into equal Parts by the Centers of the Cells; the Terms x , x^2 , x^3 , &c. whose Exponents are Integers, placed in the Points of Division, and the

Terms, as $x^{1:2}$, $x^{1+1:3}$, $x^{1+3:4}$, &c. whose Exponents are Fractions or mixed Numbers, placed in the Points which divides the Intervals between the Centers in the same Ratio, as Unity is divided by the Fraction, which forms or concurs to form the Exponent of x . Hence

Exponent of
the order of
the Terms
that are in
the same
straight line.

we are to conceive $x^{1:2}$ placed precisely in the Middle between x^0 or 1, and x^1 , and $x^{1+1:3}$ placed in the first Point of the Subdivision of the Interval between x^1 and x^2 into three equal Parts, consequently

if the Ruler passes, for Example, between the Centers of x^2 and of x^3 , but three Times nearer the first than the last, that is, through the

Point where $x^{2+3:4}$ is conceived to be placed, we are to conclude that the Dimensions of x in all the Terms under the Ruler is $2\frac{3}{4}$.

The Exponent of the Dimensions of x in the Terms under the Ruler is negative when it cuts the first Column produced below the first Line, which

happens when $m < nk:l$, or $p < qk:l$, that is, when $\frac{k}{l} > \frac{m}{n}$ or $\frac{p}{q}$,

then the Exponent $m - nk:l$ or $p - qk:l$ is negative. In this Case, the first Column is to be conceived as a straight Line produced below the Point, and divided into Parts equal to those that are above the Point,

The terms
that are a-
bove this
straight line
are of a su-
perior or-
der: those
that are be-
low it of an
inferior or-
der.

and the Terms whose Exponents are negative, as x^{-1} , x^{-2} , x^{-3} , &c. placed in the Points of Division.

LXXVI.

Since therefore of two parallel straight Lines drawn on the Surface of the analytick Triangle, that which is above the other cuts the first Column in a higher Point, the Terms in this Line are all of higher Dimensions of x than those in the Line below it. Thus the Ruler passing

through the Terms $x^m y^n$, $x^{m+k} y^{n+1}$, $x^{m+2k} y^{n+2}$, &c. which are all of $m - nk:l$ Dimensions of x . Any other Term is of higher or lower Dimensions of x , according as it is placed in a Cell whose Center is above or below the Ruler.

LXXVII.

This is the Principle which determines the Comparisons to be made for finding the greatest Terms of an Equation. When x is supposed

indefinitely great, it is manifest that it is useless to consider two Terms as being the greatest of the Equation, if the Ruler, applied to the Centers of their Cells, leaves above them any other Term, for this Term being of more Dimensions of x than those through which the Ruler passes, it would therefore be indefinitely greater than those which were supposed to be the greatest, which is absurd.

When x is supposed indefinitely little, it is useless to suppose any two Terms to be the greatest of the Equation, if the Ruler, applied to the Centers of their Cells leaves below them any Term of the Equation, for this Term being of lower Dimensions of x than those through which the Ruler passes, it would be indefinitely greater than those which were supposed to be the greatest, which is absurd.

LXXVIII.

Whence is deduced the following Rule, for discovering what Terms ought to be assumed from an Equation, in order to give a Value for x , or y , which shall make the other Terms, all of higher, or all of lower Dimensions of x or of y , than the assumed Terms.

After having traced the analytic Triangle, range each Term of the Equation in its proper Cell, or what in Practice is more commodious, form a Triangle with Points disposed in Quincunx, then change into an Asterisk each Point that represents a Square, which contains the same Dimensions of x and y , as the Terms in the Equation. Or a Triangle may be constructed of Wood or of Ivory, pierced with small Holes and ranged at equal Distances, and parallel to the Sides of the Triangle, and the Holes which represent the Cells in which the Terms of the Equation are placed, may be stopped with Pegs.

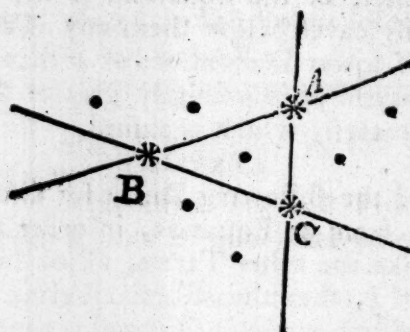
Rule deriv-
ed from the
foregoing
principles,
for discover-
ing the great-
est terms of
an equation.

Then the Triangle being couched on the Line without x , if x be supposed indefinitely great or indefinitely small, or on the Line without y , if y be supposed indefinitely great or indefinitely small, assume when the variable Quantity is supposed indefinitely great, such Terms as lie in a straight Line, so that the other Terms fall all below the straight Line. This straight Line which thus determines the greatest Terms of the Equation, is called by the Analysts, a *Superior Determinator*. The same Operation may furnish several of them.

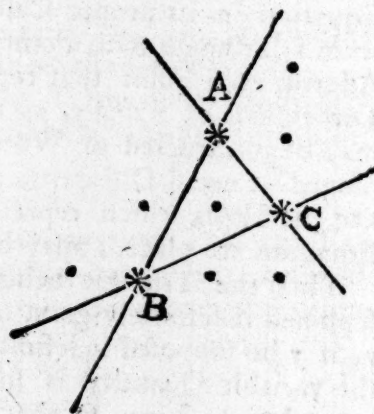
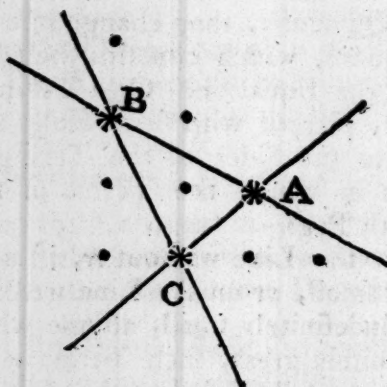
But if x or y be supposed indefinitely little, assume such Terms as lie in a straight Line, so that the other Terms lie all above the straight Line, this straight Line or those straight Lines, for there may be several of them, are called by the Analysts, *Inferior Determinators*, because they determine the greatest Terms of the Equation, being those which are placed in the Cells through whose Centers they pass.

LXXIX.

Let the Equation proposed be $x^2 y + a y^2 - a^2 x = 0$, having described the Triangle with Points, and converted into Afterisks the Points which represent the Cells $x^2 y$, y^2 and x , it will appear



Application
of the fore-
going rule
to an exam-
ple



that there are only three Determinators, AB , BC , CA , of which, when the Triangle is couched on the Line without x , two AB , AC , are superior, and one inferior BC ; but when the Triangle is couched on the Line without y , the Determinator AB is a superior one, and AC and BC are inferior ones.

The Determinator AB gives the Equation $x^2 y + a y^2 = 0$, or $xx + ay = 0$, the Determinator AC gives $x^2 y - a^2 x = 0$, or $xy - aa = 0$; and the Determinator BC gives $ay^2 - a^2 x = 0$, or $yy - ax = 0$.

Whence x being supposed indefinitely great, the proposed Equation is reduced to those two $xx + ay = 0$, and $xy - aa = 0$, given by the Determinators AB , AC , which are superior ones, when the Triangle is couched on the Line without x .

But if x is supposed indefinitely little, the Equation is reduced to $yy - ax = 0$, given by the Determinator BC , which is an inferior one, in the same Position of the Triangle.

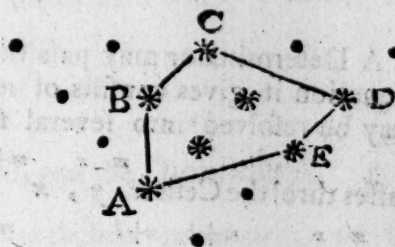
If y be supposed indefinitely great, the Equation is reduced to $xx + ay = 0$, given by the Determinator AB , which is a superior one, when the Triangle is couched on the Line without y .

But if y be supposed indefinitely little, the Equation will be reduced to $xy - aa = 0$, and $yy - ax = 0$, given by the Determinators AC , BC , which are inferior ones in this same Position of the Triangle.

LXXX.

Let $xxyy + axy^2 + bx^2y + cx^3 + ddx + eexx + f^3y = 0$, be proposed.

After having formed the Triangle with Points, and converted into Asterisks the Points which correspond to the Terms of the Equation, it will appear that all the Asterisks may be included in the Pentagon $ABCDE$, there are therefore five Determinators, which give the five following Equations.



Application of the foregoing rule to another example.

AB gives $f^3y + axy^2 = 0$, or, dividing by ay , $xy + \frac{f^3}{a} = 0$.

BC gives $axy^2 + x^2y^2 = 0$, or, dividing by xy^2 , $x + a = 0$.

CD gives $x^2y^2 + cx^3 = 0$, or, dividing by x^2 , $yy + cx = 0$.

DE gives $cx^3 + eex^2 = 0$, or, dividing by cx^2 , $x + \frac{ee}{c} = 0$.

And EA gives $eex^2 + f^3y = 0$, or, dividing by ee , $xx + \frac{f^3}{ee}y = 0$.

If x be supposed indefinitely great, the Triangle being couched on the Line without x , then CD is the only superior Determinator. Wherefore this Supposition reduces the proposed Equation to $yy + cx = 0$.

If x be supposed indefinitely little, the Triangle remaining in the same Position, the inferior Determinators are AB and AE , which give the

Equations $xy + \frac{f^3}{a} = 0$, and $xx + \frac{f^3}{e^e} y = 0$, to which two Equations the proposed one is reduced by supposing x indefinitely small.

If y be supposed indefinitely great, the Triangle being couched on the Line without y , there are three superior Determinators AB , BC , CD , which give the Equations $xy + \frac{f^3}{a} = 0$, $x + a = 0$, $yy + cx = 0$, to which the proposed Equation is reduced by supposing y indefinitely great.

But if y be supposed indefinitely little, there are two inferior Determinators, AE , ED , which give the Equations, $xx + \frac{f^3}{e^e} y = 0$, and $x + \frac{e^e}{c} = 0$, to which the proposed Equation is reduced by the Supposition of y indefinitely little.

LXXXI.

A Determinator may pass through more than two Cells, and then the Equation it gives consists of more than two Terms, but this Equation may be resolved into several simple Equations. If the Determinator

passes thro' the Cells $x^m y^n$, $x^{m+k} y^{n+l}$, $x^{m+2k} y^{n+2l}$, &c. it will give
 Roots of the equation given by a determinator. $a x^m y^n + b x^{m+k} y^{n+l} + c x^{m+2k} y^{n+2l} + d x^{m+3k} y^{n+3l}$, &c. = 0, in which Equation, the Terms corresponding to the empty Cells will have their Coefficients a , b , c , d , &c. = 0. All the Terms of

this Equation being divisible by $x^m y^n$, it may be reduced to $a + b x^k y^l + c x^{2k} y^{2l} + d x^{3k} y^{3l}$, &c. = 0, or supposing $x^k y^l = z$, to $a + bz + cz^2 + dz^3$, &c. = 0.

Let, R , r , p , &c. express the Roots of this Equation, it may therefore be resolved into those Equations $z - R = 0$, $z - r = 0$, $z - p = 0$, &c.

That is, $x^k y^l - R = 0$, $x^k y^l - r = 0$, $x^k y^l - p = 0$, &c. which are reduced to $y = R x^{-k/l}$, $y = r x^{-k/l}$, $y = p x^{-k/l}$, &c. or infine to $y = R x^{-k/l}$, $y = r x^{-k/l}$, $y = p x^{-k/l}$, &c.

LXXXII.

Those Coefficients R, r, p , &c. of the Equations $y = R^{\frac{1:l}{1:l} - \frac{k:l}{1:l}} x^{\frac{1:l}{1:l} - \frac{k:l}{1:l}}$, $y = r^{\frac{1:l}{1:l} - \frac{k:l}{1:l}} x^{\frac{1:l}{1:l} - \frac{k:l}{1:l}}$, $y = p^{\frac{1:l}{1:l} - \frac{k:l}{1:l}} x^{\frac{1:l}{1:l} - \frac{k:l}{1:l}}$, &c. may be imaginary, they are all so, when

the Roots of the Equation $a + bz + cz^2 + dz^3$, &c. = 0, are imaginary, which may happen as often as the Equation is of an even Number of Dimensions, when the compleat Number of its Terms is odd, They may be imaginary.

when the Determinator passes through an odd Number of Cells, reckoning from the first full Cell to the last. But when this Number of Cells is even, the Equation $a + bz + cz^2$, &c. = 0, being of an odd Number of Dimensions, has at least one real Root. And in particular its Roots cannot be imaginary, when the Determinator traverses only two full Cells, which are in two contiguous horizontal or vertical

Columns. For in this Case, the Equation being $a x^{\frac{m}{l}} y^{\frac{n}{l}} + b x^{\frac{m+k}{l}} y^{\frac{n+l}{l}} = 0$, (k or l being equal to Unity, on account of the Contiguity of the Columns) we will have $a + b x^{\frac{m}{l}} y^{\frac{n}{l}} = 0$, or $a + b x^{\frac{m}{l}} y^{\frac{n}{l}} = 0$, that is, $x = -\frac{a}{b} y^{-\frac{n}{l}}$, or $y = -\frac{a}{b} x^{-\frac{m}{l}}$.

LXXXIII.

The Coefficients R, r, p , &c. may be real, and the Quantities $R^{\frac{1:l}{1:l} - \frac{k:l}{1:l}} x^{\frac{1:l}{1:l} - \frac{k:l}{1:l}}$, $r^{\frac{1:l}{1:l} - \frac{k:l}{1:l}} x^{\frac{1:l}{1:l} - \frac{k:l}{1:l}}$, $p^{\frac{1:l}{1:l} - \frac{k:l}{1:l}} x^{\frac{1:l}{1:l} - \frac{k:l}{1:l}}$, &c. nevertheless imaginary, or half

imaginary. A Root is said to be *half Imaginary*, such as $\sqrt{ax}$, which is real when x is positive, and imaginary when x is negative, and is said to be entirely imaginary, as $\sqrt{-xx}$, which is imaginary, whatever Value positive or negative is given to x . Or half imaginary.

Since the even Powers of a positive or negative Root, are necessarily positive, but the odd Powers are positive, if the Root is positive, and negative if the Root is negative, it is manifest that an odd Root is always real, whatever is the Power of the Quantity whose Root is extracted, but that an even Root can only be real when the Power is positive. Therefore if this Power is an even Power of a variable Quantity, the even Root will be real or imaginary, according as the Power is taken positively or negatively, that is, according as it is affected with a positive or negative Coefficient, but if the Power, whose even Root is extracted is an odd Power of a variable Quantity, the Root is half imaginary.

Thus in the Equation $y = R^{\frac{1:l}{1:l} - \frac{k:l}{1:l}} x^{\frac{1:l}{1:l} - \frac{k:l}{1:l}} = \sqrt[l]{R x^{\frac{1:l}{1:l} - \frac{k:l}{1:l}}}$, if l is an odd Num-

ber, y is always a real Quantity, but if l is an even Number, y is half imaginary, k being odd; and k being even, y is real when R is positive, and imaginary when R is negative.

LXXXIV.

We are further to observe with respect to the Exponent $-\frac{k}{l}$ of x ,

in the Equation $y = R^{1:l - k:l} x$ given by the Determinator.

10. That it is negative, when k and l have the same Sign, which happens when the Arithmetical Progressions $m, m+k, m+2k, \&c. n, n+l, n+2l, \&c.$ of the Exponents of x and of y in the Terms which are under the Ruler, (Art. LXXI.) are both increasing or both decreasing. Then the Ruler recedes at the same Time from the first horizontal Line, and from the first vertical Line; it only cuts one of those two Lines, or sets out from the Point of the analytick Triangle. In this Case the Supposition of x indefinitely great, renders $y = R^{1:l - k:l} x$ or $\left(\frac{R}{x}\right)^{1:l}$

indefinitely little, and the Supposition of x indefinitely little, renders y indefinitely great.

2°. That this Exponent $-\frac{k}{l}$ is positive, when l and k have contrary Signs, which happens as often as one of the two Arithmetical Progressions $m, m+k, m+2k, \&c. n, n+l, n+2l, \&c.$ is increasing and the other decreasing, when the Ruler approaches one of the Sides of the Triangle whilst it recedes from the other, when it cuts them both any where else, but in the Point of the Triangle.

In this Case, the Supposition of x indefinitely great or indefinitely little, will render also $y = R^{1:l - k:l} x$ or $(Rx)^{1:l}$ indefinitely great or indefinitely little. They are of the same Number of Dimensions if $\frac{k}{l} = 1$, if $k = l$, if the Determinator is equally inclined to the

two Lines. But if $\frac{k}{l} > 1$, if $k > l$, if the Determinator is more inclined to the vertical Columns than to the horizontal ones, retrenches a greater Portion of the Line without y , than of the Line without x , then $y = R^{1:l - k:l} x$ is of a higher Order than x , and of a lower Order than x , if $\frac{k}{l} < 1$, if $k < l$, if the Determinator retrenches a

Observations respecting the exponent of those Roots.

less Portion of the Side of the Triangle without y , than of the Side without x .

3°. If $k = 0$, which happens when the Determinator is parallel to the Line without x , then $y = R^{\frac{l}{1-l} - \frac{k}{1-l}} x^{\frac{l}{1-l}}$ is reduced to $y = R^{\frac{l}{1-l}} x^0 = R^{\frac{l}{1-l}}$. wherefore x being supposed indefinitely great, gives only finit Values for y .

4°. And for a similar Reason, when the Determinator is parallel to the Side of the Triangle without y , reduces l to 0, it is obvious that y being supposed indefinitely great, gives for x only finit Values deter-

mined by the Roots of the Equation $a x^{\frac{m}{n}} y^{\frac{m+k}{n+l}} + b x^{\frac{m+k}{n+l}} y^{\frac{m+2k}{n+2l}} + c x^{\frac{m+2k}{n+2l}} y^{\frac{m+3k}{n+3l}} + \dots = 0$, which since $l = 0$, is reduced, by dividing by $x^{\frac{m}{n}} y^{\frac{m}{n}}$ to $a + b x^{\frac{k}{n}} + c x^{\frac{2k}{n}} + \dots = 0$.

LXXXV.

Those Particulars respecting the Method of finding the greatest terms of an Equation being premised, let the Value of y deduced from any proposed

Equation involving x and y be expressed by the Series $Ax^b + Bx^i + Cx^k + Dx^l + \dots$, in which the Exponents $b, i, k, l, \dots$ increase or decrease according as the Series is an ascending or descending one. $A, B, C, D, \dots$ are the Coefficients of the successive Terms, and as it is possible that one or other of them may vanish with all those that follow, it may happen that the Number of Terms of the Series is finite, and then it expresses the exact Value of y .

LXXXVI.

To determine as many Terms of the Series as you please, proceed thus. To find the first Term. Suppose x indefinitely great, if a descending Series is required, but indefinitely small, if an ascending Series

is required. This Supposition reduces the Series to its first Term Ax^b , for if the Series is a descending one, the Exponent b is greater than i, k or $l, \dots$ and x being supposed indefinitely great, the Power x^b is inde-

finitely greater than the others $x^i, x^k, x^l, \dots$ (Art. LXXXVI.) which may be neglected without Error, and if the Series is an ascending one, the Exponent b is the least, and x being indefinitely small, the Power x^b will

Investigation of the successive terms of a series.

make all the others vanish, therefore the Supposition of x indefinitely great, in a descending Series, and that of x indefinitely little, in an as-

cending Series, reduces it to $y = Ax^b$. But those same Suppositions reduce the proposed Equation to one or several Equations, such as $y =$

$R x^{1:l - k:l}$ given by the super. or infer. Determinators, (Art. LXXVIII.)

wherefore $A = R^{1:l}$ and $b = -\frac{k}{l}$, the Determinators therefore serve to determine the first Term of the Series, or Series, when the proposed Equation furnish several.

The following Terms are found after the same Manner. Let u ex-

press the Sum of the Terms $Bx^i + Cx^k + Dx^l$, &c. which succeed the

first, consequently $y = Ax^b + u$, this Value of y , substituted in the proposed Equation, transforms it into another, including the variable Quantities u and x ; let x be supposed in this Equation indefinitely great, when the Series required is a descending one, and indefinitely little, when the Series required is an ascending one: and the superior or inferior Determinators will give one or several Equations, such as

$u = R x^{1:l - k:l}$ (Art. LXXIX.) but the same Suppositions of x inde-

finitely great or indefinitely small, reduces the Series $u = Bx^i + Cx^k$ &c.

to $u = Bx^i$, therefore $B = R^{1:l}$ and $i = -\frac{k}{l}$, hence the Determinators of this first transformed Equation will give the second Term Bx^i of the Series.

Transforming a new the Equation, supposing $u = Bx^i + t$, where t expresses all the Terms $Cx^k + Dx^l$, &c. which follow the second Term of the Series; and the Determinators of this second transformed Equ-

ation will give the third Term Cx^k of the Series, and proceeding after the same Manner, the fourth Term will be obtained, and the following Terms to the last, if the Number of Terms is finite, or at least as many as the End proposed may require, if the Number of Terms of of the Series be infinite.

But we are to observe in the Course of those Operations, that the Nature of ascending Series require that the Exponents of x should continually increase, and in descending Series those Exponents should continually decrease. Wherefore, tho' at the first Operation performed on the proposed Equation, all the superior Determinators are to be taken into Consideration, to obtain all the descending Series, or all the inferior Determinators to obtain all the ascending Series: In the subsequent Operations, no Attention is to be paid to the superior Determinators which will give the same or a greater Exponent than was obtained by the precedent Operation, nor to the inferior Determinators which will give the same or a less Exponent than what was obtained by the foregoing Operation. If there are no other Determinators, the Course of the Operations is finished and the Series is ended.

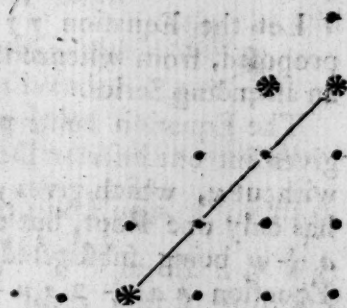
LXXXVII.

Let the Equation $ay^3 - x^3y - ax^3 = 0$, be proposed, and let it be placed on the analytic Triangle, and this Triangle being couched on the Line without x , there is but one inferior Determinator, which, passing thro' the Cells y^3 and x^3 , gives the Equation $ay^3 - ax^3 = 0$, or $y = x$, which is the first Term of an ascending Series. To find

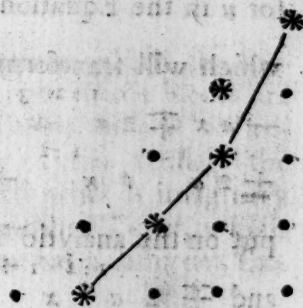
the second, I substitute $x + u$ for y , and the Equation will be transformed into $ax^3 + 3auxx + 3auux + au^3 - x^4 - x^3u - ax^3 = 0$, or $3auxx + 3auux + au^3 - x^4 - x^3u = 0$, which being placed in its Turn on the analytic Triangle, gives two inferior Denominators, one which passes thro' the Cells u^3 , uux , uxx , is to be rejected, because it would give $u = Rx$, this second Exponent of x being the same as was found by the first Operation; but the other Determinator, which passes thro' the Cells uxx and x^4 gives the Equation $3auxx - x^4 = 0$, or $u = \frac{x^2}{3a}$, which is the second Term of the Series. The third Term will be found by substituting $\frac{x^2}{3a} + t$ for u in the foregoing Equation,

which transforms it into $x^4 + 3atxx + \frac{x^5}{3a} + 2tx^3 + 3attx + \frac{x^6}{27aa} + \frac{tx^4}{3a} + ttxx + at^3 - x^4 - \frac{x^3}{3a} - tx^3 = 0$, or $3atxx + tx^3 + 3attx + \frac{x^6}{27aa} + \frac{tx^4}{3a} + ttxx + at^3 = 0$.

4 N



How to know when a series becomes imaginary, half imaginary, or forked.



which being placed on the analytic Triangle, one Determinator gives $t = Rx$, and which consequently should be rejected, but the other inferior Determinator which passes thro' $t x x$ and x^6 , gives the Equation

$$3atxx + \frac{x^6}{27aa} = 0, \text{ or } t = -\frac{x^4}{81a^3},$$

which is the third Term of the Series, for

$$y = x + u, \text{ and } u = \frac{xx}{3a} + t, \text{ and}$$

$$t = -\frac{x^4}{81a^3}, \text{ \&c. hence } y = x + \frac{xx}{3a} - \frac{x^4}{81a^3} \text{ \&c.}$$

the Law of the Progression of the Terms being evident.

LXXXVIII.

Application
of the fore-
going me-
thod to an
example.

Let the Equation $yy - 2xy + xx - 2ay + ax + aa = 0$ be proposed, from whence is to be deduced the Value of y in x expressed by an ascending Series.

The Equation being placed on the analytic Triangle, gives but one inferior Determinator couched on the Line without x , which gives $yy - 2ay + aa = 0$, which has only one Root, but double, $y - a = 0$, or $y = a$; $a + u$ being substituted for y , gives the transformed Equation $aa + 2au + uu - 2ax - 2ux + xx - 2aa - 2au + ax + aa = 0$, or $uu - ax - 2ux + xx = 0$, placing it on the analytic Triangle, I find but one inferior Determinator, which gives the Equation $uu - ax = 0$,

or $u = \pm \sqrt{ax} = \pm a x^{1:2}$. Substituting $\pm a x^{1:2} + t$ for u in the Equation $uu - ax - 2ux + x^2 = 0$,

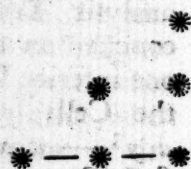
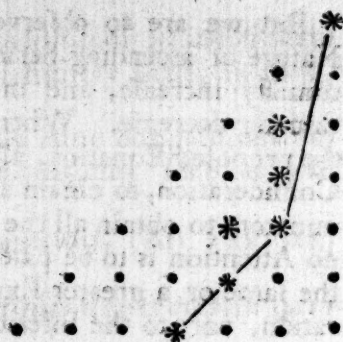
which will transform it into $ax \pm 2a t x + tt - ax \mp 2a x - 2tx + xx = 0$, or

$$\pm 2a t x + tt \mp 2a x - 2tx + xx = 0, \text{ which}$$

put on the analytic Triangle, and as two of its Terms $\pm 2a t x$ and $\mp 2a x$ have no Cells to lodge in;

let them be placed (as explained Art. LXXV.) between two Cells, viz. the first in the second Column between the Cells $t x^0$ or t , and $t x^1$, and the second in the first Column between the Cell x^1 and the Cell x^2 ; there will then be two inferior Determinators, one pas-

sing thro' tt and tx , giving $t = Rx$ is to be rejected, as having the same



Exponent as the precedent Term; the other Determinator passing thro' the

Cells $t x$ and x , gives $+2a \quad t x \quad + 2a \quad x = 0$, or $t = x$, which is the third Term of the Series.

To obtain the fourth, $x + s$ is to be substituted for t in the last Equation

$+2a \quad t x \quad + t t \quad + 2a \quad x \quad - 2 t x \quad + x x = 0$, which will transform it into $+2a \quad x \quad + 2a \quad s x \quad + x^2 \quad + 2 s x$

$+ s s \quad + 2a \quad x \quad - 2 x x \quad - 2 s x \quad + x x = 0$, or into $+2a \quad s x \quad + s s = 0$; those Terms being placed on

the analytic Triangle gives only one Determinator, which gives $s = R x$, this Exponent being less than the

preceding one, is to be rejected. Hence the Series is ended for $y = a + u = a \pm \sqrt{ax} + t = a \pm \sqrt{ax} + x$.

On examining what this last Determinator gives, or its Equation

$+2a \quad s x \quad + s s = 0$; we will find it has two Roots, $1^{\circ} s = 0$,

which terminates the Series; $2^{\circ} s = \mp 2a \quad x = \mp 2 \sqrt{ax}$; which substituted in $y = a \pm \sqrt{ax} + x + s$, gives $y = a \pm \sqrt{ax} + x \pm 2 \sqrt{ax}$, which is the same as $y = a \pm \sqrt{ax} + x$, which is therefore the exact Value of y , which may be easily verified, since by taking away the Irrationality, the Equation $y = a \pm \sqrt{ax} + x$ will be transformed into the proposed Equation $yy - 2xy + xx - 2ay + ax + aa = 0$.

LXXXIX.

It is to be observed, 1° that if among the Equations which the successive Determinators furnish, there be found any one whose Roots are all imaginary, the whole Series which the first Terms seemed to announce will become imaginary, for one imaginary Term renders the whole Sum, of which it is a Part, imaginary, unless what is imaginary in one Term be destroyed by what is imaginary in another Term, which cannot happen in this Case, where x in each Term has a different Exponent.

2° That if among the Terms of a Series there be any which are half imaginary, the Series is half imaginary, that is, imaginary when x is supposed negative, and real when x is supposed positive or reciprocally.

3° That if among the Equations which serve to determine the successive Terms of a Series, there be found any which have several real Roots;

Application of the foregoing method to another example.

then the Series becomes forked as it were, and is multiplied into as many Series as there are real Roots, as often as that happens.

XC.

Example of
an imagin-
ary series.

Let the Value of y in x , expressed by an ascending Series deduced from $x^3 + x^2 y + a y y - 2 a^2 y + a^3 = 0$ be required.

This Equation placed on the analytic Triangle couched on the Line without x , furnishes but one inferior Determinator, which gives the Equation $a y y - 2 a^2 y + a^3 = 0$, which has only one Root, but double, viz. $y - a = 0$, or $y = a$; let then $y = a + u$, and substituting this Value of y in the proposed Equation, it will transform it into $x^3 + a x^2 + u x^2 + a u u = 0$, which being placed on the analytic Triangle, furnishes but one inferior Determinator, which gives $a u u + a x x = 0$, whose Roots $u = \pm \sqrt{-x x}$ are imaginary; consequently the Value of y in x cannot be expressed by an ascending Series, because the only one which could express this Value would be $y = a + u = a \pm \sqrt{-x x}$, &c. which is imaginary.

XCI.

Example of
a series half
imaginary.

Let the Equation $x^2 y + a y y - 2 a x y + a x x = 0$ be proposed, from whence is to be deduced the Value of y in x , expressed by an ascending Series. This Equation being placed on the analytic Triangle, furnishes but one inferior Determinator, passing thro' the Cells $y y$, $x y$, $x x$, which gives the Equation $a y y - 2 a x y + a x x = 0$, which has only one Root, but double, $y = x$. Substituting therefore $x + u$ for y in the proposed Equation, it will be transformed into $x^3 + x^2 u + a u u = 0$, which being placed on the analytic Triangle, furnishes but one inferior Determinator, which gives the Equation $a u u + x^3 = 0$, which has two Roots $u = +\sqrt{-\frac{x^3}{a}}$ and $-\sqrt{-\frac{x^3}{a}}$

substituting $\pm \sqrt{-\frac{x^3}{a}} + t = a^{-1:2} \times -x^{3:2} + t$ for u in the first transformed Equation, the second will be $\pm a^{-1:2} \times -x^{3+1:2} + t x x \pm 2 a^{-1:2} t \times -x^{3:2} + a t t = 0$,

which being placed on the analytic Triangle, furnishes two inferior Determinators, one useless, because passing thro' the Cells t , t , $t x^{3:2}$, it gives

$t = R x^{3:2}$, where x has the same Exponent as in the precedent Term; the other passing thro'

$t x^{3:2}$ and $x^{3+1:2}$, gives the Equation $\pm 2 a^{1:2} t \times - x^{3:2} \pm a^{-1:2} \times - x^{3+1:2} = 0$, or $t = \frac{x x}{2 a}$

wherefore the three first Terms of the Series are $x \pm \sqrt{-\frac{x^3}{a} - \frac{x^2}{2 a}}$.

From the Inspection of which it appears, 1^o. that the Series is imaginary, when x is supposed positive, because then $\sqrt{-\frac{x^3}{a}}$ is an imaginary Quantity; but if x be supposed negative, the Series is real, and then

2^o. The Series is double, because the Term $\sqrt{-\frac{x^3}{a}}$ has equally the Sign + and the Sign -, the Equation $u u + x^3 = 0$, which furnished this Term, having two real Roots $u = + \sqrt{-\frac{x^3}{a}}$, and $u = - \sqrt{-\frac{x^3}{a}}$ wherefore in reality there are two Series, the three first Terms of one are $x + \sqrt{-\frac{x^3}{a} - \frac{x x}{2 a}}$ and of the other $x - \sqrt{-\frac{x^3}{a} - \frac{x x}{2 a}}$.

XCII.

The general Method being attended with tedious Calculations, we shall now proceed to explain how the Analysts have abridged them.

We are to observe, that the Substitution of $A x + u$ for y (Art. LXXXVI.) in any Term of the proposed Equation, transforms it into as many Terms as there are Columns, from the first to that in which it is placed inclusively; each Term being placed in a different Column, and all those Terms being situated on the same straight Line, parallel to the Determinator which gave the Equation $y = A x$.

Manner the terms of a transformed equation are placed in the analytic triangle.

For the Power n of $u + A x$ being $u + n A x u + \frac{n \cdot n - 1}{1 \cdot 2} A^2 x^2 u + \frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3} A^3 x^3 u + \dots$ to $A^n x^n$, or the last Term, which being substituted for y in a Term as $x y$

ELEMENTS OF

of the Order $m + nh$, y being of the Order h (Art. LXXXIV.) and which is placed in a Column preceded by n others, will transform it into $x^m u^n$ + $n A x^{m+b} u^{n-1}$ + $\frac{n \cdot n - 1}{1 \cdot 2} A^2 x^{m+2b} u^{n-2}$ +, &c. to $A^n x^{m+nb}$

which are all of the Order $m + nh$. But all the Terms which are of the same Order, are placed in a right Line, parallel to the Determinator hence all the Terms into which $x^m y^n$ is transformed are in a right Line parallel to the Determinator which gave $y = Ax$; and it is evident, that the Term

$x^m u^n$ occupies the Cell in which the transformed Term $x^m y^n$ was lodged in a Column preceded by n others; that the second Term $x^{m+b} u^{n-1}$ is in a Column preceded by $n - 1$ Columns; that the third $x^{m+2b} u^{n-2}$ is in the next Column, and so on to the last $A^n x^{m+nb}$ which is in the first Column, or in the Line of the Powers of x .

Example.

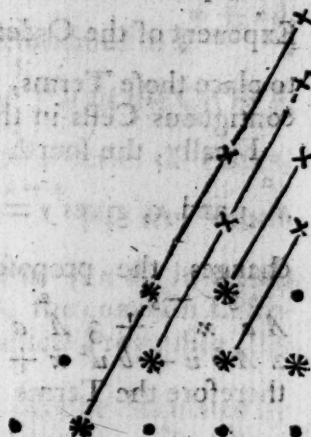
Thus when the Equation $x^2 y^2 + a y^3 + b x y^2 + c x^2 y + d d x y + f^3 x = 0$ is placed on the analytic Triangle, couched on the Line without x , it will give four Determinators; first an horizontal one, which passes thro' the Cells $x^2 y^2$ and $x^2 y$, which gives for y a constant Value (Art. LXXXIV.) & which may be expressed by A ; substituting $A + u$ for y in the proposed Equation, it will be transformed into $A^2 x^2 + 2 A u x^2 + u^2 x^2 + A^3 a + 3 A^2 a u + 3 A a u^2 + a u^3 + A^2 b x + 2 A b u x + b u^2 x + A c x^2 + c u x^2 + A d d x + d d u x + f^3 x = 0$, from whence it appears, that the Terms

$x^2 y^2$	of the proposed Equation, pro- duce in the transformed Equation	$x^2 u^2, x^2 u, x^2$
y^3		u^3, u^2, u^1, u^0
$x y^2$		$x u^2, x u, x$
$x^2 y$		$x^2 u, x^2$
$x y$		$x u, x$
x		x

Thus each Term has given one to each Column which precedes it, and those Terms are on an horizontal Line, that is, parallel to the Determinator which gave $y = A$.

The second Determinator of the proposed Equation passed thro' the Cells y^3 and $x^2 y$, and gave $y = Ax^2$: the Substitution of $Ax^2 + u$ for y , transforms the Equation into $A^2 x^6 + 2A u x^4 + u^2 x^2 + A^3 x^6 + 3A a u x^4 + 3A a u^2 x^2 + a u^3 + A^2 b x^5 + 2A b u x^3 + b u^2 x + A c x^4 + c u x^2 + A d d x^3 + d d u x + f^3 x = 0$, therefore the Terms

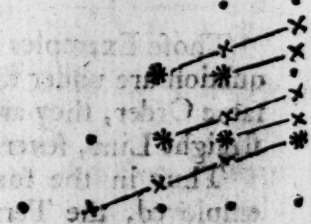
$$\left. \begin{array}{l} x^2 y^2 \\ y^3 \\ x^2 y^2 \\ x^2 y \\ x y \\ x \end{array} \right\} \text{ have given } \left\{ \begin{array}{l} x^2 u^2, x^4 u, x^6 \\ u^3, x^2 u^2, x^4 u, x^6 \\ x u^2, x^3 u, x^5 \\ x^2 u, x^4 \\ x u, x^3 \\ x \end{array} \right.$$



and placing the transformed Equation on the analytic Triangle, it will appear that all the Terms into which a Term of the proposed Equation had been transformed, are placed in the same straight Line parallel to the Determinator, which gave the Equation $y = Ax^2$, the same Thing may be verified with Respect to the two other Determinators of the proposed Equation; the third passes thro' the Cells y^3 and x , and gives an Equation of this Form $y = Ax^{1:3}$; substituting therefore $Ax^{1:3} + u$ for y ,

and the transformed Equation will be $A^2 x^{2+2:3} + 2A u x^{2+1:3} + u^2 x^{2+2:3} + A^3 a x^{2+2:3} + 3A^2 a u x^{2+1:3} + 3A a u^2 x^{2+1:3} + a u^3 + A^2 b x^{1+1:3} + 2A b u x^{1+1:3} + b u^2 x^{1+1:3} + A c x^{1+1:3} + c u x^{1+1:3} + A d d x^{1+1:3} + d d u x^{1+1:3} + f^3 x = 0$, hence the Terms

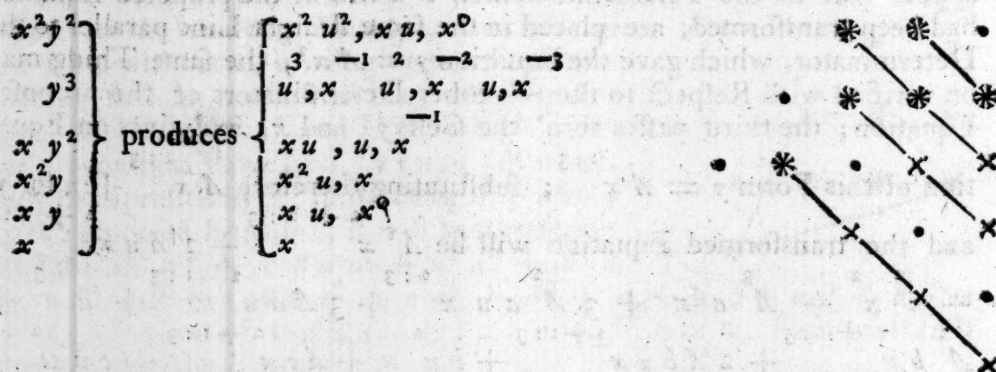
$$\left. \begin{array}{l} x^2 y^2 \\ y^3 \\ x^2 y^2 \\ x^2 y \\ x y \\ x \end{array} \right\} \text{ give } \left\{ \begin{array}{l} x^2 u^2, x^4 u, x^6 \\ u^3, x^2 u^2, x^4 u, x^6 \\ x u^2, x^3 u, x^5 \\ x^2 u, x^4 \\ x u, x^3 \\ x \end{array} \right.$$



This transformed Equation being placed on the analytic Triangle, it will appear that each Term of the proposed Equation has given a Term to each Column that precedes it; and that those Terms are on a straight

Line, parallel to the Determinator which gave $y = Ax^{\frac{1}{3}}$, but as the Exponent of the Order of y is a Fraction $\frac{1}{3}$, it was necessary, in order to place those Terms, to divide into three equal Parts the Intervals of the contiguous Cells in the same Column (Art. LXXV.)

Finally, the fourth Determinator of the proposed Equation passing thro' $x^2 y$ and x , gives $y = Ax^{-1}$, and the Substitution of $Ax^{-1} + u$ for y , changes the proposed Equation into $A^2 + 2Au x + u u x x + A^3 x^{-3} + 3A^2 a u x^{-2} + 3A a u u x^{-1} + a u + A^2 b x^{-1} + 2A b u + b u^2 x + A c x + c u x^2 + A d d + d d u x + f^3 x = 0$, therefore the Terms



Here the same Rule is observed, but as the negative Exponent (-1) of the Order of y , introduces Terms in which the Exponent of x is negative, it was necessary, in order to place those Terms, to produce the Triangle below the Line without x (Art. LXXV.)

XCIII.

Those Examples shew, that when several Terms of the proposed Equation are under the Determinator, or any of its Parallels, or are of the same Order, they are transformed into Terms, which ranged in the same straight Line, several of them sometimes lodge together in the same Cell.

Thus in the foregoing Example, if the horizontal Determinator be employed, the Terms $x^2 y^2$ and $c x^2 y$, which were under this Deter-

minator, were transformed into $x^2 u^2 + (2A + c)x^2 u + (A^2 + Ac)x^2$, and the Terms $bxy^2, ddx y, f^3 x$, which are under a straight Line, parallel to this Determinator, have given $b x u^2 + (2Ab + dd)xu + (A^2 b + Ad d + f^3)x$, which are still under the same straight Line.

Wherefore, when there are several Terms of the same Order, as $a x y + b x y + c x y$, &c. continued to x , which are all of the Order $m + nh$, since $y (= Ax)$ is of the Order h ,

the Substitution of $Ax + u$ for y transforms it into a Series of Terms, such as $P x u + Q x u + R x u$, &c. continued to $Z x$, which also are all of the Order $m + nh$, in which the Exponents of x form an arithmetical Progression, the common Difference being h , and the Exponents of u another arithmetical Progression, the common Difference being 1.

The Coefficients P, Q, R , &c. of those Terms, may be calculated by a contracted Rule, derived from the following Consideration: The Term

$a x y$, by the substitution of $u + Ax$ for y , is transformed into $a x u + n a A x u + \frac{n \cdot n - 1}{1 \cdot 2} a A^2 x u + \frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3} a A^3 x u$, &c. the Term $b x y$

is transformed into $b x u + \frac{n - 1}{1} b A x u + \frac{n - 1 \cdot n - 2}{1 \cdot 2} b A^2 x u$, &c. the Term $c x y$

into $c x u + \frac{n - 2}{1} c A x u$, &c. and the Term

$d x y$ into $d x u$, &c. Wherefore, the Sum $a x y + b x y + c x y + d x y$, &c.

is transformed into $a x u + (n a A + b) x u + \left(\frac{n \cdot n - 1}{1 \cdot 2} a A^2 + \frac{n - 1}{1} b A + c \right) x u + \left(\frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3} a A^3 + \frac{n - 1 \cdot n - 2}{1 \cdot 2} b A^2 + \frac{n - 2}{1} c A + d \right) x u$, &c. from whence

is deduced the following Rule:

Abridged method of performing the transformations requisite to be made for finding the terms of a series.

Write in the first Line all the Terms of the same Order, or even the whole Equation, distinguishing, for Conveniency's sake, the Orders of its Terms, changing, if you please, y into u ; I say if you please, for it will be found upon Trial, that it will be more commodious not to do it; but then it must be observed, that y , which denotes before the Operation the whole Series, and during the Operation its first Term only, denotes, during the second Operation, the second Term, and during the third Operation, the third Term, &c. Having wrote the proposed Equation in the first Line, and ranged the Terms according to their different Or-

ders, multiply each Term by the Exponent of y and by $A x^b$, and divide those Products by y , and this will give a second Line; multiply

each Term of this second Line by half the Exponent of y and by $A x^b$, and divide the Products by y , and a third Line will be obtained. Each Term of this Line, multiplied by one third of the Exponent of y and by

$A x^b$, and divided by y , will give a fourth Line. And this Operation is to be continued until there remains but Terms without y ; then the Sum of all those Lines will be the transformed Equation.

XCIV.

Thus in the Equation of the foregoing Article, if we employ the horizontal Determinator, which gave the Equation $x^2 y^2 + c x^2 y = 0$, or $y = -c$, there will result $A = -c$ and $h = 0$; and the Operation is performed thus:

I. Order.	H. Order.	III. Order.
$\overbrace{x^2 y^2 + c x^2 y}^{2 \quad 1}$	$+ \overbrace{b x y^2 + d d x y + f^3 x}^{2 \quad 1 \quad 0}$	$+ \overbrace{a y^3}^3$
$\times -c)$		
$\underline{-2 c x^2 y - c c x^2}$	$\underline{-2 b c x y - c d d x}$	$\underline{-3 a c y y}$
$\frac{1}{2} \quad 0$	$\frac{1}{2} \quad 0$	$\frac{2}{2}$
$\underline{+ c c x x}$	$\underline{+ b c c x}$	$\underline{-3 a c c y}$
0	0	$\frac{1}{3}$
		$\underline{- a c^3}$

The transformed Equation therefore is $x^2 y^2 + (c - 2 c) x^2 y + (c c - c c) x^2 + b x y^2 + (d d - 2 b c) x y + (f^3 - c d d + b c c) x + a y^3 - 3 a c y y + 3 a c c y - a c^3 = 0$, in which the second Term is reduced to $-c x^2 y$, and the third vanishes.

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If in the same Equation we would employ the Determinator that passes thro' the Cells $c x^2 y$ and $f^3 x$, which gives $y = -\frac{f^3}{c x}$ the Operation will be performed thus:

Application of this method to another example.

I.Order.	II.Order.	III.Order.	IV.Order.
$\overbrace{c x^2 y + f^3 x}^{1 \quad 0}$	$\overbrace{x^2 y^2 + d d x y}^{2 \quad 1}$	$\overbrace{b x y y}^{2}$	$\overbrace{a y^3}^{3}$
$\times -\frac{f^3}{c x}$			
$-f^3 x$	$-\frac{2 f^3 x y}{c}$	$-\frac{f^3 d d}{c}$	$-\frac{2 b f^3 y}{c}$
0	$\frac{\frac{1}{2}}{0}$	0	$\frac{\frac{1}{2}}{0}$
	$+\frac{f^6}{c c}$	$+\frac{b f^6}{c c x}$	$-\frac{3 a f^6 y}{c c x x}$
	0	0	$\frac{\frac{1}{3}}{0}$
			$+\frac{a f^9}{c^3 x^3}$

Whence the transformed Equation is $c x^2 y + (f^3 - f^3) x + x^2 y^2 + (d d - \frac{2 f^3}{c}) x y - (\frac{f^3 d d}{c} - \frac{f^6}{c c}) + b x y y - \frac{2 b f^3 y}{c} + \frac{b f^6}{c c x} + a y^3 - \frac{3 a f^3 y y}{c x} + \frac{3 a f^6 y}{c^2 x^2} - \frac{a f^9}{c^3 x^3} = 0$, in which the second Term disappears.

XCV.

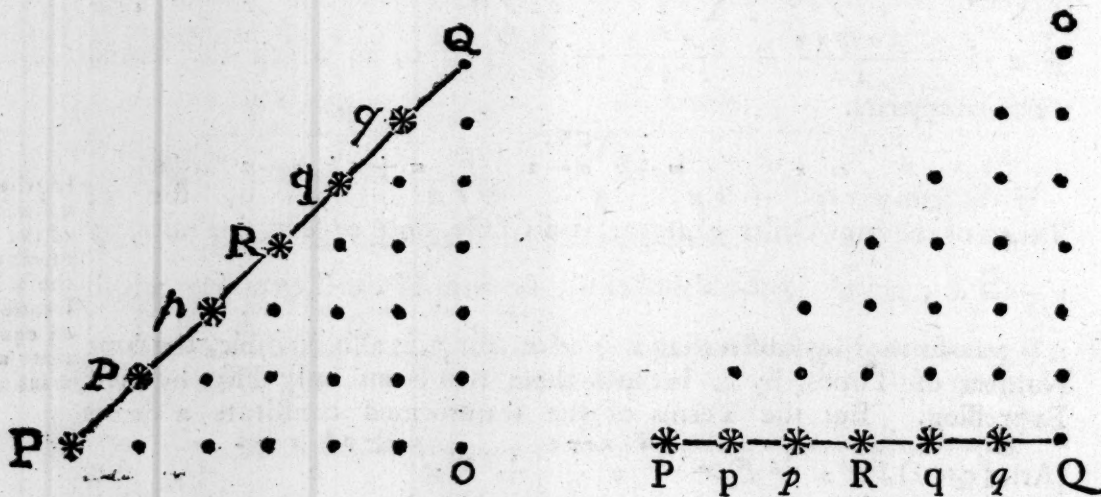
If the Sum $a x^m y^{n-b} + b x^{m+b} y^{n-1} + c x^{m+2b} y^{n-2} + \dots$ of Terms of the same Order whatever, is divisible, once or several Times, by $y - A x$, which is the Value of u , the Sum of the Terms into which it is transformed by substituting $u + A x$ for y , is also divisible, the same Number of Times, by u , because those two Sums only differ by their Expression. But the Terms of the transformed constitute a Series

In what cases some of the terms of those transformed equations are wanting.

(Art. XCIII.) $P x^m u + Q x^{m+b} u^{n-1} + R x^{m+2b} u^{n-2} + \dots$ terminated by the Terms $+ X x^{m+n} u + Y x^{m+n} u$ + $Z x^{m+n} u$. This Series cannot be divisible by u , unless the last Term $= 0$ and $Z = 0$; it cannot be divisible by $u u$, or twice by u , unless the two last Terms Y and Z be $= 0$; it cannot be divisible by u^3 , or three Times by u , unless X , Y , and Z be $= 0$. In general, as many Times

as u , or rather $y - Ax$, which is its Value, divides the Sum of the Terms of any Order, so many Terms of this Order are wanting in the transformed in the first Columns, for the Terms Z, Y, X , &c. are those which are placed in the first, second, third, &c. Columns.

Wherefore, since $y - Ax = 0$ is one of the Roots furnished by the Determinator, $y - Ax$ divides at least once, the Sum of the Terms which are under this Determinator; consequently, in the transformed Equation is necessarily wanting the Term corresponding to the Point where the Determinator cuts the first vertical Line, and in the transformed Equation, will be wanting the two Terms corresponding to the Points where the Determinator cuts the first and second Column, if $y - Ax$ divides the Sum of the Terms which are under the Determinator, twice, if $y - Ax = 0$ be a double Root of the Equation which this Determinator furnishes; but if $y - Ax = 0$ be a triple Root of this Equation, in the transformed Equation will be wanting the Terms corresponding to the Points where the Determinator crosses the three first Columns, and so on.

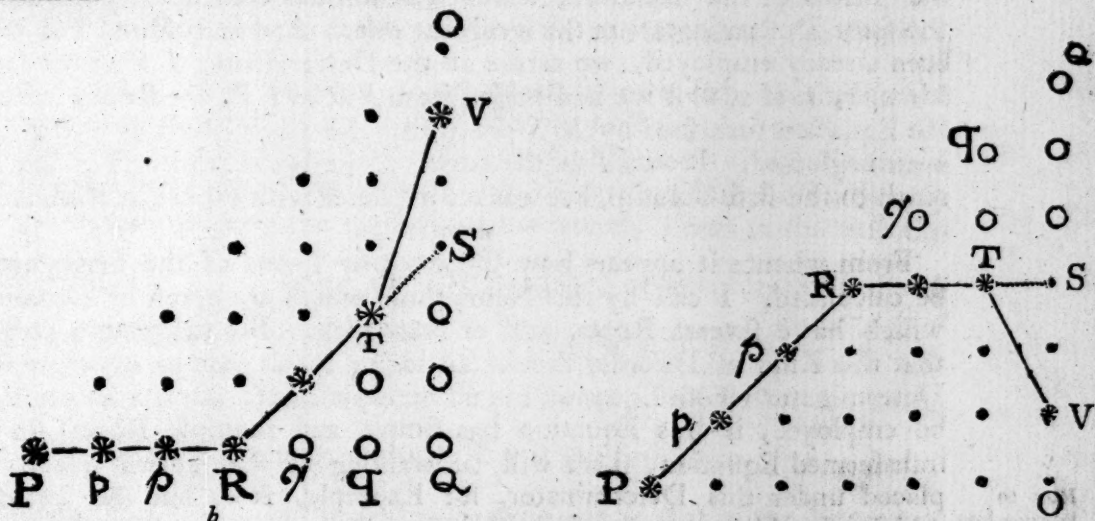


XCVI.

If therefore PQ represents a Determinator, and $y - Ax = 0$ be a simple Root of the Equation which it furnishes, in the transformed Equ-

tion under this Determinator will be wanting only the Term which should occupy the Cell Q in the first Column QO , at least the Term q in the second Column will not be wanting. Pq is also a Determinator of the transformed, for all the Terms of the transformed are below (or above) those under Pq , as were all the Terms of the proposed Equation (Art. LXXXVI.); but it is a useless Determinator, because Pq being Part of PQ has the same Inclination to the Lines and Columns as PQ .

Wherefore PQ having given $y = Ax$, Pq will give $u = Bx$ (Art. LXXII.) consequently having employed PQ , the Determinator Pq is useless; but from the Point q there parts another Determinator which passes thro' the highest or lowest full Cell of the first Column QO , and which gives an Equation by which the second Term of the Series is determined.



But if $y - Ax = 0$ be a multiple Root of the Equation furnished by the Determinator PQ , for Example, a triple Root; then in the transformed the Cells Q , q , q , remain empty, and the Determinator Pq is terminated at the Cell R on the fourth Column, it is useless, for the Reasons already given, to employ this Determinator. But there parts another (RS) from the Cell R , which may terminate in S at the first Column, and this is the second Determinator which gives an Equation by which the second Term of the Series will be found.

How the irregular terms of a series are investigated.

It may happen that the Term S is wanting, and that the Determinator RS is terminated on some other Column than the first, as in T , and then this Point T gives rise to another Determinator TV ; and if this does not reach the first Column, but is terminated sooner, there parts from its Extremity another Determinator, and perhaps from that another, &c. to that one which is terminated on the first Column. Each

of those Determinators being differently inclined to the Lines and Columns, give a different Exponent to the Power of x in the second (or third, fourth, &c.) Term; whereby the Series is forked into as many Series as there are Roots in all the Equations given by all those Determinators.

To investigate which, it suffices to consider the Determinator RT , which parts from the Point R , Extremity of the first Determinator PR which has been neglected, and to make Use of all the Roots of the Equation which it furnishes; one of those Roots is $u = 0$, the Sum of the Terms which are under this Determinator RT being divisible by u , since RT does not reach the first Column, which is the Line without u ; and this Root being employed to obtain the following transformed Equation by substituting $0 + t$ for u , which is effected by only writing t for u in all the Terms of the Equation, which transformed consequently furnishes the same Determinators as the foregoing one. And as PR and RT have been already employed, we arrive at the Determinator TV in the same Manner, as if at first we had passed from PR to TV , the Root $u = 0$ of the Equation furnished by RT producing the same Effect as if RT had been neglected. It would be the same, if the Determinator TV did not reach to the first Column, but was accompanied with other Determinators.

XCVII.

From whence it appears how the *irregular Terms* of the Series are to be calculated. I call by this Name those which are given by Equations which have several Roots, real or imaginary. But it is scarce possible that this Kind of Disorder should last long; for as soon as we come to a Determinator whose Equation has no multiple Roots (or if a simple Root be employed, if this Equation has simple and multiple Roots) in the transformed Equation, there will be wanting of the Terms which are placed under this Determinator, for Example, RS , but the Term S which should be on the first Column (Art. xcv.). The Determinator therefore of this transformed will part from the Cell T the highest (or lowest) of the second Column, and will pass thro' the Cell V , also the highest (or the lowest) of the full Cells in the first Column; wherefore, in the Equation furnished by this Determinator TV , the variable Quantity u for Example does not surpass the first Degree, because T is on the second Column, which is the Line u ; and this Equation will only have one Root, which will certainly be real; and then the Series becomes regular, because all the ensuing Determinators parting from the Point T , no Equations any more occurs which have several Roots; and all the succeeding Terms of the Series may be calculated with greater Facility by a Method we shall now proceed to explain.

How to
know when
a Series
begins to
become
regular.

xcviii.

Suppose a Determinator to be obtained whose Equation has a simple Determinator Root, and that this Root is employed; for greater Simplicity I shall call it the first Determinator, abstracting from all the foregoing ones, if there were any; I shall also call the Equation which it furnishes the proposed one, altho' it may be one of the transformed ones. Let m express the Order of the Terms thro' which this first Determinator passes, and let $y - Ax^b = 0$ be a simple Root of the Equation it furnishes. Substituting $Ax^b + u$ for y in the Sum of the Terms of the Order m , the one which should be on the first Column would be x^m , we neglect its Coefficient as being of no Consequence; but this Term is wanting, since $y - Ax^b$ is supposed to divide the Sum of the Terms of the Order m (Art. xciii.). The Term $x^{m-b}u$ which follows falls on the second Column, and the Cell which it fills is the highest (or the lowest) of the full Cells of this Column. If $m \mp n$ is the Exponent of the Terms of the second Order (the Sign $-$ is for the descending Series, the Sign $+$ for the ascending ones) the highest (or the lowest) Term in the first Column will be $x^{m \mp n}$, hence the Determinator of this transformed passing thro' the Cells $x^{m-b}u$ and $x^{m \mp n}$ will give $u = Bx^{m \mp n - m + b} = Bx^{b \mp n}$ for the second Term of the Series. Wherefore the Difference $\mp n$ of the Exponents b and $b \mp n$ of the first Term Ax^b , and of the second $Bx^{b \mp n}$, of the Series, is the same as that of the Exponents m and $m \mp n$ of the first and second Order.

In all the following Transformations, the Cell $x^{m-b}u$ will remain full, u being changed successively into t , s , r , &c. and all the following Determinators will part from this Cell to pass thro' the highest (or the lowest) of the full Cells of the first Column; they pass successively thro' the different Cells of this Column, because at each transformation, the Cell thro' which the Determinator passed is emptied; but on the other Hand, each transformation fills some new Cells of this first Column.

Hence when $Bx^{b \mp n} + t$ is substituted for u in the first transformed, the Terms of the first Order m fill the Cells $x^{m \mp 1n}$, $x^{m \mp 2n}$, $x^{m \mp 3n}$, $x^{m \mp 4n}$, &c. and the Terms of the second Order $m \mp n$ also fill the Cells

ELEMENTS OF

$x^{m-2n}, x^{m-3n}, \&c.$ in general x^{m-jn} (j expressing any Integer) but the same Substitution in the Terms of the Order $m+p$ will fill in the first Column, the Cells $x^{m-p}, x^{m-p-n}, x^{m-p-2n}, \&c.$ in general x^{m-p-jn} .

If x^{m-2n} be the highest (or the lowest) full Cell in the first Column, the third Determinator will pass thro' x^{m-b} t and x^{m-2n} , and will give $t = Cx^{b-2n}$. And if afterwards x^{m-3n} is the highest (or the lowest) of the full Cells of this Column, s will be $= Dx^{b-3n}$, and thus the successive Exponents of x in the Terms $y, u, t, s, \&c.$ of the Series, will be $h, h-2n, h-4n, h-6n, \&c.$ in arithmetical Progression, whose Difference is n . The Series would have no other Terms, if there were in the proposed Equation no other Terms but those of the Orders m and $m+n$, all the Transformations, *ad infinitum*, would give no other Terms but such as are contained under this general Expression

x^{m-jn} (j is an Integer, or sometimes $= 0$). But if there be in the proposed Equation, Terms of another Order, whose Exponent is $m+p$, the Cell x^{m+p} will be once the highest (or the lowest) of the first Column; then the Determinator, which parts always from the Cell x^{m-b} u (or x^{m-b} t , or x^{m-b} s , &c.) will give $u = Bx^{b+p-m-2n}$ $= Bx^{b+p}$ (or $t = Cx^{b+p}$ or $s = Dx^{b+p}$; the Term in which x has for Exponent $h+p$ is therefore one of the Terms of the Series.

The substitution of $Bx^{b+p} + t$ for u (or $Cx^{b+p} + s$ for t , &c.) in the Terms of the Orders $m-jn$ will fill, in the first Column, the Cells $x^{m-jn+p}, x^{m-jn+p-n}, x^{m-jn+p-2n}, \&c.$ this first Column therefore will acquire Terms included under the general Expression $x^{m-jn+p-jp}$, and the Determinator passing successively thro' those Terms, will give to the Series, Terms contained under this general Expression $Hx^{b+jn+p-jp}$.

If the proposed Equation included a fourth Order of Terms whose Exponent is $m+q$, the general Expression of the Terms of the Series would be $Hx^{b+jn+p-jp+q}$, and so on, if there be a greater Number of Orders of Terms.

XCIX.

Hence, when in the Calculation of a Series we come to the *Regular Terms*, that is, when we come to a Determinator whose Equation has no multiple Roots, or when a simple Root of the Equation furnished by a Determinator is employed; the Succession of the Exponents of x in the following Terms of the Series are easily found thus: take the Exponents $m, m + n, m + p, m + q, \&c.$ of all the Orders of the Terms of the Equation, and subtract them all from the greater m (or subtract from them all, the least m) in order to obtain the Differences $n, p, q, \&c.$ then add or subtract from h , Exponent of the first Term, which is given by the first Determinator, the multiples $n, 2n, 3n, \&c.$ of the first Difference; then add to, or subtract from, all those Terms successively, the Multiples $p, 2p, 3p, \&c.$ of the second Difference; and add to, or subtract from all those Terms, the Multiples $q, 2q, 3q, \&c.$ of the third Difference successively, and continue thus until all the Differences are exhausted. Finally, range all those Exponents according to their Magnitude.

Rule derived from the foregoing considerations for finding the succession of exponents of the regular terms of a series.

The arithmetical Progression which begins by h , and whose Difference is the greatest common Divisor of $n, p, q, \&c.$ includes those Exponents, but contains useless Terms, unless the least Difference n be a common Divisor of all the other Differences.

C.

By this Rule, the Form of the Series is obtained, that is, the Succession of the Powers of x which form those Terms; but the Coefficients of those Terms are also to be found, which is easily effected, by giving to each of the Powers of x , which enters the Series, an undetermined Coefficient $A, B, C, D, \&c.$ and substituting, in the Equation, for y this undetermined Series which expresses its Value, and determining, one after the other, each Coefficient $A, B, C, D, \&c.$ by means of the complex Coefficient of each Term of the transformed Equation, which are equal to Nothing, since otherwise the Whole could not be equal to Nothing.

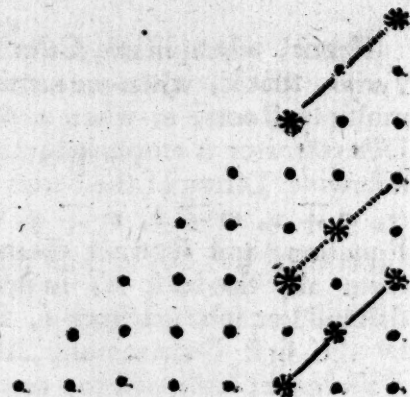
CI.

Let us apply this Method to the Equation $6x^7 - 2x^5y^2 - a^3x^2y^2 + 4a^3x^3y + 2a^5xx - 3a^5xy + a^5yy = 0$, suppose the Value of y in x , expressed by an ascending Series, is required.

Application of this rule to an example

The Equation being placed on the analytic Triangle, there is only one inferior Determinator, which gives the Equation $a^5yy - 3a^5xy + 2a^5xx = 0$, or $yy - 3xy + 2xx = 0$, which has two simple Roots $y = x$ and $y = 2x$, in general $y = Ax$, wherefore $h = 1$. Drawing straight Lines parallel to the Determinator thro' all the Terms of the Equation, it will appear that it is composed of three Orders of Terms; the first, which contains the Terms $a^5yy, 3a^5xy, 2a^5xx$, thro' which passes the Determinator, has 2 for its Exponent, because this straight Line cuts the first Column in the Centre of the Cell x^2 ;

the second includes the Terms— $a^3 x^2 y^2$ and $+4 a^3 x^3 y$, and its Exponent is 4, because the straight Line which passes thro' the Cells $x^2 y^2$ and $x^3 y$ cuts the first Column in the Centre of the Cell x^4 ; and the third Order of Terms, which is composed of the Terms— $2 x^5 y^2$ and $6 x^7$ has, for a similar Reason, 7 for its Exponent: those three Orders of Terms and their Exponents may be also discovered, by substituting in the Equation for y



its Value $Ax = Ax$, which transforms it into $6x^7 - 2A^2x^7 - A^2a^3x^4 + 4Aa^3x^4 + 2a^5xx - 3Aa^5xx + A^2a^5xx = 0$, where it evidently appears, that the two first Terms are of the Order 7, the two following of the Order 4, and the three last of the Order 2. Since an ascending Series is required, the least Exponent 2 is to be subtracted from the two others 4 and 7, by which the Differences 2 and 5 are obtained; then the Numbers contained in the general Expression $h + jn + jp$ or $1 + 2j + 5j$ is to be sought, by adding to 1 (h), the Multiples of 2 (n), and adding to all those Numbers the Multiples of 5 (p), which will give

1, 3, 5, 7, 9, 11, &c.

6, 8, 10, 12, 14, 16, &c.

11, &c.

16, &c.

21, &c.

which being ranged according to their Magnitude, constitute the Progression 1, 3, 5, 6, 7, 8, 9, 10, 11, &c. the Form of the Series will be therefore $y = Ax + Bx^3 + Cx^5 + Dx^6 + Ex^7 + Fx^8 + \&c.$ which Value of y substituted in the Equation gives

$$\begin{aligned}
 6x^7 &= \dots\dots\dots + 6x^7. \\
 -2x^5y^2 &= \dots\dots\dots - 2A^2x^7. \quad \bullet \quad - \&c. \\
 -a^3x^2y^2 &= \dots\dots\dots - a^3A^2x^4 - 2a^3ABx^6 \quad \bullet \quad - 2a^3ACx^8 - \&c. \\
 &\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad - a^3B^2x^8 - \&c. \\
 +4a^3x^3y &= \quad \quad \quad + 4a^3Ax^4 + 4a^3Bx^6 + \bullet \quad + 4a^3Cx^8 + \&c. \\
 +2a^5xx &= +2a^5xx \\
 -3a^5xy &= -3a^5Ax^4 - 3a^5Bx^6 - 3a^5Cx^8 - 3a^5Dx^7 - 3a^5Ex^8 - \&c. \\
 +a^5yy &= +a^5A^2xx + 2a^5ABx^4 + 2a^5ACx^6 + 2a^5ADx^7 + 2a^5AEx^8 + \&c. \\
 &\quad \quad \quad + a^5B^2x^6 + \bullet \quad + 2a^5BCx^8 + \&c.
 \end{aligned}$$

Now since $6x^7 - 2x^5y^2 - a^3x^2y^2 + 4a^3x^3y + 2a^5xx - 3a^5xy + a^5yy = 0$, it follows, that the Sum of these Series involving x must vanish; but that cannot be, if the Coefficient of every particular Term does not vanish. For every Term where x is infinitely little, is infinitely greater than the following Terms: So that if every Term does not vanish of itself, the addition or subtraction of the following Terms, which are infinitely less than it, or of the preceding Terms, which are infinitely greater, cannot destroy it, and therefore the Whole cannot vanish. It appears therefore, that the complex Coefficient of each Term being put equal to nothing, will furnish the following Equations:

$$\begin{aligned} 2a^5 - 3a^5A + a^5AA &= 0, \text{ or } AA - 3A + 2 = 0, \\ -a^3A^2 + 4a^3A - 3a^5B + 2a^5AB &= 0, \\ \text{or } AA - 4A &= (2aaA - 3aa)B, \\ -2a^3AB + 4a^3B - 3a^5C + 2a^5AC + a^5B^2 &= 0, \\ \text{or } 2AB - 4B - aBB &= (2aaA - 3aa)C, \\ 6 - 2A^2 - 3a^5D + 2a^5AD &= 0, \\ \text{or } 2AA - 6 &= (2aaA - 3aa)a^3D, \\ -2a^3AC - A^3B^2 + 4a^3C - 3a^5E + 2a^5AE + 2a^5BC &= 0, \\ \text{or } 2AC + B^2 - 4C - 2a^2BC &= (2a^2A - 3a^2)E, \\ \&c. & \&c. \end{aligned}$$

for determining $A, B, C, D, E, \&c.$ and give

$$\begin{aligned} A &= 1 \quad \dots \dots \dots \text{ or } A = 2 \\ B &= \frac{AA - 4A}{2aaA - 3aa} = \frac{3}{aa} \quad \dots \dots \dots B = -\frac{4}{aa} \\ C &= \frac{2AB - 4B - a^2B^2}{2aaA - 3aa} = \frac{15}{a^4} \quad \dots \dots \dots C = -\frac{16}{a^4} \\ D &= \frac{2AA - 6}{(2aaA - 3aa)a^3} = \frac{4}{a^5} \quad \dots \dots \dots D = +\frac{2}{a^5} \\ E &= \frac{2AC + B^2 - 4C - 2a^2BC}{2aaA - 3aa} = \frac{111}{a^6} \quad \dots \dots \dots E = -\frac{112}{a^6} \\ \&c. & \&c. \end{aligned}$$

There are two ascending Series, $y = x + \frac{3x^3}{aa} + \frac{15x^5}{a^4} + \frac{4x^6}{a^5} + \frac{111x^7}{a^6}, \&c.$ and $y = 2x - \frac{4x^3}{aa} - \frac{16x^5}{a^4} + \frac{2x^6}{a^5} - \frac{112x^7}{a^6}, \&c.$

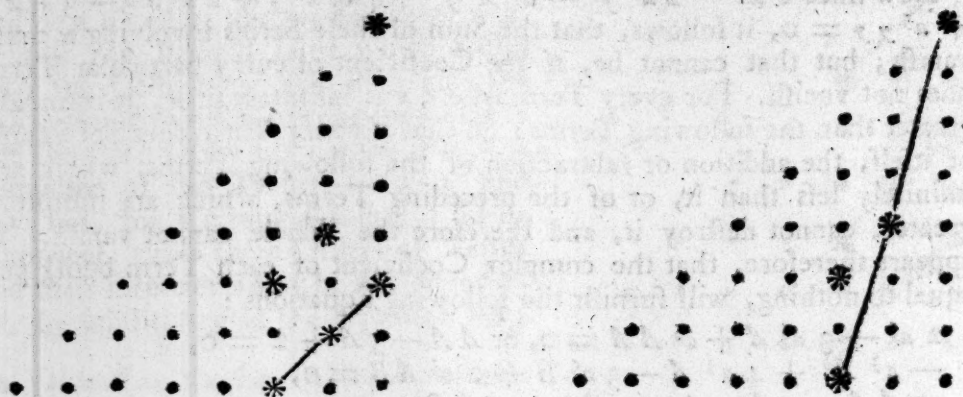
CII.

Let the Equation $x^7 - a^3x^3y + a^3x^2y^2 + a^5yy - 2a^5xy + a^5xx = 0$ be proposed, from which the Value of y in x , expressed by an ascending Series, is required to be deduced.

Application of this rule to another example.

This Equation being placed on the analytic Triangle, there will be found but one inferior Determinator, which gives the Equation $a^5yy - 2a^5xy + a^5xx = 0$ or $y^2 - 2xy + xx = 0$, which has

ELEMENTS OF



two equal Roots, or one double Root, $y = x$, whence $x + u$ being substituted for y in the proposed Equation, it will be transformed into $x^7 + a^3 x^3 u + a^3 x^2 u^2 + a^5 u u = 0$, which being in its turn placed on the analytic Triangle, there are two inferior Determinators, one of which gives the Equation $a^5 u u + a^3 x^3 u = 0$, or $u = -\frac{x^3}{a^2}$, and the other gives the Equation $a^3 x^3 u + x^7 = 0$, or $u = -\frac{x^4}{a^3}$, both one and the other Exponent of x exceeds the preceding one, consequently those two Determinators are useful; but it suffices to employ the first, which gives $u = -\frac{x^3}{a^2} = A x^3$, that is, $h = 3$ and $A = -\frac{1}{a^2}$, and substituting $A x^3$, or simply x^3 , in the Equation $x^7 + a^3 x^3 u + a^3 x^2 u^2 + a^5 u u = 0$, it will be changed into $x^7 + a^3 x^6 + a^3 x^8 + a^5 x^6 = 0$, which, it is manifest, includes three Orders of Terms, whose Exponents are 6, 7, 8, and the Differences 1, 2. Since the least divides the greatest, the Succession of the Exponents of x will be the arithmetical Progression 3, 4, 5, 6, &c. whose first Term is 3 (h), and the Difference 1 (n), the Form of the Series will be then $u = A x^3 + B x^4 + C x^5 + D x^6$, &c. and this Value of x substituted in the Equation gives

$$\begin{aligned}
 x^7 &= & + x^7 \\
 + a^3 x^3 u &= & + a^3 A x^6 + a^3 B x^7 + a^3 C x^8 + a^3 D x^9 + a^3 E x^{10} + \&c. \\
 + a^3 x^2 u^2 &= & + a^3 A^2 x^8 + 2 a^3 A B x^9 + 2 a^3 A C x^{10} + \&c. \\
 & & + a^3 B B x^{10} + \&c. \\
 + a^5 u u &= & - a^5 A^2 x^6 + 2 a^5 A B x^7 + 2 a^5 A C x^8 + 2 a^5 A D x^9 + 2 a^5 A E x^{10} + \&c. \\
 & & + a^5 B B x^8 + 2 a^5 B C x^9 + 2 a^5 B D x^{10} + \&c. \\
 & & + a^5 C C x^{10} + \&c.
 \end{aligned}$$

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From whence is deduced, by putting each complex Coefficient equal to Nothing,

$$a^3 A + a^5 A A = 0 \quad \text{that is, } A = -\frac{1}{a^2} \text{ or } A = 0$$

$$1 + a^3 B + 2 a^5 A B = 0 \quad B = \frac{1}{a^3} \quad B = -\frac{1}{a^3}$$

$$a^3 C + a^3 A A + 2 a^5 A C + a^5 B B = 0 \quad . . C = \frac{2}{4a^4} \quad C = -\frac{1}{a^4}$$

$$a^3 D + 2 a^3 A B + 2 a^5 A D + 2 a^5 B C = 0 \quad D = \frac{2}{a^5} \quad D = -\frac{2}{a^5}$$

$$a^3 E + 2 a^3 A C + a^3 B B + 2 a^5 A E + 2 a^5 B D + a^5 C C = 0. E = -\frac{7}{a^6} \quad E = -\frac{4}{a^6}$$

&c.

&c.

&c.

Hence u is expressed by two Series, to each of which adding x , we will have two Values of $y = x - \frac{x^3}{a^2} + \frac{x^4}{a^3} + \frac{2x^5}{a^4} + \frac{2x^6}{a^5} + \frac{7x^7}{a^6}$, &c.

and $y = x - \frac{x^4}{a^3} - \frac{x^5}{a^4} - \frac{2x^6}{a^5} - \frac{4x^7}{a^6}$, &c. this last Series is precisely

the same as that which would be obtained by the Means of the Determinator which gave the Equation $u = -\frac{x^4}{a^2}$ for substituting x^4 for u in

the Equation $x^7 + a^3 x^3 u + a^3 x^2 u^2 + a^5 u u = 0$, it will be changed into $x^7 + a^3 x^7 + a^3 x^{10} + a^5 x^8 = 0$, in which, it is manifest, there are three Orders of Terms, whose Exponents are 7, 8, 10, the Differences are therefore 1, 3; and as the least divides the greatest, the Succession of the Exponents of x is 4, 5, 6, 7, &c. and the Form of the Series $y = A x^4 + B x^5 + C x^6$, &c. whose Coefficients being determined, is

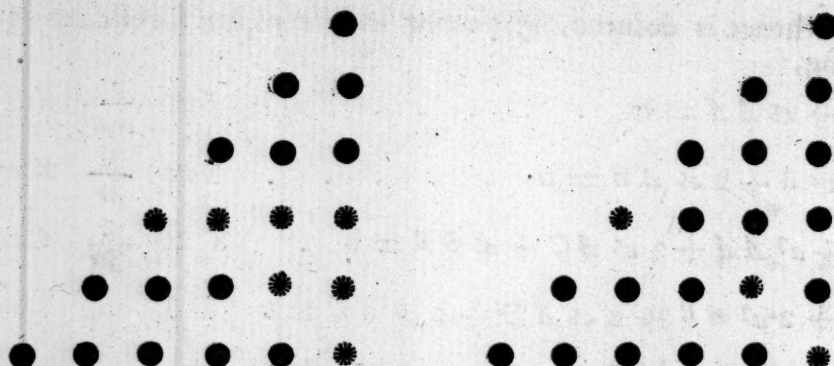
converted into $-\frac{x^4}{a^3} - \frac{x^5}{a^4} - \frac{2x^6}{a^5} - \frac{4x^7}{a^6}$, &c. hence this Determinator gives only the second of the two Series which the other had furnished.

CIII.

A descending Series is required, expressing the Value of y in x , deduced from this Equation $x^2 y^3 + 3 a x^2 y^2 + 3 a^2 x^2 y + a^3 x x - a^3 x y - a^4 x + a^5 = 0$.

This Equation being placed on the analytic Triangle, furnishes but one superior Determinator, which gives the Equation $x^2 y^2 + 3 a x^2 y^3 + 3 a^2 x^2 y + a^3 x^2 = 0$, or dividing by x^2 , $y^3 + 3 a y^2 + 3 a^2 y + a^3 = 0$, which has only one Root, but triple, $y = a$; substituting therefore $a + u$ for y in the proposed Equation, it will be transformed into $x^2 u^3 - a^3 x u + a^5 = 0$, which being placed on the analytic Triangle, furnishes two superior Determinators, one of which gives $x^2 u^3 - a^3 x u = 0$, or

$u = \pm a \quad x$, and the other $-a^3 x u + a^5 = 0$, or $u = a^2 x$.



But it suffices (Art. xciv.) to employ only the first, which gives $h = -\frac{1}{2}$, and substituting $x^{-1:2}$ for u in the transformed Equation, it

will change it into $x^{1:2} - a^3 x^{1:2} + a^5 x^0 = 0$, in which, it is manifest, there are only two Orders of Terms whose Exponents are $\frac{1}{2}$ and 0;

there is therefore only one Difference $\frac{1}{2}$, the Succession of Exponents are therefore $-\frac{1}{2}, -1, -\frac{3}{2}, -2$, &c. and the Form of the Series

$$Ax^{-1:2} + Bx^{-1} + Cx^{-3:2}, \text{ \&c. this Series substituted for } u \text{ in}$$

$$x^2 u^3 = A^3 x^{\frac{3}{2}} + 3AABx^0 + 3AACx^{-\frac{1}{2}} + 3AADx^{-1} + 3AAEx^{-\frac{3}{2}} + \&c.$$

$$+ 3ABB + 6ABC + 6ABD + 3ACC + 3BBC$$

$$- a^3 x u = -a^3 Ax^{\frac{1}{2}} - a^3 Bx^0 - a^3 Cx^{-\frac{1}{2}} - a^3 Dx^{-1} - a^3 Ex^{-\frac{3}{2}} - \&c.$$

$$+ a^5 = +a^5$$

furnishes the following Equations:

$$A^3 - a^3 A = 0 \quad \dots \quad \text{that is } A = \pm a^{3:2} \text{ or } A = 0$$

$$3AAB - a^3 B + a^5 = 0 \quad \dots \quad B = -\frac{1}{2}aa \quad B = aa$$

$$3AAC + 3ABB - a^3 C = 0 \quad \dots \quad C = \mp \frac{3}{8}a^{5:2} \quad C = 0$$

$$3AAD + 6ABC + B^3 - a^3 D = 0 \quad \dots \quad D = -\frac{1}{2}a^3 \quad D = a^3$$

$$3AAE + 6ABD + 3ACC + 3BBC - a^3 E = 0 \quad E = \pm \frac{105}{128}a^{7:2} \quad E = 0$$

$$\&c. \quad \&c. \quad \&c.$$

Hence we have three descending Series expressing the Value of u in x ,

$$\begin{aligned} \text{viz. } u &= a^{\frac{3:2}{x}} - \frac{1}{2} a a x^{-1} - \frac{3}{8} a^{\frac{5:2}{x}} - \frac{1}{2} a^3 x^{-2} \\ &+ \frac{105}{128} a^{\frac{7:2}{x}} - \frac{5:2}{x}, \text{ \&c. } u = -a^{\frac{3:2}{x}} - \frac{1}{2} a a x^{-1} + \\ &\frac{3}{8} a^{\frac{5:2}{x}} - \frac{1}{2} a^3 x^{-2} - \frac{105}{128} a^{\frac{7:2}{x}} - \frac{5:2}{x}, \text{ \&c. } u = a a x^{-1} \\ &+ a^3 x^{-2}, \text{ \&c. this last is precisely the same as would be obtained by the} \end{aligned}$$

Means of the second Determinator, which gave $u = a^2 x^{-1}$; for x substituted for u in the Equation $x^2 u^3 - a^3 x u + a^5$ transforms it into

$x^{-1} - a^3 + a^5 = 0$, which shews that there are two Orders of Terms, whose Exponents are -1 and 0 ; there is therefore only one Difference 1 , and the Form of the Series is $A x^{-1} + B x^{-2} + C x^{-3} + D x^{-4} + \text{\&c.}$ which Value being substituted for u in the Equation

$$x^2 u^3 = A^3 x^{-1} + 3 A A B x^{-2} + 3 A A C x^{-3} + 3 A A D x^{-4} + \text{\&c.}$$

$$\begin{array}{r} 3 A B B \\ + 6 A B C \\ + B^3 \end{array}$$

$$-a^3 x u = a^3 A - a^3 B x^{-1} - a^3 C x^{-2} - a^3 D x^{-3} - a^3 E x^{-4} - \text{\&c.}$$

$$+ a^5 = a^5 + \text{\&c.}$$

&c.
gives the following Equations:

$$\begin{array}{ll} -a^3 A + a^5 = 0 & \text{that is } A = a^2 \\ A^3 - a^3 B = 0 & B = a^3 \\ 3 A A B - a^3 C = 0 & C = 3 a^4 \\ 3 A A C + 3 A B B - a^3 D = 0 & D = 10 a^5 \\ 3 A A D + 6 A B C + B^3 - a^3 E = 0 & E = 49 a^6 \\ & \text{\&c.} \end{array}$$

Wherefore $u = a^2 x^{-1} + a^3 x^{-2} + 3 a^4 x^{-3} + 10 a^5 x^{-4} + 49 a^6 x^{-5} + \text{\&c.}$ thus the Value sought of y in x is expressed by three descending

Series, $y = a + a \sqrt{\frac{a}{x}} - \frac{a a}{2 x} - \frac{3 a a}{8 x} \sqrt{\frac{a}{x}} - \frac{a^3}{2 x x} + \frac{105 a^3}{128 x x} \sqrt{\frac{a}{x}} + \text{\&c.}$

$$y = a - a \sqrt{\frac{a}{x}} - \frac{a a}{2 x} + \frac{3 a a}{8 x} \sqrt{\frac{a}{x}} - \frac{a^3}{2 x x} - \frac{105 a^3}{128 x x} \sqrt{\frac{a}{x}} + \text{\&c.}$$

$$y = a + \frac{a a}{x} + \frac{a^3}{x x} + \frac{3 a^4}{x^3} + \frac{10 a^5}{x^4} + \frac{49 a^6}{x^5} + \text{\&c.}$$

When in an indeterminate Equation, the Value of one of the variable Quantities (y) is found expressed by an infinite Series, including in its Terms the Powers of the other variable Quantity (x), and it is required to find the Value of x in y , expressed by an infinite Series, whose Terms contain the Powers of y , this Method is called the Reversion of Series, to which the preceding Doctrine may be conveniently applied.

Reversion
of Series.

Suppose we had found $y = ax + bx^2 + cx^3 + dx^4 + ex^5$, &c. whose Coefficients a, b, c, d , &c. are known, and the Value of x in y is required, I put $0 = y - ax - bx^2 - cx^3 - dx^4$, &c. in which x being supposed very small, it will appear (Art. CXIX) that the Form of the Series must be $x = ly + my^2 + ny^3 + py^4$, &c. substituting this Value of x in the Equation, we will have

$$\begin{aligned}
 -y &= -y \\
 +ax &= +aly + amy^2 + any^3 + apy^4 + aqy^5 + ar y^6 \text{ \&c.} \\
 +bx^2 &= . . +bll y^2 + 2bml y^3 + bm^2 y^4 + 2blpy^5 + bbn y^6 \text{ \&c.} \\
 &\quad + 2blny^4 + 2bmny^5 + 2blqy^6 \text{ \&c.} \\
 &\quad + 2bm p y^6 \text{ \&c.} \\
 +cx^3 &= . . . +cl^3 y^3 + 3cllmy^4 + 3clm^2 y^5 + cm^3 y^6 \text{ \&c.} \\
 &\quad + 3clln y^5 + 6clmny^6 \text{ \&c.} \\
 &\quad + 3cllp y^6 \text{ \&c.} \\
 +dx^4 &= . . . +dl^4 y^4 + 4dl^3my^5 + 6dllmm y^6 \text{ \&c.} \\
 &\quad + 4dl^3ny^6 \text{ \&c.} \\
 +ex^5 &= . . . +el^5 y^5 + 5el^4my^6 \text{ \&c.} \\
 +fx^6 &= . . . + fl^6 y^6 \text{ \&c.}
 \end{aligned}$$

which gives the the following Equations:

$$\begin{aligned}
 alx - x &= 0 \quad . . . \quad \text{or } l = \frac{1}{a} \\
 am + bll &= 0 \quad . . . \quad m = -\frac{b}{a^3} \\
 an + 2bml + cl^3 &= 0 \quad . . . \quad n = \frac{2bb - ac}{a^6} \\
 ap + bm^2 + 2bln + 3cllm + dl^4 &= 0, p = \frac{5abc - 5b^3 - a^2d}{a^7} \\
 &\quad \text{\&c.} \quad \quad \quad \text{\&c.}
 \end{aligned}$$

and substituting those Values in $x = ly + my^2 + ny^3 + py^4$, &c. there will result $x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{2bb - ac}{a^5} y^3 + \frac{5abc - 5b^3 - a^2d}{a^7} y^4 + \frac{14b^4 - 21acbb + 6a^2bd + 3a^2cc - ca^3}{a^9} y^5 + \text{\&c.}$ which is the Value required of y .

When a Series is regular, that is, when a simple Root of the Equation furnished by the Determinator is employed, the Difference (n) of the

Exponents of x in the first and the second (Ax^b, Bx^{b+n}) Terms of the Series, is equal to the Difference of the Exponents ($m, m+n$) of the first and second Orders of Terms of the Equation; but it is not so if the Root $y - Ax^b = 0$ be Multiple. If the Degree of its Multiplicity be expressed by j , that is, if $y - Ax^b$ divides j Times, the Sum of the Terms of the first Order m , provided it does not divide the Sum of the Terms of the second Order $m+n$, the Difference $n - j$ of the Exponents of the first and second Terms of the Series $Ax^b + Bx^{b+n}$, &c. will be equal to $\frac{n}{j}$, which is the Difference n of the Exponents of the Orders m and $m+n$ divided by j , so that in the second Term, the Exponent of x will be ($i =$) $b + \frac{n}{j}$.

Observation whereby may be discovered in many cases if a Series is half imaginary.

For since $y - Ax^b = 0$ is a Root whose Degree of Multiplicity is j , when $Ax^b + u$ is substituted for y , there will be wanting in the transformed Equation the Terms corresponding to the Points where the Determinator cuts the j first Columns, (Art. xciv.) that is, the Terms

$x^m, x^{m-b}, x^{m-2b}, \dots, x^{m-jb}$, &c. continued to the Term x^{m-jb} , which will be the Term corresponding to the Extremity of this first Determinator; it is therefore from this Term that the second Determinator parts,

which will pass thro' the first Term of the second Order x^{m+n} whose Cell will not be empty, since $y - Ax^b$ is not a Divisor of the Sum of the Terms of the second Order. Hence the Equation which this second

Determinator furnishes, will be of this Form $x^{m-jb} u = Bx^{m+n}$

or $u = Bx^{m+n-m+jb}$, or $u = Bx^{b \pm \frac{n}{j}}$, the Exponent therefore of the second Term is $b \pm \frac{n}{j}$.

Hence we may discover, without Calculation, whether a Series whose first Term is real, be half imaginary; when the Root $y - Ax^b = 0$, furnished by the first Term, dividing several Times the Terms of the first Order m , does not divide those of the second Order $m \mp n$, which is a very ordinary Case. For if j , Degree of the multiplicity of this Root, is an even Number, and n , Difference of the Exponents of the Orders, an odd

Number; the second Term $(Bx^{b \pm \frac{n}{j}})$ of the Series is half imaginary (Art. LXXXIII.) and it is real if j be odd, but if n and j be both even, it will

be real or imaginary, according as the Coefficients of the Terms x^{m-jb} u

and $x^{m \mp n}$ have different Signs or the same Sign, but it is on this second Term it depends that the Series be real, or entirely or half imaginary, for the Equation which in the present Case furnishes this Term, consisting only of two Terms, will have no multiple Roots; from hence forth, the Series will become regular, and all its Terms, counting from the third, will be real.

END OF SPECIOUS ARITHMETICK.

E R R A T A.

Page 2. Art. II. Line 25. for, its triple more 410 makes 490, read, its triple more 410 makes 890.

P. 40. Art. LXIV, l. 9. for $ce = 18$, $bf = -12$, read, $ce = -12$, $bf = 18$.

P. 104. Art. LIX. l. 5. for $-3 (BA' - AB') (EB' - BE')$ read, $-3 (BA' - AB') (ED' - DE')$

P. 124. Art. x. l. 6. for, $\sqrt[3]{\frac{a^3 c^5}{b}} x$, read, $\sqrt[3]{\frac{a^2 c^5}{b}} x$. Line 7.

for, $= a^{\frac{3}{2}}$, read, $a^{\frac{2}{3}}$. For, $b^{\frac{3}{2}}$, read, $c^{\frac{3}{2}}$.

Page 149. for, $\sqrt[3]{5} \times \sqrt[3]{20} \times \sqrt[3]{10}$, read, $\sqrt[3]{5} + \sqrt[3]{20} + \sqrt[3]{10}$.

P. 160. last Line, for, $ap - a = zp^n$, read, $ap - a = zp^n - a$, for $a +$, read $a \times$.

P. 223. l. 7. for, $\frac{AdA + BdB}{AA + BB} x$, read, $\frac{AdA + BdB}{AA + BB} +$.

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CHAP. IV.

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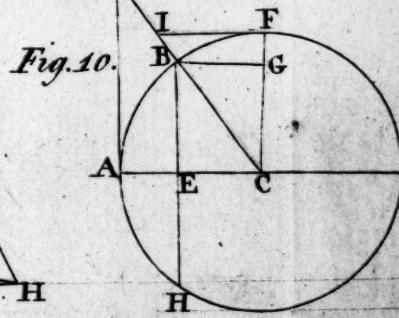
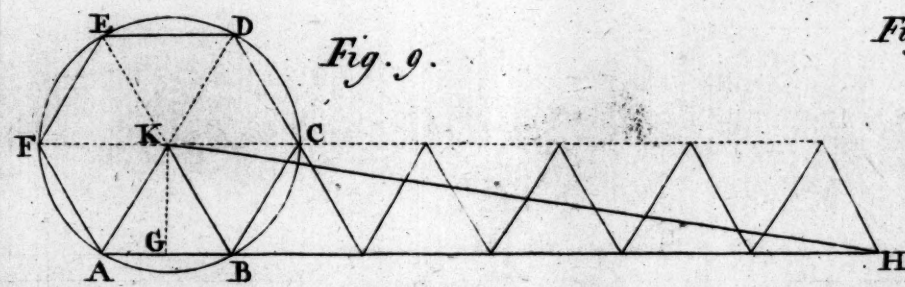
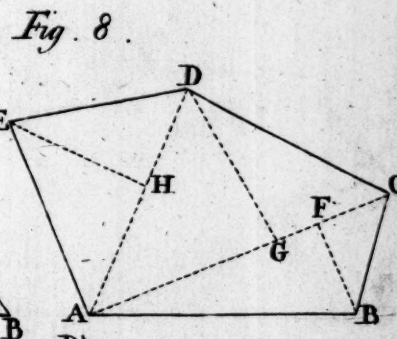
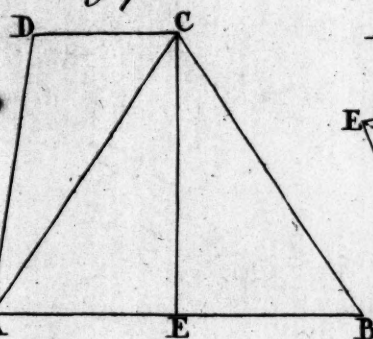
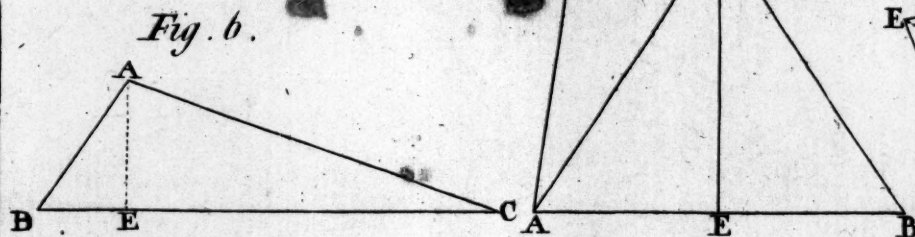
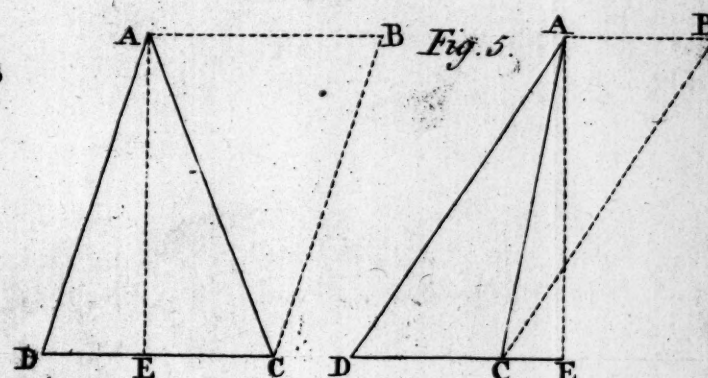
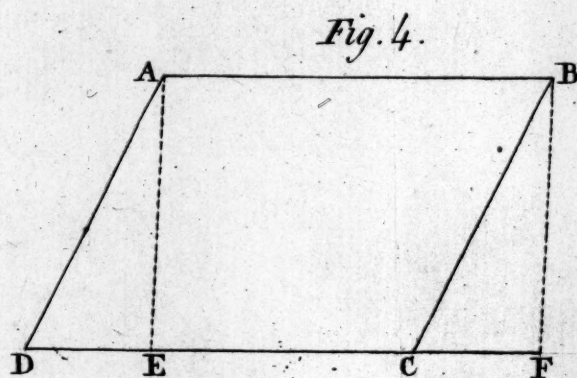
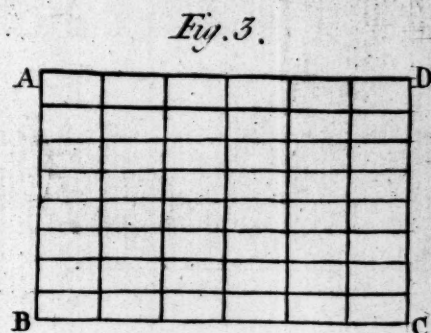
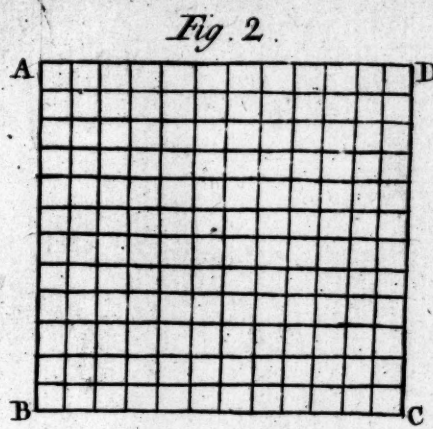
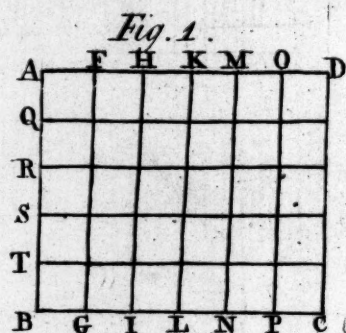


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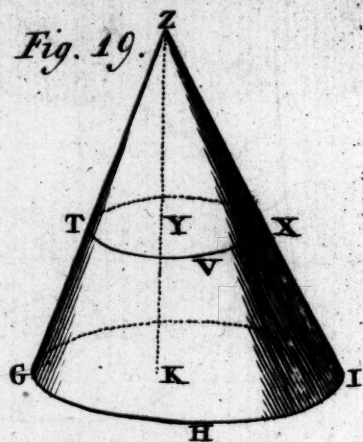


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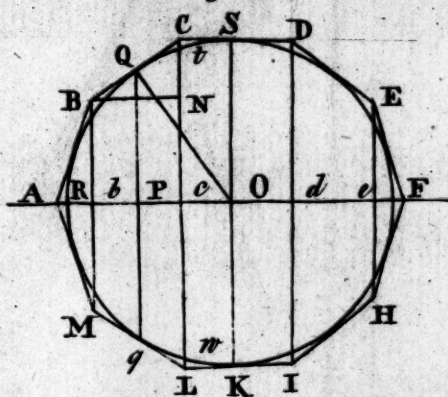


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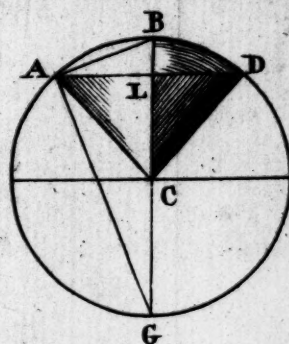


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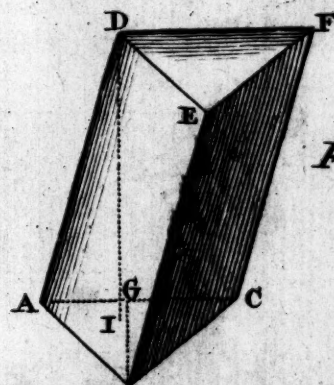
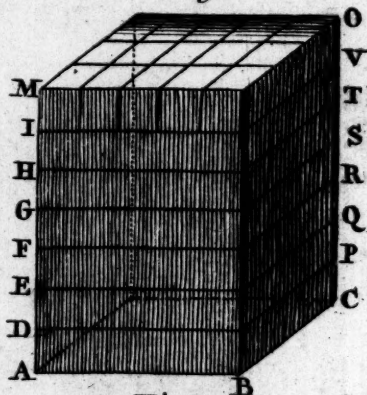


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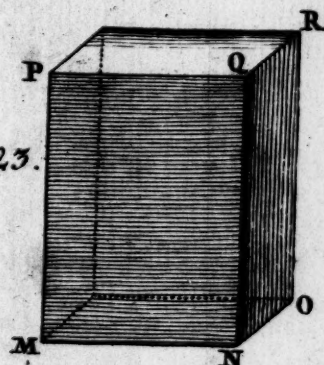


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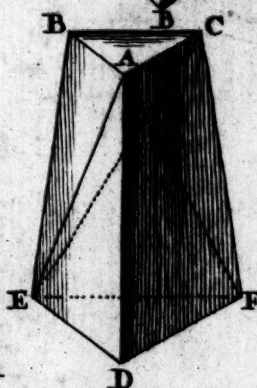
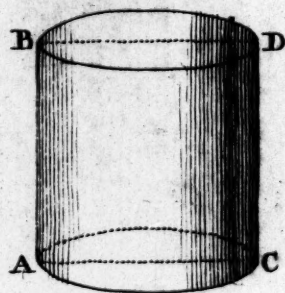


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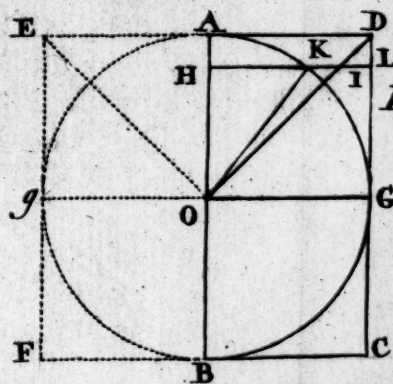
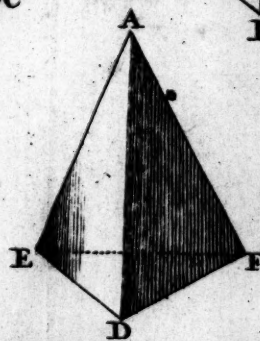


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